2

Radial Sound Speeds

In this chapter we will discuss geometric inverse problems in a ball with radial sound speed. The fact that the sound speed is radial is a strong symmetry condition, which allows one to determine the behaviour of geodesics and solve related inverse problems quite explicitly. We will restrict our attention to the two-dimensional case, since the general case of a ball with radial sound speed in $\mathbb{R}^n$ reduces to this by looking at two-dimensional slices through the origin.

We first discuss geodesics of a radial sound speed satisfying the important *Herglotz condition*, using the Hamiltonian approach to geodesics and Cartesian coordinates. We then prove the classical result of Herglotz (1907) that travel times uniquely determine a radial sound speed of this type. Next we switch to polar coordinates and study geodesics of a rotationally symmetric metric, and prove that the geodesic X-ray transform is injective. The main point is that the geodesic equation can be integrated explicitly by quadrature, and a function can be determined from its integrals over geodesics using suitable changes of coordinates and inverting Abel-type transforms. Finally, we give examples of manifolds (surfaces of revolution) where the geodesic X-ray transform is injective or is not injective.

2.1 Geodesics of a Radial Sound Speed

The fact that the geodesics of a radial sound speed can be explicitly determined is related to the existence of multiple conserved quantities in the Hamiltonian approach to geodesics. We first recall this approach.

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2.1.1 Geodesics as a Hamilton Flow

Let $M \subset \mathbb{R}^n$, let x be the standard Cartesian coordinates in $\mathbb{R}^n$, and let $g = (g_{jk}(x))_{j,k=1}^n$ be a Riemannian metric on M. A curve $x(t) = (x^1(t), \ldots, x^n(t))$ is a geodesic if and only if it satisfies the geodesic equations

$$\ddot{x}^l(t) + \Gamma^l_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \tag{2.1}$$

where Γ^l_{jk} are the Christoffel symbols given by

$$\Gamma^l_{jk} = \frac{1}{2} g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}).$$

Recall that (g^{lm}) is the inverse matrix of (g_{jk}), and that we are using the Einstein summation convention where a repeated index in upper and lower position is summed. We will assume that all geodesics have unit speed, i.e.

$$|\dot{x}(t)|_g = \sqrt{g_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t)} = 1.$$

In this section we will also use the Euclidean length of vectors $x \in \mathbb{R}^n$, written as

$$|x|_e = \sqrt{x_1^2 + \cdots + x_n^2}.$$

We recall that the geodesic equations are often derived by using the *Lagrangian approach* to classical mechanics: they arise as the Euler–Lagrange equations that are satisfied by critical points of the length functional $L(x) = \int_a^b |\dot{x}(t)|_g \, dt$. We will now switch to the *Hamiltonian approach*, which considers the position $x(t)$ and momentum $\xi(t)$, where $\xi(t)$ is the covector corresponding to $\dot{x}(t)$, simultaneously.

Writing

$$\xi_j(t) := g_{jk}(x(t))\dot{x}^k(t), \qquad f(x,\xi) := \sqrt{g^{jk}(x)\xi_j\xi_k},$$

a short computation shows that the geodesic equations (for unit speed geodesics) are equivalent with the Hamilton equations

$$\begin{cases} \dot{x}(t) &= \nabla_\xi f(x(t), \xi(t)), \\ \dot{\xi}(t) &= -\nabla_x f(x(t), \xi(t)). \end{cases} \tag{2.2}$$

Here $f(x,\xi) = |\xi|_g$ (*speed*, or square root of kinetic energy) is called the *Hamilton function*, and it is defined on the cotangent space

$$T^*M = \{(x,\xi)\,;\, x \in M,\, \xi \in \mathbb{R}^n\} = M \times \mathbb{R}^n \subset \mathbb{R}^{2n}.$$

The operators ∇_x and ∇_ξ are the standard (Euclidean) gradient operators with respect to the x and ξ variables.

Exercise 2.1.1 Show that (2.1) is equivalent with (2.2).

Writing $\gamma(t) = (x(t), \xi(t))$ and using the Hamilton vector field H_f on T^*M, defined by

$$H_f := \nabla_\xi f \cdot \nabla_x - \nabla_x f \cdot \nabla_\xi = (\nabla_\xi f, -\nabla_x f),$$

we may write the Hamilton equations as

$$\dot{\gamma}(t) = H_f(\gamma(t)).$$

Definition 2.1.2 A function $u = u(x, \xi)$ is called a *conserved quantity* or a *first integral* if it is constant along the Hamilton flow, i.e. $t \mapsto u(x(t), \xi(t))$ is constant for any curve $(x(t), \xi(t))$ solving (2.2).

Now (2.2) implies that

$$\begin{aligned} & u \text{ is conserved,} \\ \iff \quad & \frac{d}{dt} u(x(t), \xi(t)) = 0, \\ \iff \quad & H_f u(x(t), \xi(t)) = 0. \end{aligned}$$

Since

$$H_f f = (\nabla_\xi f, -\nabla_x f) \cdot (\nabla_x f, \nabla_\xi f) = 0,$$

the Hamilton function f (speed) is always conserved.

Let now $M \subset \mathbb{R}^2$, and consider a metric of the form

$$g_{jk}(x) = c(x)^{-2} \delta_{jk},$$

where $c \in C^\infty(M)$ is positive. Then $f(x, \xi) = c(x)|\xi|_e$ and, writing $\hat{\xi} = \frac{\xi}{|\xi|_e}$,

$$H_f = c(x)\hat{\xi} \cdot \nabla_x - |\xi|_e \nabla_x c(x) \cdot \nabla_\xi.$$

Define the *angular momentum*

$$L(x, \xi) = \xi \cdot x^\perp, \qquad x^\perp = (-x_2, x_1).$$

When is L conserved? We compute

$$H_f L = c(x)\hat{\xi} \cdot (-\xi^\perp) - |\xi|_e \nabla_x c(x) \cdot x^\perp = -|\xi|_e \nabla_x c(x) \cdot x^\perp.$$

Thus $H_f L = 0$ if and only if $\nabla c(x) \cdot x^\perp = 0$, which is equivalent with the fact that c is radial.

Lemma 2.1.3 *The angular momentum L is conserved if and only if*

$$c = c(r), \qquad r = |x|_e.$$

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If $M \subset \mathbb{R}^2$ and $c(x)$ is radial, then the Hamilton flow on T^*M (a four-dimensional manifold) has two independent conserved quantities (the speed f and angular momentum L). One says that the flow is *completely integrable*, which implies that the geodesic equations can be solved quite explicitly by quadrature using the conserved quantities f and L. See e.g. (Taylor, 2011, chapter 1) for more details on these facts.

2.1.2 Geodesics of a Radial Sound Speed

We will now begin to analyze geodesics in this setting, following the presentation in Bal (2019). Let $M = \overline{\mathbb{D}} \setminus \{0\}$ where $\mathbb{D}$ is the unit disk in $\mathbb{R}^2$. Assume that

$$g_{jk}(x) = c(r)^{-2}\delta_{jk}, \qquad r = |x|_e,$$

where $c \in C^\infty([0,1])$. Note that the origin is a special point and $g_{jk}(x)$ is not necessarily smooth there; hence we will consider geodesics only away from the origin.

We write

$$r(t) = |x(t)|_e, \qquad \hat{x} = \frac{x}{|x|_e}.$$

Then $f(x,\xi) = c(r)|\xi|_e$ and the Hamilton equations (2.2) become

$$\begin{cases} \dot{x}(t) = c(r(t))\hat{\xi}(t), \\ \dot{\xi}(t) = -|\xi(t)|_e c'(r(t))\hat{x}(t). \end{cases} \tag{2.3}$$

Consider geodesics starting on $\partial\mathbb{D}$, i.e. $r(0) = 1$, and write

$$\xi(0) = \frac{1}{c(1)}\left(-\sqrt{1-p^2}x(0) + px(0)^\perp\right), \qquad 0 < p < 1. \tag{2.4}$$

Note that $\xi(0)$ points inward, and hence also $\dot{x}(0) = c(1)^2\xi(0)$ points inward. The normalization yields $|\dot{x}(0)|_g = |\xi(0)|_g = 1$, so that the geodesic has unit speed.

