Preface

This book is an introduction to the theory of categories, together with applications of this theory to some constructions with rings and modules. We start with a discussion of categories in general, and then concentrate on the types of category – additive, abelian – which enjoy to an increasing extent the properties of categories of modules. Our applications are the Morita theory, the localization and completion of rings and modules, and finally some 'local-global' methods, in which the properties of a module are compared to those of its localizations and completions. We also develop the tools that we need for these applications, namely, the tensor product and limits, both direct and inverse.

The selection and presentation of our material is motivated by the needs of algebraic K-theory. Indeed, this book started out as some preliminary remarks within a text on that subject. Thus the categorical foundations are those needed to set up algebraic K-theory, and our applications are chosen since they underly some fundamental results in algebraic K-theory. However, the content of this text will be, we hope, of interest to a wider readership than potential K-theorists.

Here is a more detailed survey of the material that we cover. Our first chapter sets out the basics of category theory. There are three fundamental definitions, those of a category, a functor, and a natural transformation, and we show how to use these notions to define universal objects and universal constructions. An understanding of these is important for two reasons. On the one hand, many definitions in module theory (kernels, cokernels, ...) can be reinterpreted in the language of universal objects and so extended to more general situations; on the other, many of the objects in K-theory itself arise as universal objects in one category or another.

A category of modules has a richer structure than an abstract category, since the additive structure on modules extends to their homomorphisms.

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The second chapter shows how this extra structure can be axiomatized and thus imposed on abstract categories. We first analyse the properties of the groups of homomorphisms in a module category to obtain a list of axioms that define additive and abelian categories; these are abstract categories that share most of the important properties of module categories. The final topic in this chapter stems from the fact that modules can be arranged in short exact sequences, which is basic to the definitions in algebraic K-theory. We are thus led to the idea of an exact category, which is a category together with a nicely behaved collection of short exact sequences. Such categories are the domains on which K-theory operates.

The larger part of this text is more directly concerned with categories of modules, rather than categories in the abstract. In the third chapter, we introduce the tensor product and use it to change the ring of scalars of a module. This leads to the Morita theory (Chapter 4), which tells us when two rings have, in some sense, the same module theory.

In Chapters 5 and 6, we look at several related methods for constructing new rings and categories from old, by taking direct and inverse limits and by performing localizations and completions.

Finally, we exploit these techniques to obtain some local-global results, mainly in the theory of orders. These tackle the problem of gaining information about the projective modules over a ring from knowledge of the modules over its localizations and completions. Roughly speaking, the structure of a projective module is transparent locally, that is, over the completions of a ring. The game is to assemble this information to give the structure over the original ring. The full story requires K-theory for its telling.

Although the novelty of a book such as this lies mainly in its selection and presentation of material, we do make some innovations and give some new arguments, including the following.

- In our treatment of modules, we deal both with categories of left modules and with categories of right modules, and it seems expedient to have separate systems of notation for their homomorphisms, according to the rule that homomorphisms should act 'opposite scalars'. This separation leads us to two types of abstract category, namely a left category and a right category, in which the rules for composition of morphisms follow those of the corresponding module category. These two types do not seem to have been distinguished explicitly before. We introduce the term *chirality* for the 'handedness' of a category, and with each category we associate a formal *mirror* category of opposite chirality.
- In K-theory, the term 'exact category' is used for a category together with

some specified exact sequences, which is different to the usage of the term by people who are primarily concerned with category theory. To complicate the issue a bit more, there are two types of exact category in K-theory. The first type has sufficient structure for the purposes of 'lower' K-theory, while the second type has extra structure that is needed to support 'higher' K-theory. We therefore coin the terms 'G-exact category' and 'Q-exact category' for the two types of exact category used in K-theory.

- We give direct proofs of some results that are normally obtained by using the methods of homological algebra, which appears in this text only incidentally. These results mainly concern flat modules (particularly section 3.2), and the local-global properties of modules over orders (section 7.3).
- Recently, Quillen has developed a module theory for nonunital rings. We incorporate some of his results, mostly as exercises.
- The treatments of localization of categories (section 6.3), projective module lifting (section 7.2) and local-global methods (section 7.3) use the minimum of apparatus, so as to make them more accessible to a beginning graduate student.

This text requires the reader to have prior knowledge of some basic results in ring theory, module theory and number theory, since these provide the examples that motivate and illustrate our discussions. These prerequisites are generally met in the first year of graduate studies, although it is unlikely that any one graduate course includes all of them. We have therefore collected them together in a companion volume, [BK: IRM]. Much of the ring theory and module theory can also be found in the expository works [Cohn 1979], [Cohn 1982], [Jacobson 1985] and [Jacobson 1980], with some extra topics that can be met in [Cohn 1985], [McConnell & Robson 1987] or [Rowen 1988]. Sources for the number theory are [Fröhlich & Taylor 1991] and [Marcus 1977]. However, for brevity, we usually give references only to [BK: IRM].

We try to keep to a fairly leisurely pace throughout this text, and, when in doubt, we tend to include an explanation rather than exclude it. Some of the more frequently used prerequisites are recapitulated for convenience.

The symbol □ indicates either the end of a proof, or that a proof is immediate from the preceding discussion. The symbol ○ after a statement indicates that the assertion is quoted from another source, rather than being a consequence of arguments in this text.

There are about two hundred exercises for more ambitious or enthusiastic readers. Some of them (among the exercises!) are fairly routine, others are rather deeper and explore further developments of our results or give brief introductions to topics that are not covered in the body of the text. We also use the exercises to provide examples and counterexamples that indicate the limitations of our results.

A number of historical remarks are distributed within the text. Readers in search of further details may wish to consult [Bourbaki 1994], [Dieudonné 1989], [Mac Lane 1975] and [Weibel].

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