11

Locally Analytic Representations of p-adic Groups

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11.1 Introduction

The p-adic representation theory of p-adic groups is the subject of much ongoing research. It is primarily motivated by the desire to better understand the conjectural p-adic Langlands correspondence, but it is also an interesting branch of representation theory in its own right. The purpose of this chapter, which is based on a series of lectures given at the LMS 'Autumn Algebra School', is to give an introduction to Schneider and Teitelbaum's theory of admissible locally analytic representations to people with an algebra/representation theory background without assuming any familiarity with the Langlands program or the number theoretic motivations for it. We therefore begin by briefly recalling what the conjectured p-adic Langlands correspondence is.

Fix two primes ℓ and p. When $\ell \neq p$, there is a classical Local Langlands Correspondence, which can be roughly stated as follows: if $n \geqslant 1$ there is an injection

$$\left\{ \begin{array}{c} \text{Frobenius-semisimple, continuous} \\ \text{representations of } \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \text{ on} \\ n\text{-dimensional } \overline{\mathbb{Q}_\ell}\text{-vector spaces,} \\ \text{up to isomorphism} \end{array} \right\} \to \left\{ \begin{array}{c} \text{Irreducible, admissible,} \\ \text{smooth representations of} \\ \mathrm{GL}_n(\mathbb{Q}_p) \text{ on } \overline{\mathbb{Q}_\ell}\text{-vector} \\ \text{spaces, up to isomorphism} \end{array} \right\}.$$

In fact, one usually considers the larger class of Frobenius-semisimple representations of the Weil–Deligne group on the left in order to get a bijection. This correspondence can be uniquely characterised by various properties (e.g. equivalence of L- and ε-factors of pairs), and can be stated more generally by replacing $\mathbb{Q}_p$ with some fixed finite extension F of $\mathbb{Q}_p$. This correspondence was first established by Harris–Taylor [23] and Henniart [24], and more recently by Scholze [40].

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When $\ell = p$ one would like to obtain an analogue of the above correspondence, often referred to as the 'p-adic Local Langlands Correspondence'. However, the situation is trickier as there are now 'more' Galois representations on the left, due to the fact that the topologies of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and of $\overline{\mathbb{Q}_p}$ are compatible. As a result one would need to enlarge the right-hand side, and to this end Breuil [9] proposed to consider certain (not necessarily irreducible) Banach space representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ over some fixed finite extension of $\mathbb{Q}_p$. In the case $n = 2$, a correspondence was established using the fact that both sides can be classified very explicitly in terms of (φ, Γ)-modules (see [14], [15], [12], [13] amongst others). We note that this correspondence was only established for $\mathrm{GL}_2(\mathbb{Q}_p)$ and not for $\mathrm{GL}_2(F)$, where F is more generally a finite extension of $\mathbb{Q}_p$. The general situation (working with $n > 2$ or over some non-trivial finite extension of $\mathbb{Q}_p$) is still very mysterious.

In the early 2000s, Schneider and Teitelbaum started the systematic study of locally analytic representations of p-adic groups in a series of papers [34, 35, 36, 37, 38]. These locally analytic representations arise naturally in various geometric constructions, and they are related to Banach space representations by the process of taking locally analytic vectors. Part of the challenge in working with $\mathrm{GL}_n(\mathbb{Q}_p)$-representations over topological vector spaces is that one needs to find the right finiteness condition to replace the notion of admissibility that occurred in the $\ell \neq p$ case, in order to get a well-behaved theory which still retains all the important examples that arose in the Langlands program. The class of locally analytic representations that Schneider and Teitelbaum introduced to serve that purpose are called admissible, and they generalise the above notion of admissible smooth representations. The main aim of this chapter is to explain their theory in detail.

In Section 11.2, we recall all the basic definitions required in order to define locally analytic representations and we provide examples. In Section 11.3, we explain the construction of the distribution algebra and give some of its properties. In Section 11.4 we explain the notion of a Fréchet–Stein algebra and define admissible representations. We also give a short account of some further developments in this theory. Finally, to make the material more accessible we also include an appendix summarising all the notions from non-archimedean functional analysis that we require.

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11.2 Basic Definitions

We begin by recalling the definitions of p-adic Lie groups and locally analytic representations.

11.2.1 Non-archimedean Fields

Throughout, L will denote a field.

Definition 11.1 A non-archimedean absolute value (NAAV) on L is a function $|\cdot| : L \to \mathbb{R}$ such that for all $a, b \in L$:

(i) $|a| \geqslant 0$;
(ii) $|a| = 0 \iff a = 0$;
(iii) $|a \cdot b| = |a| \cdot |b|$; and
(iv) $|a+b| \leqslant \max\{|a|, |b|\}$.