We wish to study how deep the geodesic goes into M, which boils down to understanding $r(t)$. Computing the derivative of $r(t)$ gives

$$\dot{r} = \frac{x \cdot \dot{x}}{|x|_e} = \frac{c(r)}{r|\xi|_e}(x \cdot \xi). \tag{2.5}$$

In particular, we see that $\dot{r}(t)$ has the same sign as $x(t)\cdot\xi(t)$. The latter quantity can be analyzed by (2.3). We compute

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$$\begin{aligned}\frac{d}{dt}(x \cdot \xi) &= \dot{x} \cdot \xi + x \cdot \dot{\xi} = |\xi|_e (c - rc'(r)) \\ &= c^2 |\xi|_e \frac{d}{dr}\left(\frac{r}{c(r)}\right)\Big|_{r=r(t)}. \end{aligned} \tag{2.6}$$

Next we make use of the conserved quantities:

$$f \text{ conserved} \implies c(r(t))|\xi(t)|_e = 1 \implies |\xi(t)|_e = \frac{1}{c(r(t))}, \tag{2.7}$$

$$L \text{ conserved} \implies \xi(t) \cdot x(t)^{\perp} = \xi(0) \cdot x(0)^{\perp}. \tag{2.8}$$

Then (2.6) becomes

$$\frac{d}{dt}(x \cdot \xi) = c(r) \frac{d}{dr}\left(\frac{r}{c(r)}\right)\Big|_{r=r(t)}. \tag{2.9}$$

Remark 2.1.4 We note that one can derive a useful ordinary differential equation (ODE) for $r(t)$. By (2.5) one has $\dot{r} = c(\hat{x} \cdot \hat{\xi})$. Decompose $\hat{\xi} = (\hat{\xi} \cdot \hat{x})\hat{x} + (\hat{\xi} \cdot \hat{x}^{\perp})\hat{x}^{\perp}$. Noting that $|\hat{x} \cdot \hat{\xi}| = \sqrt{1 - (\hat{\xi} \cdot \hat{x}^{\perp})^2} = \sqrt{1 - \left(\frac{pc(r)}{rc(1)}\right)^2}$ by (2.7), (2.8), and (2.4), we see that $r(t)$ solves the equation

$$\dot{r} = \pm c(r) \sqrt{1 - \left(\frac{pc(r)}{rc(1)}\right)^2}, \qquad \pm \xi \cdot \hat{x} \geq 0. \tag{2.10}$$

This is an autonomous ODE for $r(t)$ (all other dependence on t has been eliminated).

To simplify the behaviour of geodesics we would like that $\dot{r}(t)$ has a unique zero at some $t = t_p$, is negative for $t < t_p$, and is positive for $t > t_p$. This means that geodesics curve back toward the boundary after they reach their deepest point. Since $\dot{r}(t)$ has the same sign as $x(t) \cdot \xi(t)$, the identity (2.9) shows that this is guaranteed by the following important condition.

Definition 2.1.5 We say that a radial sound speed $c \in C^{\infty}([0, 1])$ satisfies the *Herglotz condition* if

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0, \qquad r \in [0, 1]. \tag{2.11}$$

Assuming this condition we can describe the behaviour of geodesics.

Theorem 2.1.6 *Assume that* $c \in C^{\infty}([0, 1])$ *satisfies the Herglotz condition. Let* $0 < p < 1$, *and consider the geodesic with* $x(0) \in \partial\mathbb{D}$ *and* $\xi(0)$ *given by* (2.4). *There is a unique time* $t_p > 0$ *such that*

$$\dot{r}(t) < 0 \;\; \textit{for} \;\; 0 \leq t < t_p, \qquad \dot{r}(t_p) = 0, \qquad \dot{r}(t) > 0 \;\; \textit{for} \;\; t_p < t \leq 2t_p.$$

One has $0 < r(t) < 1$ *for* $0 < t < 2t_p$ *and* $r(0) = r(2t_p) = 1$. *Moreover, the geodesic is symmetric with respect to* $t = t_p$ *so that* $x(t_p + s) = R_p(x(t_p - s))$ *where* R_p *is reflection about* $\hat{x}(t_p)$.

Proof By (2.4) one has

$$x(0) \cdot \xi(0) = -c(1)^{-1}\sqrt{1 - p^2} < 0, \qquad (2.12)$$

and (2.5) implies that $\dot{r}(0) < 0$. Thus $x(t)$ stays in $\overline{\mathbb{D}} \setminus \{0\}$ at least for a short time. Note also that by (2.7) (conservation of f) and the positivity of c, one has $|\xi(t)|_e \geq \varepsilon_0 > 0$ whenever the geodesic is defined.

Let T be the maximal time of existence of the geodesic $x(t)$, i.e.

$$T = \sup\left\{\bar{t} > 0\,;\, x|_{[0,\bar{t})} \text{ stays in } \overline{\mathbb{D}} \setminus \{0\}\right\}.$$

There are two ways that $x(t)$ can exit $\overline{\mathbb{D}} \setminus \{0\}$: either $x(t)$ can go to 0, or $x(t)$ can go to $\partial\mathbb{D}$. Let us show that the first case cannot happen. If $x|_{[0,\bar{t})}$ stays in $\overline{\mathbb{D}} \setminus \{0\}$ and $x(t_j) \to 0$ as $t_j \to \bar{t}$, then (2.8) implies that $\xi(0) \cdot x(0)^{\perp} = 0$. But (2.4) gives that $\xi(0) \cdot x(0)^{\perp} = p/c(1)$, which is impossible since we assumed that $0 < p < 1$. This shows that either $T = \infty$, or T is finite and $x(T) \in \partial\mathbb{D}$.

Now we go back to (2.9) and note that the positivity of c and the Herglotz condition (2.11) imply that

$$\frac{d}{dt}(x(t) \cdot \xi(t)) \geq \varepsilon_0 > 0, \qquad t \in [0, T).$$

Thus $x(t) \cdot \xi(t)$ is strictly increasing. By (2.12) one has $x(0) \cdot \xi(0) < 0$ and

$$x(t) \cdot \xi(t) \geq x(0) \cdot \xi(0) + \varepsilon_0 t, \qquad t \in [0, T). \qquad (2.13)$$

Now if $x(t) \cdot \xi(t)$ were negative for $t \in [0, T)$, then by (2.5) $r(t)$ would be strictly decreasing for $t \in [0, T)$, and the maximal time would be $T = \infty$ since $x(t)$ could not go to $\partial\mathbb{D}$. This is a contradiction with (2.13), hence there must be a unique $t_p > 0$ with $x(t_p) \cdot \xi(t_p) = 0$. By (2.5) one has $\dot{r}(t) < 0$ for $t < t_p$, $\dot{r}(t_p) = 0$, and also $\dot{r}(t) > 0$ for $t > t_p$ since $x(t) \cdot \xi(t)$ is strictly increasing.

The other claims follow if we can show the symmetry $x(t_p + s) = R_p(x(t_p - s))$. Since everything is rotationally symmetric, we may assume that $\hat{x}(t_p) = (1, 0)$ and $R_p(x_1, x_2) = (x_1, -x_2)$. Define $\eta(s) = (x(t_p + s), \xi(t_p + s))$ and $\zeta(s) = (R_p(x(t_p - s)), -R_p(\xi(t_p - s)))$. Then both $\eta(s)$ and $\zeta(s)$ satisfy the Hamilton equations (2.3) with the same initial data when $s = 0$ (since $x(t_p) \cdot \xi(t_p) = 0$), and the symmetry condition follows by uniqueness for ODEs. $\square$

2.2 Travel Time Tomography

We will now consider a variant of the travel time tomography problem discussed in the introduction, and prove the classical result of Herglotz (1907) showing that travel times uniquely determine a radial sound speed satisfying the Herglotz condition.

If $c \in C^\infty([0,1])$ satisfies the Herglotz condition, then by Theorem 2.1.6 the unit speed geodesic starting at $x(0) \in \partial\mathbb{D}$ having codirection $\xi(0) = \frac{1}{c(1)}(-\sqrt{1-p^2}x(0) + px(0)^\perp)$ where $0 < p < 1$ returns to $\partial\mathbb{D}$ after time $2t_p$. Note that the travel time $2t_p$ does not depend on the choice of $x(0) \in \partial\mathbb{D}$ because of radial symmetry. Thus we may define the travel time function

$$T_c(p) = 2t_p, \qquad 0 < p < 1.$$

Theorem 2.2.1 (Travel time tomography) *Assume that $c \in C^\infty([0,1])$ is positive and satisfies the Herglotz condition. From the knowledge of the value $c(1)$ and the travel times*

$$T_c(p), \qquad 0 < p < 1,$$

one can determine $c(r)$ for $r \in (0,1]$.

Remark 2.2.2 The problem of determining a radial sound speed from travel time measurements was known to geophysicists in the early twentieth century. A mathematical treatment based on inverting Abel integrals was given in Herglotz (1907) and independently in Bateman (1910), and the problem was further analyzed in Wiechert and Geiger (1910). In geophysics the approach based on these ideas goes by the names of Herglotz, Wiechert, and Bateman.

To prove this theorem, we start with the ODE (2.10), which gives that

$$\frac{dr}{dt} = c(r)\sqrt{1 - \left(\frac{pc(r)}{rc(1)}\right)^2}, \qquad t_p \leq t \leq 2t_p.$$

We use this fact and a change of variables to obtain

$$T_c(p) = 2t_p = 2\int_{t_p}^{2t_p} dt = 2\int_{r_p}^{1} \frac{1}{c(r)\sqrt{1 - \left(\frac{pc(r)}{rc(1)}\right)^2}}\, dr, \tag{2.14}$$

where $r_p = r(t_p)$. Thus, from the measurements $T_c(p)$ with $0 < p < 1$ we know the integrals (2.14) involving $c(r)$. We wish to recover $c(r)$ from these integrals.

