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On Some Convexity Questions of Handelman

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Abstract. **We resolve some questions posed by Handelman in 1996 concerning log convex L^1 functions. In particular, we give a negative answer to a question he posed concerning the integrability of $h^2(x)/h(2x)$ when h is L^1 and log convex and $h(n)^{1/n} \to 1$.**

1 Introduction

In [1], Handelman investigated eventual positivity of power series and deduced its existence for a wide variety of functions by appealing to a particular maximal function. If $h : (0, \infty) \to (0, \infty)$ is continuous, then he defined the maximal function

$$H_h(a) = \max_{b \geq a} \frac{h(b)}{h(a+b)}.$$

Note that H takes values in $[0, \infty]$. This maximal function was introduced in [1], where some of its properties are discussed. In particular, it is meaningful to have an understanding of when hH_h is integrable on $(0, \infty)$. It is easy to see that if h is log convex, then $H_h(x) = h(x)/h(2x)$. This led Handelman to ask the following question (see [1, page 338]):

Question 1 If $h : (0, \infty) \to (0, \infty)$ is a log convex function that is integrable on $(0, \infty)$ and satisfies $\limsup_{n\to\infty} h(n)^{1/n} = 1$, then is it true that $h(x)^2/h(2x)$ is also integrable on $(0, \infty)$?

One of our main results is a demonstration that the answer to Question 1 is "no." In fact, we will prove the following result:

***Theorem* 1.1** *There is a function $h : (0, \infty) \to (0, \infty)$ that is log convex, integrable on $(0, \infty)$, satisfies $\lim_{n\to\infty} h(n)^{1/n} = 1$, and is such that $h^r(x)/h(rx)$ is not integrable on $(0, \infty)$ for any $r > 0$.*

Our proof of Theorem 1.1 is constructive in that we will show how to actually create a counterexample.

Also in [1], Handelman made the following conjecture (see the discussion following [1, Theorem 9]):

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Suppose $h : (0,\infty) \longrightarrow (0,\infty)$ is a log convex function that satisfies

$$\lim_{n\to\infty} \frac{h(n)}{h(n+1)} = 1 \tag{1.1}$$

and $h^2(x)/h(2x)$ is integrable on $(0,\infty)$. Then h is also integrable on $(0,\infty)$.

Our second main result is the following:

Theorem **1.2** *Conjecture 1 is true.*

Remark. We should point out that the hypothesis (1.1) is essential to proving Theorem 1.2, for otherwise one could take $h(x) = x^x$ as a counterexample. We also remind the reader of an observation made by Handelman in [1, page 338], which is that the condition (1.1) for a log convex function h is equivalent to the condition that the series $\sum h(j)t^j$ has radius of convergence exactly 1.

In addition to above Question and Conjecture, Handelman also asked: if a function h is log convex and integrable on $(0,\infty)$, then is $h^2(x)/h(2x)$ also integrable on $(0,\infty)$ (see [1, page 331])? We believe this question is still open.

The remainder of the paper is devoted to the proofs of Theorem 1.1 and Theorem 1.2. Our methods are elementary and require only basic facts about convex functions (see [2] for a discussion of many tools in convexity theory).

2 Proofs

The purpose of this section is to prove all of the results discussed in the introduction.

2.1 The Construction

In this section, we will resolve Question 1. Let us write $h(x) = \exp(f(x))$ where $f(x)$ is convex. Since $h \in L^1(\mathbb{R}^+)$, it must be that $\lim_{x\to\infty} f(x) = -\infty$. The remaining condition on h implies $\lim_{n\to\infty} f(n)/n = 0$, or equivalently (by the convexity of f) $\lim_{x\to\infty} f'(x) = 0$ provided $f'(x)$ exists. In fact, the function f we construct will be piecewise linear and hence $f'(x)$ will be undefined on a discrete set. We will choose sequences $\{a_n\}_{n=0}^{\infty}$, $\{m_n\}_{n=0}^{\infty}$, and $\{b_n\}_{n=0}^{\infty}$ so that

$$f(x) = m_n x + b_n \qquad , \qquad x \in [a_n, a_{n+1}], \tag{2.1}$$

and f is continuous.

To begin our construction, let $\{m_n\}_{n=0}^{\infty}$ be a fixed sequence of negative real numbers that monotonically increases to 0. With this fixed sequence in hand, we will construct the sequence $\{a_n\}_{n=0}^{\infty}$ inductively, and the sequence $\{b_n\}_{n=0}^{\infty}$ will then be defined implicitly in order to make f continuous.