To simplify (2.14), we make the change of variables

$$u = \left(\frac{c(1)r}{c(r)}\right)^2. \tag{2.15}$$

This is a valid change of variables by the Herglotz condition (2.11). Note that since $\dot{r}(t_p) = 0$, the ODE (2.10) shows that $r_p = r(t_p)$ satisfies

$$\frac{r_p}{c(r_p)} = \frac{p}{c(1)}.$$

Hence $r = r_p$ corresponds to $u = p^2$. Then $T_c(p)$ becomes

$$T_c(p) = \frac{2}{c(1)} \int_{p^2}^1 \frac{dr}{du} \frac{u}{r} \frac{1}{\sqrt{u - p^2}} \, du. \tag{2.16}$$

This is an *Abel integral*, of the kind encountered in Abel (1826) when determining the profile of a hill by measuring the time it takes for a particle with different initial positions to roll down the hill. This work of Abel is considered to be the first appearance of an integral equation in mathematics.

These Abel integrals can be inverted by the following result, where we also pay attention to various mapping properties of the Abel transform. See Gorenflo and Vessella (1991) for a detailed treatment of Abel integral equations.

Theorem 2.2.3 (Abel transform) *Let $\alpha < \beta$, and define the* Abel transform

$$Au(x) := \int_x^\beta \frac{1}{(y - x)^{1/2}} u(y) \, dy, \qquad \alpha < x \leq \beta.$$

The Abel transform takes $L^1_{\mathrm{loc}}((\alpha, \beta])$ to itself. Define the space

$$\mathcal{A}((\alpha, \beta]) := \left\{ f \in L^1_{\mathrm{loc}}((\alpha, \beta]) \,;\, Af \in W^{1,1}_{\mathrm{loc}}((\alpha, \beta]) \right\}.$$

The Abel transform is a bijective map between the following spaces:

$$A\colon\ L^1_{\mathrm{loc}}((\alpha, \beta]) \to \mathcal{A}((\alpha, \beta]), \tag{2.17}$$

$$A\colon\ \mathcal{A}((\alpha, \beta]) \to \{f \in W^{1,1}_{\mathrm{loc}}((\alpha, \beta]) \,;\, f(\beta) = 0\}, \tag{2.18}$$

$$A\colon\ C^\infty((\alpha, \beta]) \to \{(\beta - x)^{1/2} h(x) \,;\, h \in C^\infty((\alpha, \beta])\}. \tag{2.19}$$

Given any $f \in \mathcal{A}((\alpha, \beta])$, the equation $Au = f$ has a unique solution $u \in L^1_{\mathrm{loc}}((\alpha, \beta])$ given by the formula

$$u(y) = -\frac{1}{\pi} \frac{d}{dy} \int_y^\beta \frac{f(x)}{(x - y)^{1/2}} \, dx. \tag{2.20}$$

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If additionally $f \in W^{1,1}_{\mathrm{loc}}((\alpha,\beta])$ *with* $f(\beta) = 0$*, one has the alternative formula*

$$u(y) = -\frac{1}{\pi}\int_y^\beta \frac{f'(x)}{(x-y)^{1/2}}\,dx. \tag{2.21}$$

Remark 2.2.4 Here

$$L^1_{\mathrm{loc}}((\alpha,\beta]) = \{u\,;\, u|_{[\gamma,\beta]} \in L^1([\gamma,\beta]) \text{ whenever } \alpha < \gamma < \beta\},$$

and similarly for $W^{1,1}_{\mathrm{loc}}((\alpha,\beta])$. Recall that in one dimension $W^{1,1}$ coincides with the space of absolutely continuous functions, and hence functions in $W^{1,1}_{\mathrm{loc}}((\alpha,\beta])$ can be evaluated pointwise at β.

Proof If $\alpha < \gamma < \beta$, we may use Fubini's theorem to show that

$$\begin{aligned}\int_\gamma^\beta |Au(x)|\,dx &\le \int_\gamma^\beta\int_x^\beta \frac{|u(y)|}{(y-x)^{1/2}}\,dy\,dx = \int_\gamma^\beta\int_\gamma^y \frac{|u(y)|}{(y-x)^{1/2}}\,dx\,dy\\ &= 2\int_\gamma^\beta (y-\gamma)^{1/2}|u(y)|\,dy \le 2(\beta-\gamma)^{1/2}\int_\gamma^\beta |u(y)|\,dy.\end{aligned}$$

This shows that A maps $L^1_{\mathrm{loc}}((\alpha,\beta])$ to itself. We use the definition of A and Fubini's theorem to compute

$$\begin{aligned}A^2u(z) &= \int_z^\beta \frac{Au(x)}{(x-z)^{1/2}}\,dx = \int_z^\beta\int_x^\beta \frac{u(y)}{(x-z)^{1/2}(y-x)^{1/2}}\,dy\,dx\\ &= \int_z^\beta\int_z^y \frac{u(y)}{(x-z)^{1/2}(y-x)^{1/2}}\,dx\,dy.\end{aligned}$$

The last quantity may be written as $\int_z^\beta k(z,y)u(y)\,dy$ where, using the change of variables $x = z + (y-z)w$,

$$k(z,y) = \int_z^y \frac{1}{(x-z)^{1/2}(y-x)^{1/2}}\,dx = \int_0^1 \frac{1}{w^{1/2}(1-w)^{1/2}}\,dw.$$

Thus $k(z,y)$ is a constant, given by the beta function $B(\frac{1}{2},\frac{1}{2}) = \pi$. The constant can be computed directly as follows: changing variables $w = \frac{1}{2} + \frac{1}{2}v$ and $v = \sin\theta$ gives

$$\int_0^1 \frac{1}{w^{1/2}(1-w)^{1/2}}\,dw = \int_{-1}^1 \frac{1}{\sqrt{1-v^2}}\,dv = \int_{-\pi/2}^{\pi/2} d\theta = \pi.$$

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This shows that for any $u \in L^1_{\mathrm{loc}}((\alpha,\beta])$, one has

$$A^2 u(z) = \pi \int_z^\beta u(y)\,dy. \tag{2.22}$$

Thus $(A(Au))'(z) = -\pi u(z)$, so A maps $L^1_{\mathrm{loc}}((\alpha,\beta])$ into $\mathcal{A}((\alpha,\beta])$.

We next show that the map (2.17) is bijective. By (2.22), if $Au = 0$ it follows that $u \equiv 0$, so A is injective. Now let $f \in \mathcal{A}((\alpha,\beta])$. Setting $u := -\frac{1}{\pi}\frac{d}{dx}Af$, we have $u \in L^1_{\mathrm{loc}}((\alpha,\beta])$ and

$$\pi \int_z^\beta u(y)\,dy = Af(z),$$

since one always has $Af(\beta) = 0$. Combining this with (2.22) we get $Af = A(Au)$, and since A is injective we have $Au = f$. We have proved that (2.17) is bijective and that one has the inversion formula (2.20).

Next let $f \in W^{1,1}_{\mathrm{loc}}((\alpha,\beta])$ with $f(\beta) = 0$, and integrate by parts to obtain

$$\begin{aligned} Af(x) &= \int_x^\beta f(y)\frac{d}{dy}\big(2(y-x)^{1/2}\big)\,dy \\ &= -2\int_x^\beta (y-x)^{1/2} f'(y)\,dy. \end{aligned}$$

It follows that $Af \in L^1_{\mathrm{loc}}((\alpha,\beta])$ and

$$(Af)'(x) = \int_x^\beta \frac{f'(y)}{(y-x)^{1/2}}\,dy = A(f')(x).$$

By (2.20) the function $u := -\frac{1}{\pi}(Af)'$ satisfies $Au = f$. But now one also has $u = -\frac{1}{\pi}A(f')$, which proves the second inversion formula (2.21). The fact that (2.18) is a bijective map follows immediately.

Finally, if $u \in C^\infty((\alpha,\beta])$ we change variables $y = x + (\beta - x)s$ and obtain

$$Au(x) = \int_x^\beta \frac{u(y)}{(y-x)^{1/2}}\,dy = (\beta - x)^{1/2}\int_0^1 \frac{u(x + (\beta - x)s)}{s^{1/2}}\,ds.$$

Since u is smooth, one has $Au(x) = (\beta - x)^{1/2}h(x)$ where $h \in C^\infty((\alpha,\beta])$. Conversely, if $f(x) = (\beta - x)^{1/2}h(x)$ where $h \in C^\infty((\alpha,\beta])$, the change of variables $x = y + (\beta - y)s$ gives

$$\int_y^\beta \frac{f(x)}{(x-y)^{1/2}}\,dx = (\beta - y)\int_0^1 \frac{(1-s)^{1/2}h(y + (\beta - y)s)}{s^{1/2}}\,ds.$$

If u is defined by (2.20), we see that $u \in C^\infty((\alpha,\beta])$ and u solves $Au = f$. Thus (2.19) is a bijective map. □

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We now return to (2.16). Since the value $c(1)$ is known, using (2.16) and Theorem 2.2.3 we can determine the function $f(u) := \frac{dr}{du}\frac{u}{r(u)}$ from the knowledge of $T_c(p)$ for $0 < p < 1$. We rewrite this as $\frac{d}{du}\log r(u) = \frac{f(u)}{u}$, which shows that we can recover the function

$$r(u) = \exp\left(-\int_u^1 \frac{f(v)}{v}\,dv\right).$$

By taking the inverse function, we can determine $u(r)$. By (2.15), we have determined the function $c(r) = c(1)r/\sqrt{u(r)}$. This completes the proof of Theorem 2.2.1.

Remark 2.2.5 If we assume that the sound speed extends smoothly to $M := \overline{\mathbb{D}}$, then Theorem 2.2.1 can be reformulated using the notation of Chapter 3 as follows: if g_1 and g_2 are two Riemannian metrics on M corresponding to radial sound speeds satisfying the Herglotz condition, if $g_1|_{\partial M} = g_2|_{\partial M}$ and if $\tau_{g_1}|_{\partial_+ SM} = \tau_{g_2}|_{\partial_+ SM}$ (the travel times of maximal geodesics for g_1 and g_2 agree), then $g_1 = g_2$.

In the boundary rigidity problem, one considers measurements given by the boundary distance function $d_g|_{\partial M\times\partial M}$ instead of the travel time function τ_g. It follows from equation (11.2) that if $d_{g_1}|_{\partial M\times\partial M} = d_{g_2}|_{\partial M\times\partial M}$ and the boundary is strictly convex, then $g_1|_{\partial M} = g_2|_{\partial M}$. Moreover, if the manifolds are simple then by Proposition 11.3.2 one has $\tau_{g_1}|_{\partial_+ SM} = \tau_{g_2}|_{\partial_+ SM}$. Thus, in the setting of simple metrics, Theorem 2.2.1 also solves the boundary rigidity problem for radial sound speeds.

Remark 2.2.6 Theorem 2.2.1 assumes that $c(1)$, i.e. $g|_{\partial M}$, is known. Often one can determine $g|_{\partial M}$ by looking at short geodesics. However, in the present setting one gets something slightly different. In (2.16), write $f(u) = \frac{dr}{du}\frac{u}{r(u)}$ and note that f is smooth in $[p^2, 1]$. The change of variables $u = p^2+(1-p^2)s$ yields

$$\int_{p^2}^1 \frac{f(u)}{\sqrt{u-p^2}}\,du = \left(1-p^2\right)^{1/2}\int_0^1 \frac{f\left(p^2+(1-p^2)s\right)}{s^{1/2}}\,ds.$$

Thus we obtain

$$\lim_{p\to 1}\frac{T_c(p)}{\sqrt{1-p^2}} = \frac{4f(1)}{c(1)}.$$

From (2.15) we see that $\frac{du}{dr} = c(1)^2\left(\frac{2r}{c(r)^2} - \frac{2r^2c'(r)}{c(r)^3}\right)$. This implies that $f(1) = \frac{dr}{du}(1) = (2 - \frac{2c'(1)}{c(1)})^{-1} = \frac{c(1)}{2(c(1)-c'(1))}$. Hence, by looking at travel times of short geodesics, one recovers the quantity $c(1) - c'(1)$ from $T_c(p)$.

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2.3 Geodesics of a Rotationally Symmetric Metric

For the rest of this chapter, it will be convenient to switch from Cartesian coordinates (x_1, x_2) to polar coordinates (r, θ), where $x = (r\cos\theta, r\sin\theta)$. Recall that the Euclidean metric $g = dx_1^2 + dx_2^2$ looks like $g = dr^2 + r^2\, d\theta^2$ in polar coordinates. Hence the metric $g = c(r)^{-2}(dx_1^2 + dx_2^2)$ with radial sound speed $c(r)$ becomes

$$g = c(r)^{-2}\, dr^2 + (r/c(r))^2\, d\theta^2. \tag{2.23}$$

We will work in the region $M = \{(r,\theta)\,;\, r_0 < r \le r_1\}$ where $r_0 < r_1$ (note that r_0 is not necessarily required to be positive), and consider metrics of the form

$$g = a(r)^2\, dr^2 + b(r)^2\, d\theta^2, \tag{2.24}$$

where $a, b \in C^\infty([r_0, r_1])$ are positive. Clearly this includes metrics (2.23) with radial sound speed, with $a(r) = 1/c(r)$ and $b(r) = r/c(r)$. However, the two forms turn out to be equivalent:

Exercise 2.3.1 Show that a metric of the form (2.24) can be put in the form (2.23) by a change of variables.

Working with the form (2.24) will be useful in view of the following example.

Example 2.3.2 (Surfaces of revolution) Let r be the z-coordinate in $\mathbb{R}^3$, and let $h\colon [r_0, r_1] \to \mathbb{R}$ be a smooth positive function. Let S be the surface of revolution obtained by rotating the graph of $r \mapsto h(r)$ about the z-axis. The surface S is given by $S = \{q(r,\theta)\,;\, r \in (r_0, r_1],\, \theta \in [0, 2\pi]\}$ where

$$q(r,\theta) = (h(r)\cos\theta, h(r)\sin\theta, r).$$

Then S has tangent vectors

$$\partial_r = (h'(r)\cos\theta, h'(r)\sin\theta, 1),$$
$$\partial_\theta = (-h(r)\sin\theta, h(r)\cos\theta, 0).$$

Equip S with the metric g induced by the Euclidean metric in $\mathbb{R}^3$. Since $\partial_r \cdot \partial_r = 1 + h'(r)^2$, $\partial_r \cdot \partial_\theta = 0$ and $\partial_\theta \cdot \partial_\theta = h(r)^2$, one has

$$g = (1 + h'(r)^2)\, dr^2 + h(r)^2\, d\theta^2.$$

Thus, surfaces of revolution have metrics of the form (2.24), where $a(r) = \sqrt{1 + h'(r)^2}$ and $b(r) = h(r)$.

The geodesic equations for the metric (2.24) can be determined by computing the Christoffel symbols

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$$\Gamma^l_{jk} = \frac{1}{2} g^{lm}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}).$$

A direct computation shows that

$$\Gamma^1_{11} = \partial_r a/a, \quad \Gamma^1_{12} = \Gamma^1_{21} = 0, \quad \Gamma^1_{22} = -b\partial_r b/a^2,$$
$$\Gamma^2_{11} = 0, \quad \Gamma^2_{12} = \Gamma^2_{21} = \partial_r b/b, \quad \Gamma^2_{22} = 0.$$

Thus the geodesic equations are

$$\ddot{r} + \frac{\partial_r a}{a}(\dot{r})^2 - \frac{b\partial_r b}{a^2}(\dot{\theta})^2 = 0, \tag{2.25}$$
$$\ddot{\theta} + \frac{2\partial_r b}{b}\dot{r}\dot{\theta} = 0. \tag{2.26}$$

The conserved quantities (speed and angular momentum) corresponding to (2.7) and (2.8) are given as follows:

$$(a(r)\dot{r})^2 + (b(r)\dot{\theta})^2 \text{ is conserved,} \tag{2.27}$$
$$b(r)^2\dot{\theta} \text{ is conserved.} \tag{2.28}$$

In fact, the first quantity is conserved since geodesics have constant speed, and the fact that the second quantity is conserved follows directly by taking its t-derivative and using the second geodesic equation.

As in Theorem 2.1.6, we would like that when a geodesic reaches its deepest point where $\dot{r} = 0$, it turns back toward the surface (i.e. $\ddot{r} > 0$). Now (2.25) implies that

$$\dot{r} = 0 \implies \ddot{r} = \frac{b\partial_r b}{a^2}(\dot{\theta})^2.$$

Thus, when $\dot{r} = 0$, one has $\ddot{r} > 0$ if and only if $b' > 0$. This is the analogue of the Herglotz condition. For a radial sound speed as in (2.23), one has $b(r) = r/c(r)$ and the condition $b' > 0$ is equivalent with $\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0$.

Definition 2.3.3 A metric $g = a(r)^2\,dr^2 + b(r)^2\,d\theta^2$, where $a, b \in C^\infty([r_0, r_1])$ are positive, satisfies the *Herglotz condition* if

$$b'(r) > 0, \qquad r \in [r_0, r_1].$$

The following result is the analogue of Theorem 2.1.6.