We begin our construction of the sequence $\{a_n\}_{n\geq 0}$ by defining $a_0 = 0$ and we also define $b_0 = 0$. Now, choose a_1 large enough so that

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$$-\frac{e^{m_0 a_1}}{m_1} < \frac{1}{2}$$

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$$a_1 > 1 = a_0 + e^{-b_0}.$$

Now set $b_1 = m_0 a_1 - m_1 a_1$ and observe that $m_1 a_1 + b_1 = m_0 a_1$.

Now let us assume that $\{a_j\}_{j=0}^n$ and $\{b_j\}_{j=0}^n$ have already been defined. We will now show how one can choose a_{n+1} and then b_{n+1} to complete the construction. Indeed, as above, we will choose a_{n+1} large enough so that

$$-\frac{e^{m_n a_{n+1}+b_n}}{m_{n+1}} < \frac{1}{2^{n+1}},$$

$$a_{n+1} > \max\left\{a_n + e^{(1-t)b_n} : t \in [0, n]\right\}.$$

Then define

$$b_{n+1} = m_n a_{n+1} + b_n - m_{n+1} a_{n+1}$$

and observe that

$$m_{n+1} a_{n+1} + b_{n+1} = m_n a_{n+1} + b_n. \tag{2.2}$$

Proceeding inductively, we arrive at two sequences $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$. It is clear from our construction that $a_{n+1} > a_n + 1$ (since $1 \in [0, n]$) and so each a_n is positive (except a_0) and $\lim_{n\to\infty} a_n = \infty$. Therefore this procedure defines f on all of $(0, \infty)$ if we define f by (2.1). Notice also that $a_n > 0$ and $m_n > m_{n-1}$ inductively implies that each $b_n < 0$ when $n > 0$.

Now let us check that this function has the desired properties. First of all, since $m_n \to 0$ monotonically, it is clear that $f(n)/n \to 0$ and also that f is convex. Now we calculate

$$\begin{aligned}
\int_0^\infty h(x)dx = \int_0^\infty e^{f(x)} dx &= \sum_{n=0}^\infty \int_{a_n}^{a_{n+1}} e^{m_n x + b_n} dx \\
&\leq \sum_{n=0}^\infty \int_{a_n}^\infty e^{m_n x + b_n} dx \\
&= \sum_{n=0}^\infty -\frac{e^{m_n a_n + b_n}}{m_n} \\
&= -\frac{1}{m_0} + \sum_{n=1}^\infty -\frac{e^{m_{n-1} a_n + b_{n-1}}}{m_n} \qquad \text{(we use (2.2) here)} \\
&< -\frac{1}{m_0} + \sum_{n=1}^\infty 2^{-n},
\end{aligned}$$

which is clearly finite. Therefore, $h \in L^1(\mathbb{R}^+)$ as desired.

Finally, fix $r \in (0, \infty)$ and choose $N \in \mathbb{N}$ so that $r < N$. Notice that

$$\frac{h^r(x)}{h(rx)} = e^{(r-1)b_n} \qquad , \qquad x \in [a_n, a_{n+1}].$$

Therefore,

$$\int_0^\infty \frac{h^r(x)}{h(rx)}dx = \sum_{n=0}^{\infty} e^{(r-1)b_n}(a_{n+1}-a_n) > \sum_{n=N}^{\infty} e^{(r-1)b_n}(a_{n+1}-a_n) > \sum_{n=N}^{\infty} 1,$$

by construction, so $\frac{h^r(x)}{h(rx)} \notin L^1(\mathbb{R}^+)$. This completely answers Question 1.

2.2 The Conjecture

In this section, we will prove Theorem 1.2. The log convexity of h implies that h is either monotone increasing on (A,∞) for some $A \geq 0$ or monotone decreasing on $(0,\infty)$. In the latter case, we have $h(x) \geq h(2x)$ and so

$$\int_0^\infty h(x)dx \leq \int_0^\infty \frac{h^2(x)}{h(2x)}dx < \infty,$$

so h is integrable.

If h is monotone increasing on (A,∞), then $\log(h(x))$ is also increasing on (A,∞). Note that the convexity implies that we can choose A so that $\log(h(x))$ is strictly monotone increasing on (A,∞), for otherwise h would be constant on some interval $[B,\infty)$. This would contradict the assumption that $h^2(x)/h(2x)$ is integrable on $(0,\infty)$. Since $\log(h(x))$ is convex, it must be that there is some constant $c > 0$ so that

$$\log(h(n+1)) - \log(h(n)) \geq c, \qquad n > A.$$

This implies $h(n+1)/h(n) \geq e^c$, which means h cannot satisfy (1.1). Therefore, this case cannot occur, and we have proven Theorem 1.2.

References

[1] D. Handelman, *Eventual positivity for analytic functions*, Math. Ann., 304 (1996), no. 2, 315–338.
[2] B. Simon, *Convexity: An Analytic Viewpoint*, Cambridge Tracts in Mathematics, 187. Cambridge University Press 2011.

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