Theorem 2.3.4 (Geodesics) *Let g satisfy the Herglotz condition as in Definition 2.3.3. Let $(r(t), \theta(t))$ be a unit speed geodesic with $r(0) = r_1$ and $\dot{r}(0) < 0$. There are two types of geodesics: either $r(t)$ strictly decreases to $\{r = r_0\}$ in finite time, or the geodesic stays in M and goes back to $\{r = r_1\}$*

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in finite time. Geodesics of the second type have a unique closest point (ρ, α) *to the origin, and they consist of two symmetric branches where first* $r(t)$ *strictly decreases from* r_1 *to* ρ*, and then* $r(t)$ *strictly increases from* ρ *to* r_1*. Moreover, for any* $(\rho, \alpha) \in M$ *there is a unique such geodesic* $\gamma_{\rho,\alpha}(t) = (r(t), \theta(t))$ *with* $\dot{\theta}(0) > 0$*, and it satisfies*

$$\dot{r} = \mp \frac{1}{a(r)b(r)} \sqrt{b(r)^2 - b(\rho)^2}, \tag{2.29}$$

$$\theta(t) = \alpha \mp b(\rho) \int_{\rho}^{r(t)} \frac{a(r)}{b(r)} \frac{1}{\sqrt{b(r)^2 - b(\rho)^2}}\, dr, \tag{2.30}$$

where $-$ *corresponds to the first branch where* $r(t)$ *decreases, and* $+$ *corresponds to the second branch where* $r(t)$ *increases.*

Proof Since the geodesic has unit speed, (2.27) implies that

$$(a(r)\dot{r})^2 + (b(r)\dot{\theta})^2 = 1. \tag{2.31}$$

Moreover, (2.28) implies that

$$b(r)^2\dot{\theta} = p \tag{2.32}$$

for some constant p. Combining the (2.31) and (2.32) gives that $(a(r)\dot{r})^2 + (p/b(r))^2 = 1$, and thus

$$(a(r)\dot{r})^2 = 1 - \frac{p^2}{b(r)^2}. \tag{2.33}$$

Let I be the maximal interval of existence of the geodesic $(r(t), \theta(t))$ in M, so I is of the form $[0, T)$, $[0, T]$, or $[0, \infty)$ for some $T > 0$. Now, since $\dot{r}(0) < 0$, there are two possible cases: either $\dot{r}(t) < 0$ for all $t \in I$, or $\dot{r}(\bar{t}) = 0$ for some $\bar{t} \in I$. Assume that we are in the first case. Taking the t-derivative in (2.33) gives

$$2a(r)\dot{r}\frac{d}{dt}(a(r)\dot{r}) = 2p^2 b(r)^{-3} b'(r)\dot{r}, \qquad t \in I.$$

Since $\dot{r}(t) < 0$ for all $t \in I$, we may divide by $\dot{r}$ and obtain

$$\frac{d}{dt}(a(r)\dot{r}) = \frac{p^2 b(r)^{-3} b'(r)}{a(r)}, \qquad t \in I.$$

Using the Herglotz condition we have $b'(r) > 0$ for all $r \in [r_0, r_1]$. Thus there are $\varepsilon_0 > 0$ and $c_0 \in \mathbb{R}$ so that

$$a(r)\dot{r} \geq c_0 + \varepsilon_0 t, \qquad t \in I. \tag{2.34}$$

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Now if $T = \infty$ one would get $\dot r(\bar t) = 0$ for some $\bar t \in I$, which is a contradiction. Hence in the first case where $\dot r(t) < 0$ for all $t \in I$, the geodesic must reach $\{r = r_0\}$ in finite time and $r(t)$ is strictly decreasing.

Assume now that we are in the second case where $\dot r(t) < 0$ for $0 \le t < \bar t$ and $\dot r(\bar t) = 0$ for some $\bar t \in I$. Let $\rho = r(\bar t)$ and $\alpha = \theta(\bar t)$. Since both $\eta(s) = (r(\bar t + s), \theta(\bar t + s))$ and $\zeta(s) = (r(\bar t - s), 2\alpha - \theta(\bar t - s))$ solve the geodesic equations with the same initial data when $s = 0$, the geodesic has two branches that are symmetric with respect to $t = \bar t$. Note that we must have $p = \pm b(\rho)$ upon evaluating (2.33) at $t = \bar t$. If additionally $\dot\theta(0) > 0$ then by (2.32) one has $p > 0$, so in fact $p = b(\rho)$.

Moreover, given any $(\rho, \alpha) \in M$, we may consider the geodesic with $(r(0), \theta(0)) = (\rho, \alpha)$ and $(\dot r(0), \dot\theta(0)) = (0, 1/b(\rho))$ where the value for $\dot\theta(0)$ is obtained from (2.31) (the geodesic must have unit speed). The earlier arguments show that this geodesic has two symmetric branches, and reaches $\{r = r_1\}$ in finite time by (2.34). The required geodesic $\gamma_{\rho,\alpha}$ is obtained from $(r(t), \theta(t))$ after a translation in t.

The equation for $\dot r(t)$ follows from (2.33), where $p = b(\rho)$. Finally, (2.32) with $p = b(\rho)$ gives

$$\theta(t') = \alpha + b(\rho) \int_{\bar t}^{t'} \frac{1}{b(r(t))^2}\, dt.$$

We change variables $t = t(r)$ and use that (2.29) gives

$$\frac{dt}{dr}(r) = \frac{1}{\dot r(t(r))} = \mp \frac{a(r)b(r)}{\sqrt{b(r)^2 - b(\rho)^2}}.$$

This proves (2.30). □

2.4 Geodesic X-ray Transform

In this section we prove the result of Romanov (1967) (see also Romanov (1987); Sharafutdinov (1997)) showing invertibility of the geodesic X-ray transform for rotationally symmetric metrics satisfying the Herglotz condition. Let

$$g = a(r)^2\, dr^2 + b(r)^2\, d\theta^2$$

be a metric in $M = \{(r, \theta)\,;\, r_0 < r \le r_1\}$ satisfying the Herglotz condition $b'(r) > 0$ for $r \in [r_0, r_1]$. For $f \in C^\infty(M)$, we wish to study the problem of recovering f from its integrals over maximal geodesics starting from $\{r = r_1\}$. By Theorem 2.3.4 there are two types of geodesics: those that go to $\{r = r_0\}$ in finite time, and those that never reach $\{r = r_0\}$ and curve back to $\{r = r_1\}$

in finite time. We only consider integrals of f over geodesics of the second type. This corresponds to having measurements only on $\{r = r_1\}$ and not on $\{r = r_0\}$, which is relevant for instance in seismic imaging where $\{r = r_1\}$ corresponds to the surface of the Earth.

By Theorem 2.3.4, for any $(\rho,\alpha) \in M$ there is a unique unit speed geodesic $\gamma_{\rho,\alpha}(t)$ joining two points of $\{r = r_1\}$ and having (ρ,α) as its closest point to the origin. Denote by $\tau(\rho,\alpha)$ the length of this geodesic. Given $f \in C^\infty(M)$, we define its *geodesic X-ray transform* by

$$If(\rho,\alpha) = \int_0^{\tau(\rho,\alpha)} f(\gamma_{\rho,\alpha}(t))\,dt, \qquad (\rho,\alpha) \in M.$$

The main result in this section shows that under the Herglotz condition the geodesic X-ray transform is injective, i.e. f is uniquely determined by If.

Theorem 2.4.1 (Injectivity) *Let g satisfy the Herglotz condition in Definition 2.3.3. If $f \in C^\infty(M)$ satisfies $If(\rho,\alpha) = 0$ for all $(\rho,\alpha) \in M$, then $f = 0$.*

To prove the theorem, we first note that by Theorem 2.3.4 one has

$$\gamma_{\rho,\alpha}(t) = (r(t), \alpha \mp \psi(\rho, r(t))),$$

where

$$\psi(\rho, r(t)) := b(\rho) \int_\rho^{r(t)} \frac{a(r)}{b(r)} \frac{1}{\sqrt{b(r)^2 - b(\rho)^2}}\,dr. \tag{2.35}$$

Moreover,

$$\frac{dr}{dt} = \mp \frac{1}{a(r)b(r)} \sqrt{b(r)^2 - b(\rho)^2}.$$

Here the sign $-$ corresponds to the first branch of the geodesic where $r(t)$ decreases from r_1 to ρ, and $+$ corresponds to the second branch where $r(t)$ increases.

Changing variables $t = t(r)$, we have

$$\begin{aligned} If(\rho,\alpha) &= \int_0^{\tau(\rho,\alpha)} f(r(t), \theta(t))\,dt \\ &= \int_0^{\frac{1}{2}\tau(\rho,\alpha)} f(r(t), \alpha - \psi(\rho, r(t)))\,dt \\ &\quad + \int_{\frac{1}{2}\tau(\rho,\alpha)}^{\tau(\rho,\alpha)} f(r(t), \alpha + \psi(\rho, r(t)))\,dt \\ &= \int_\rho^{r_1} \frac{a(r)b(r)}{\sqrt{b(r)^2 - b(\rho)^2}} f(r, \alpha - \psi(\rho, r))\,dr \\ &\quad + \int_\rho^{r_1} \frac{a(r)b(r)}{\sqrt{b(r)^2 - b(\rho)^2}} f(r, \alpha + \psi(\rho, r))\,dr. \end{aligned} \tag{2.36}$$

Assume for the moment that f is radial, $f = f(r)$. This is analogous to the result in Theorem 2.2.1 of determining a radial sound speed $c(r)$ from travel times, and the proof will use a similar method. If $f = f(r)$, we obtain

$$If(\rho,\alpha) = 2\int_{\rho}^{r_1} \frac{a(r)b(r)}{\sqrt{b(r)^2 - b(\rho)^2}} f(r)\,dr. \tag{2.37}$$

We change variables

$$s = b(r)^2. \tag{2.38}$$

This is a valid change of variables since $b(r)$ is strictly increasing by the Herglotz condition. One has

$$If(\rho,\alpha) = 2\int_{b(\rho)^2}^{b(r_1)^2} \frac{a(r(s))b(r(s))r'(s)}{(s - b(\rho)^2)^{1/2}} f(r(s))\,ds.$$

This is an Abel transform as in Theorem 2.2.3, where x corresponds to $b(\rho)^2$. If $If(\rho,\alpha) = 0$ for $r_0 < \rho < r_1$, it follows from Theorem 2.2.3 that

$$a(r(s))b(r(s))r'(s)f(r(s)) = 0, \qquad b(r_0)^2 < s < b(r_1)^2.$$

Since a, b, and r' are positive, we get $f(r(s)) = 0$ for all s and thus $f(r) = 0$ for $r_0 < r < r_1$ as required.

We next consider the general case where $f = f(r,\theta) \in C^\infty(M)$. For any fixed r, the function $f(r,\cdot\,)$ is a smooth 2π-periodic function in $\mathbb{R}$, and it has the Fourier series

$$f(r,\theta) = \sum_{k=-\infty}^{\infty} f_k(r)e^{ik\theta}. \tag{2.39}$$

Here the Fourier coefficients $f_k(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\theta)e^{-ik\theta}\,d\theta$ are smooth functions in $(r_0,r_1]$, and the Fourier series converges absolutely and uniformly in $\{\bar{r} \leq r \leq r_1\}$ whenever $r_0 < \bar{r} < r_1$.

Inserting (2.39) in (2.36), we have

$$If(\rho,\alpha) = \sum_{k=-\infty}^{\infty}\left[\int_{\rho}^{r_1} \frac{a(r)b(r)}{\sqrt{b(r)^2 - b(\rho)^2}} f_k(r)2\cos(k\psi(\rho,r))\,dr\right]e^{ik\alpha}.$$

Denote the expression in brackets by $A_k f_k(\rho)$. Thus, if $If(\rho,\alpha) = 0$ for $(\rho,\alpha) \in M$, then the Fourier coefficients $A_k f_k(\rho)$ vanish for each k and for $r_0 < \rho < r_1$. It remains to show that each generalized Abel transform A_k is injective. Note that if $k = 0$, then A_0 is exactly the Abel transform in (2.37) and this was already shown to be injective.

For $k \neq 0$, we make the same change of variables as in (2.38) and write

$$g_k(s) = 2a(r(s))b(r(s))r'(s)f_k(r(s)).$$

Then $A_k f_k(\rho) = T_k g_k(b(\rho)^2)$, where

$$T_k g_k(x) = \int_x^{b(r_1)^2} \frac{K_k(x,s)}{(s-x)^{1/2}} g_k(s)\,ds,$$

where $x = x(\rho) = b(\rho)^2$ takes values in the range $b(r_0)^2 < x \leq b(r_1)^2$, and

$$K_k(x,s) = \cos(k\psi(\rho(x), r(s))).$$

Since a, b, and r' are positive, the injectivity of A_k is equivalent with the injectivity of T_k.

We now record some properties of the functions K_k.

Lemma 2.4.2 *For any $k \in \mathbb{Z}$, $K_k(x,s)$ is smooth in $\{b(r_0)^2 \leq x \leq s \leq b(r_1)^2\}$ and satisfies $K_k(x,x) = 1$ for all x.*

Proof Changing variables $s = b(r)^2$, we have

$$\psi(\rho, r) = b(\rho) \int_{b(\rho)^2}^{b(r)^2} \frac{q(s)}{(s - b(\rho)^2)^{1/2}}\,ds,$$

where $q(s) = \frac{a(r(s))r'(s)}{b(r(s))}$ is smooth. We further make another change of variables $s = b(\rho)^2 + (b(r)^2 - b(\rho)^2)t$ to obtain that

$$\psi(\rho, r) = \left(b(r)^2 - b(\rho)^2\right)^{1/2} G(\rho, r),$$

where

$$G(\rho, r) = b(\rho) \int_0^1 \frac{q\left(b(\rho)^2 + (b(r)^2 - b(\rho)^2)t\right)}{t^{1/2}}\,dt.$$

Here G is smooth since q and b are smooth. Using that $\cos x = \eta(x^2)$ where $\eta(t)$ is smooth on $\mathbb{R}$ (this can be seen by looking at the Taylor series of $\cos x$), it follows that $K_k(x,s) = \eta(k^2\psi(\rho(x), r(s))^2)$ is smooth. Finally, note that $x = s$ corresponds to $\rho = r$, which shows that $K_k(x,x) = \cos(k\psi(\rho(x), \rho(x))) = 1$. □

The equation $T_k g_k = F$ is a *singular Volterra integral equation of the first kind* (see Gorenflo and Vessella (1991) for a detailed treatment of such equations). The injectivity of T_k now follows from the next result that extends Theorem 2.2.3 (which considers the special case $K \equiv 1$). This concludes the proof of Theorem 2.4.1.

Theorem 2.4.3 *Let $K \in C^1(T)$ where $T := \{(x,t)\,;\, \alpha \leq x \leq t \leq \beta\}$, and assume that $K(x,x) = 1$ for $x \in [\alpha, \beta]$. Given any $f \in \mathcal{A}((\alpha, \beta])$, there is a unique solution $u \in L^1_{\mathrm{loc}}((\alpha, \beta])$ of*

$$\int_x^\beta \frac{K(x,t)}{(t-x)^{1/2}} u(t)\,dt = f(x). \tag{2.40}$$

Moreover, if $K \in C^\infty(T)$ *and if* $f(x) = (\beta - x)^{1/2} h(x)$ *for some* $h \in C^\infty((\alpha,\beta])$*, then* $u \in C^\infty((\alpha,\beta])$.

Proof We define

$$H(x,t) := K(x,t) - 1.$$

Note that $H(x,x) = 0$ by the assumption on K. The equation (2.40) may be written as

$$Au + Bu = f, \tag{2.41}$$

where $Au(x) = \int_x^\beta \frac{u(t)}{(t-x)^{1/2}}\,dt$ is the Abel transform, and

$$Bu(x) := \int_x^\beta \frac{H(x,t)}{(t-x)^{1/2}} u(t)\,dt.$$

If $B \equiv 0$ then (2.41) is a standard Abel integral equation and it can be solved using Theorem 2.2.3. More generally, we will show that the perturbation B can be handled by a Volterra iteration.

We first show that B maps any function $u \in L^1_{\mathrm{loc}}((\alpha,\beta])$ into $\mathcal{A}((\alpha,\beta])$, i.e. that $ABu \in W^{1,1}_{\mathrm{loc}}((\alpha,\beta])$. We use Fubini's theorem and the change of variables $s = x + (t-x)r$ to compute

$$\begin{aligned} ABu(x) &= \int_x^\beta \int_s^\beta \frac{H(s,t)}{(s-x)^{1/2}(t-s)^{1/2}} u(t)\,dt\,ds \\ &= \int_x^\beta \int_x^t \frac{H(s,t)}{(s-x)^{1/2}(t-s)^{1/2}} u(t)\,ds\,dt \\ &= \int_x^\beta \left[\int_0^1 \frac{H(x+(t-x)r,t)}{r^{1/2}(1-r)^{1/2}}\,dr \right] u(t)\,dt. \end{aligned}$$

Thus $ABu(x) = \int_x^\beta G(x,t)u(t)\,dt$ where $G \in C^1(T)$ since $K \in C^1(T)$. It follows that $ABu \in W^{1,1}_{\mathrm{loc}}((\alpha,\beta])$. By Theorem 2.2.3 we may write

$$Bu = ARu, \qquad u \in L^1_{\mathrm{loc}}((\alpha,\beta]),$$

where $Ru = -\frac{1}{\pi}\frac{d}{dx} ABu$. Since $H(x,x) = 0$ we have $G(x,x) = 0$, and thus using the above formula for ABu we have

$$Ru(x) = -\frac{1}{\pi} \int_x^\beta \partial_x G(x,t) u(t)\,dt.$$

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In particular, the integral kernel of R is in $C^0(T)$, and it follows that

$$|Ru(x)| \leq C \int_x^\beta |u(t)|\, dt. \tag{2.42}$$

Since $Bu = ARu$, (2.41) is equivalent with

$$A(u + Ru) = f.$$

Since $f \in \mathcal{A}((\alpha, \beta])$, one has $f = Au_0$ for some $u_0 \in L^1_{\text{loc}}((\alpha, \beta])$ by Theorem 2.2.3. Because A is injective, (2.41) is further equivalent with the equation

$$u + Ru = u_0. \tag{2.43}$$

It is enough to show that (2.43) has a unique solution $u \in L^1_{\text{loc}}((\alpha, \beta])$ for any $u_0 \in L^1_{\text{loc}}((\alpha, \beta])$. For uniqueness, if $u + Ru = 0$, then (2.42) implies that

$$|u(x)| \leq C \int_x^\beta |u(t)|\, dt.$$

Gronwall's inequality implies that $u \equiv 0$. To prove existence, we iterate the bound (2.42) that yields

$$\begin{aligned} |R^j u(x)| &\leq C \int_x^\beta |R^{j-1} u(t_1)|\, dt_1 \leq \cdots \\ &\leq C^j \int_x^\beta \int_{t_1}^\beta \cdots \int_{t_{j-1}}^\beta |u(t_j)|\, dt_j \cdots dt_1 \\ &\leq C^j \frac{(\beta - x)^{j-1}}{(j-1)!} \|u\|_{L^1([x,\beta])}. \end{aligned}$$

Thus, whenever $\alpha < \gamma < \beta$ one has

$$\|R^j u\|_{L^1([\gamma,\beta])} \leq \frac{(C(\beta - \gamma))^j}{j!} \|u\|_{L^1([\gamma,\beta])}. \tag{2.44}$$

The series

$$u := \sum_{j=0}^\infty (-R)^j u_0$$

converges in $L^1_{\text{loc}}((\alpha, \beta])$ by (2.44), and the resulting function u solves (2.43).

We have proved that given any $f \in \mathcal{A}((\alpha, \beta])$, (2.40) has a unique solution $u \in L^1_{\text{loc}}((\alpha, \beta])$. Let now $K \in C^\infty(T)$ and $f(x) = (\beta - x)^{1/2} h(x)$ for some $h \in C^\infty((\alpha, \beta])$. By Theorem 2.2.3 one has $f = Au_0$ for some $u_0 \in C^\infty((\alpha, \beta])$, and it is enough to show that the solution u of (2.43) is smooth.

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But if $K \in C^\infty(T)$ the operator R above has C^∞ integral kernel, hence Ru is smooth, and thus also $u = -Ru + u_0$ is smooth. This concludes the proof of the theorem. $\square$

2.5 Examples and Counterexamples

In this section we give some examples of manifolds where the geodesic X-ray transform is injective, and some examples where it is not injective. We first begin with some remarks on the Herglotz condition.

Let $g = a(r)^2\,dr^2 + b(r)^2\,d\theta^2$ be a metric in $M = \{r_0 < r \le r_1\}$, where $a, b \in C^\infty([r_0, r_1])$ are positive. We first give a definition.

Definition 2.5.1 The circle $\{r = \bar{r}\}$ is *strictly convex* (respectively *strictly concave*) as a submanifold of (M, g) if for any geodesic $(r(t), \theta(t))$ with $r(0) = \bar{r}$, $\dot{r}(0) = 0$ and $\dot{\theta}(0) \neq 0$, one has $\ddot{r}(0) > 0$ (respectively $\ddot{r}(0) < 0$).

Strict convexity means that any tangential geodesic to the circle $\{r = \bar{r}\}$ curves away from this circle toward $\{r = r_1\}$, with exactly first order contact with the circle when $t = 0$. More precisely, we should say that the circle is strictly convex when viewed from $\{r = r_1\}$ (there is a choice of orientation involved). Strict convexity is equivalent to the fact that $\{r = \bar{r}\}$ has positive definite second fundamental form in (M, g). Conversely, strict concavity means that tangential geodesics to the circle $\{r = \bar{r}\}$ have first order contact and curve toward $\{r = r_0\}$.

Lemma 2.5.2 *Let $r_0 < \bar{r} \le r_1$.*

(a) $\{r = \bar{r}\}$ is strictly convex as a submanifold of (M, g) if and only if $b'(\bar{r}) > 0$.
(b) The circle $t \mapsto (\bar{r}, t)$ is a geodesic of (M, g) if and only if $b'(\bar{r}) = 0$.
(c) $\{r = \bar{r}\}$ is strictly concave as a submanifold of (M, g) if and only if $b'(\bar{r}) < 0$.

Proof If $(r(t), \theta(t))$ is a geodesic with $r(0) = \bar{r}$ and $\dot{r}(0) = 0$, then by (2.25)

$$\ddot{r}(0) = \frac{b(\bar{r})b'(\bar{r})}{a(\bar{r})^2}(\dot{\theta}(0))^2. \tag{2.45}$$

If $\dot{\theta}(0) \neq 0$, then $\ddot{r}(0)$ has the same sign as $b'(\bar{r})$ since b is positive. This proves parts (a) and (c). For part (b), if $b'(\bar{r}) = 0$, then $t \mapsto (\bar{r}, t)$ satisfies the geodesic equations (2.25)–(2.26). Conversely, if $t \mapsto (\bar{r}, t)$ satisfies the geodesic equations, then $\ddot{r}(0) = 0$ and (2.45) implies that $b\partial_r b/a^2|_{r=\bar{r}} = 0$. One must have $b'(\bar{r}) = 0$. $\square$

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Thus, if the Herglotz condition is violated, either $b' = 0$ somewhere and there is a *trapped geodesic* (one that never reaches the boundary), or $b' < 0$ somewhere and tangential geodesics curve toward $\{r = r_0\}$. We also obtain the following characterization of the Herglotz condition.

Corollary 2.5.3 *The following conditions are equivalent.*

(a) The circles $\{r = \bar{r}\}$ *are strictly convex for* $r_0 < \bar{r} \leq r_1$.
(b) $b'(1) > 0$ *and no circle* $\{r = \bar{r}\}$ *is a trapped geodesic for* $r_0 < \bar{r} \leq r_1$.
(c) $b'(r) > 0$ *for* $r \in (r_0, r_1]$.

We now go back to Example 2.3.2 and surfaces of revolution. Recall the setup: r corresponds to the z-coordinate in $\mathbb{R}^3$, $h\colon [r_0, r_1] \to \mathbb{R}$ is a smooth positive function, and S is the surface of revolution obtained by rotating the graph of $r \mapsto h(r)$ about the z-axis. The surface S is given by

$$S = \{(h(r)\cos\theta, h(r)\sin\theta, r)\,;\, r \in (r_0, r_1],\, \theta \in [0, 2\pi]\}.$$

The metric on S induced by the Euclidean metric on $\mathbb{R}^3$ has the form

$$g = (1 + h'(r)^2)\, dr^2 + h(r)^2\, d\theta^2.$$

Thus $a(r) = \sqrt{1 + h'(r)^2}$ and $b(r) = h(r)$.

Finally we give five illustrative examples: two examples where the geodesic X-ray transform is injective, two examples where it fails to be injective, and one example related to Eaton lenses.

Example 2.5.4 (Small spherical cap) Let $h\colon [r_0, r_1] \to \mathbb{R}$, $h(r) = \sqrt{1 - r^2}$ where $r_0 = -1$ and $r_1 = -\alpha$ where $0 < \alpha < 1$. Then $S = S_\alpha$ corresponds to a punctured spherical cap strictly contained in a hemisphere (cf. Figure 2.1):

$$S_\alpha = \{x \in S^2\,;\, x_3 \leq -\alpha\} \setminus \{-e_3\}.$$

Clearly $h' > 0$ in $[r_0, r_1]$. Thus the Herglotz condition is satisfied, and by Theorem 2.4.1 the geodesic X-ray transform on S_α is injective whenever $0 < \alpha < 1$. More precisely, a function f can be recovered from its integrals over geodesics that start and end on the boundary $\{x_3 = -\alpha\}$, with the geodesics going through the south pole excluded. Of course, geodesics in S_α are segments of great circles.

Example 2.5.5 (Large spherical cap) Let $h\colon [r_0, r_1] \to \mathbb{R}$, $h(r) = \sqrt{1 - r^2}$ where $r_0 = -1$ and $r_1 = \beta$ where $0 < \beta < 1$. Then $S = S_\beta$ corresponds to a punctured spherical cap that is larger than a hemisphere:

$$S_\beta = \{x \in S^2\,;\, x_3 \leq \beta\} \setminus \{-e_3\}.$$

Now the Herglotz condition is violated: one has $h'(r) > 0$ for $r < 0$, but $h'(0) = 0$ and $h'(r) < 0$ for $r > 0$. In particular, the geodesic $\{r = 0\}$, which

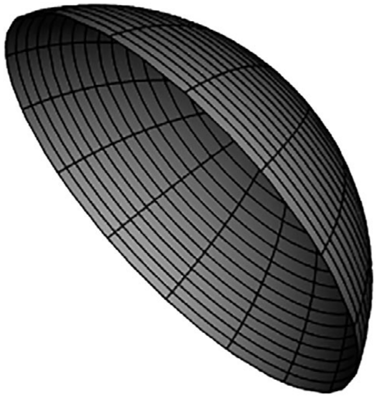

Figure 2.1 Small spherical cap.

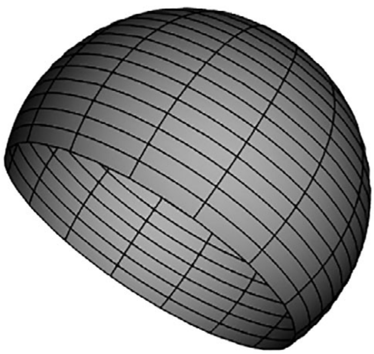

Figure 2.2 Large spherical cap.

is just the equator, is a trapped geodesic in S_β. The great circles close to the equator are also trapped geodesics, and S_β is an example of a manifold with strong trapping properties (cf. Figure 2.2).

In fact the geodesic X-ray transform is *not injective* on S_β (even if the south pole is included). To see this, let $f\colon S^2 \to \mathbb{R}$ be an odd function with respect to the antipodal map, i.e. $f(-x) = -f(x)$, and assume f is supported in $\{-\beta < x_3 < \beta\}$. For example, one can take $f(x) = \varphi(x) - \varphi(-x)$ where φ is a C^∞ function supported in a small neighbourhood of e_1 with $\varphi > 0$ near e_1.

Using the support condition for f, the integral of f over a maximal geodesic in (M, g) (a segment of a great circle C in S^2) is equal to the integral of f over the whole great circle C. But since f is odd, its integral over any great circle

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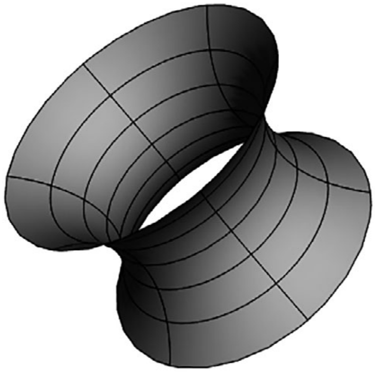

Figure 2.3 Catenoid.

is zero. This shows that the geodesic X-ray transform If of f in S_β vanishes, but f is not identically zero.

Example 2.5.6 (Catenoid) Let $h\colon [-1,1] \to \mathbb{R}$, $h(r) = \cosh(r) = \frac{e^r+e^{-r}}{2}$. The corresponding surface of revolution is the *catenoid* (cf. Figure 2.3)

$$S = \{(\cosh(r)\cos(\theta), \cosh(r)\sin(\theta), r)\,;\, r \in [-1,1], \theta \in [0,2\pi]\}.$$

One has $h'(r) = \sinh(r) = \frac{e^r-e^{-r}}{2}$. Thus in particular $h'(0) = 0$ and $h'(r) > 0$ for $r > 0$. Define

$$S_\pm = \{x \in S\,;\ \pm x_3 > 0\}.$$

Then S_+ corresponds to $h\colon (r_0, r_1] \to \mathbb{R}$ with $r_0 = 0$ and $r_1 = 1$. By Theorem 2.4.1 the geodesic X-ray transform in S_+ is injective, when considering geodesics that start and end on $S_+ \cap \{x_3 = 1\}$. By symmetry, also the geodesic X-ray transform on S_- is injective for geodesics that start and end on $S_- \cap \{x_3 = -1\}$. Since $S = S_+ \cup S_- \cup S_0$ where $S_0 = S \cap \{x_3 = 0\}$ has zero measure, it follows that also the geodesic X-ray transform on S is injective (any smooth function on S can be recovered from its integrals starting and ending on ∂S).

Note that since $h'(0) = 0$, the geodesic S_0 is a trapped geodesic in S. The manifold S has also other trapped geodesics that start on ∂S and orbit S_0 for infinitely long time. The catenoid is an example of a negatively curved manifold with weak trapping properties (the trapped set is hyperbolic). Because the trapping is weak, the geodesic X-ray transform is still invertible in this case.

Example 2.5.7 (Catenoid type surface with flat cylinder glued in the middle) Let $h\colon [-1,1] \to \mathbb{R}$ with $h(r) = 1$ for $r \in [-\frac{1}{2}, \frac{1}{2}]$, $h'(r) > 0$ for $r > \frac{1}{2}$, and $h'(r) < 0$ for $r < -\frac{1}{2}$, and let S be the surface of revolution obtained by rotating $h|_{[-1,1]}$. Then $S \cap \{-\frac{1}{2} \leq x_3 \leq \frac{1}{2}\}$ is a flat cylinder.

Consider a smooth function f in S given by

$$f(h(r)\cos\theta, h(r)\sin\theta, r) = \eta(r),$$

where $\eta \in C_c^\infty(-\frac{1}{2}, \frac{1}{2})$ is nontrivial and satisfies $\int_{-1/2}^{1/2} \eta(r)\,dr = 0$. Then f integrates to zero over any geodesic starting and ending on ∂S. To see this, note that f vanishes outside the flat cylinder, and any geodesic that enters the flat cylinder must be a geodesic of the cylinder. Since $h \equiv 1$ in the cylinder, the metric is $dr^2 + d\theta^2$, one has $a = b = 1$, the geodesic equations are $\ddot{r} = \ddot{\theta} = 0$, and unit speed geodesics are of the form $\zeta(t) = (r(t), \theta(t)) = (\alpha t + \beta, \gamma t + \delta)$ where $(\dot{r})^2 + (\dot{\theta})^2 = \alpha^2 + \gamma^2 = 1$. Thus it follows that

$$\int_\zeta f\,dt = \int \eta(\alpha t + \beta)\,dt = 0.$$

Thus S is an example of a manifold that has a large flat part (the cylinder) with many trapped geodesics, and the geodesic X-ray transform is not injective. The reason for non-injectivity is that S contains part of $\mathbb{R} \times S^1$, and the X-ray transform on $\mathbb{R}$ is not injective (there are nontrivial functions that integrate to zero on $\mathbb{R}$).

Example 2.5.8 (Eaton lenses) Geodesics of a sound speed may also be interpreted as the paths followed by light rays when a suitable index of refraction n is introduced. According to Fermat's principle light rays propagate along geodesics of the metric $g_{jk} = n^2\delta_{jk}$ and thus by setting $c = 1/n$ our previous analysis applies. Let us consider an index of refraction n, which is radial and work in polar coordinates, so that the metric is $n^2(dr^2 + r^2 d\theta^2)$ and hence $a(r) = n(r)$ and $b(r) = rn(r)$. Besides travel times between boundary points, we might also be interested in how incoming light rays come out after traversing through our Riemannian surface (*the lens*) determined by $n(r)$. From this point of view there are choices of n that produce interesting effects. We mention here two noteworthy instances depicted in Figures 2.4 and 2.5.

The original Eaton lens (Figure 2.4) is given by

$$n(r) = \sqrt{\frac{2}{r} - 1},$$

while for the invisible Eaton lens (Figure 2.5), n is determined by

$$\sqrt{n} = \frac{1}{nr} + \sqrt{\frac{1}{n^2 r^2} - 1}.$$

In both cases $n(r)$ is defined for $r \in (0, 1]$ and in the second case n is given intrinsically as the solution of the equation above. In the first case we see light

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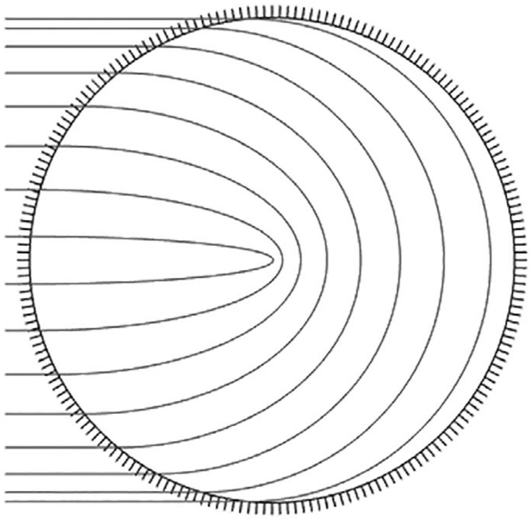

Figure 2.4 Original Eaton lens.

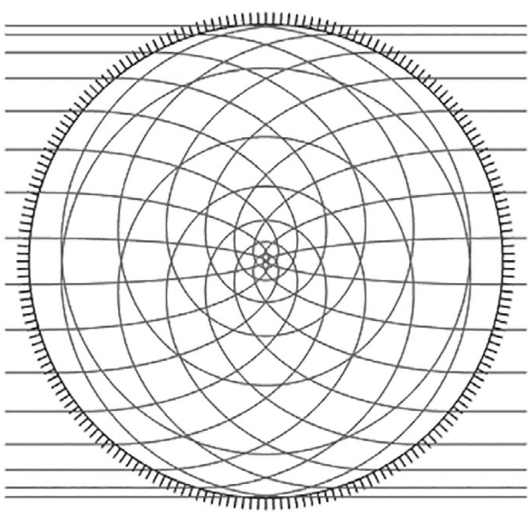

Figure 2.5 Invisible Eaton lens.

rays rotating by π and in the second case we see light rays rotating by 2π and hence becoming indistinguishable from the light rays of $n = 1$, hence the name invisible Eaton lens. The index of refraction becomes infinite (in both cases) at the origin.

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Exercise 2.5.9 Show that in both Eaton lenses, the Herglotz condition is satisfied for all $r \in (0, 1)$ but the circle $\{r = 1\}$ at the boundary is a trapped light ray. Moreover, show that the geodesics behave as depicted in the pictures (use Theorem 2.3.4). Can you design a lens so that lights rays come out of the lens experiencing a rotation of $\pi/2$? (See Leonhardt and Philbin (2010) for details on these lenses.)

Exercise* 2.5.10 Investigate if the X-ray transform is injective for the Eaton lenses and for the case $\alpha = 1$ in Example 2.5.4.

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