This gives a metric on L via $d(a,b) := |a-b|$, making L into a topological field. The unit ball $\mathcal{O}_L := \{a \in L : |a| \leqslant 1\}$ is a subring.

From now on, we assume that L is equipped with a NAAV and that it is *complete*, i.e. Cauchy sequences converge. It is possible to develop a whole theory of functional analysis over fields equipped with such absolute values, see the Appendix.

Remark We can more generally topologise L^n for any n by equipping it with the norm $||(a_1, \dots, a_n)|| := \max\{|a_1|, \dots, |a_n|\}$. It then becomes an L-Banach space.

Example 11.2 Let p be a prime number and let $a \in \mathbb{Q}$. Define $|a|_p := p^{-r}$ if $a = p^r \cdot \frac{m}{n}$ where $(m,p) = (n,p) = 1$. This is a NAAV on $\mathbb{Q}$ and the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$, the *field of p-adic numbers*, and the unit ball of $\mathbb{Q}_p$ is denoted by $\mathbb{Z}_p$, the ring of *p-adic integers*. Moreover, the NAAV on $\mathbb{Q}_p$ extends uniquely to a NAAV on L for any finite field extension $L/\mathbb{Q}_p$.

Concretely, elements of $\mathbb{Z}_p$ are 'infinite base p expansions', i.e. can be represented uniquely as a series

$$a_0 + a_1 p + a_2 p^2 + \dots + a_n p^n + \dots,$$

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where $a_i \in \{0,1,\ldots,p-1\}$ for all i. We then have $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$.

Going back to general L, convergence of series will be central to the basic definitions of locally analytic functions. Part (iv) in the definition of a NAAV has the following immediate and useful consequence in that regard: if (a_n) is a sequence in L, then

$$\sum_{n\geqslant 0} a_n \text{ converges} \iff a_n \to 0 \text{ as } n \to \infty.$$

In particular, as a function, a power series $f(x) = \sum_{n\geqslant 0} a_n x^n$ will converge on a ball $B_{0,\varepsilon} := \{a \in L : |a| \leqslant \varepsilon\}$ if and only if $\varepsilon^n |a_n| \to 0$ as $n \to \infty$.

11.2.2 *p*-adic Lie Groups

Given $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$, we adopt the notation $|\alpha| := \alpha_1 + \ldots + \alpha_n$ and $\underline{X}^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$.

Definition 11.3 Let V be an L-Banach space and fix $n \geqslant 1$.

(i) If $U \subseteq L^n$ is open, then a function $f : U \to V$ is *locally analytic* if for all $x_0 \in U$, there exists $\varepsilon > 0$ and a power series $F(\underline{X}) = \sum_{\alpha\in\mathbb{N}^n} v_\alpha \underline{X}^\alpha$, where $v_\alpha \in V$ and $\varepsilon^{|\alpha|} \cdot ||v_\alpha|| \to 0$ as $|\varepsilon| \to \infty$, such that for all $x \in U$ with $||x - x_0|| \leqslant \varepsilon$ we have $f(x) = F(x - x_0)$.

(ii) Given $x_0 \in L^n$ and $\varepsilon > 0$, write $B(x_0,\varepsilon) = \{x \in L^n : ||x - x_0|| < \varepsilon\}$. We say that a map $f : B(x_0,\varepsilon) \to V$ is *holomorphic* if $f(x) = F(x - x_0)$ for all $x \in B(x_0,\varepsilon)$, with F as in (i). The vector space

$$\mathcal{F}(x_0,\varepsilon,V) := \{f : B(x_0,\varepsilon) \to V | f \text{ holomorphic}\}$$

is then an L-Banach space, with norm $||\sum_{\alpha\in\mathbb{N}^n} v_\alpha \underline{X}^\alpha|| = \sup_\alpha \varepsilon^{|\alpha|} ||v_\alpha||$.

This definition can be extended to the case where V is a Hausdorff locally convex space. In that case, we say $f : U \to V$ is locally analytic if it factors as a composite $U \to W \to V$, where W is an L-Banach space, the map $U \to W$ is locally analytic as defined above and $W \to V$ is a continuous injective L-linear map.

Next we can now introduce manifolds:

Definition 11.4 Let M be a Hausdorff topological space and let $n \geqslant 1$. An *atlas of dimension n* on M is a set $\mathcal{A} = \{(U_i,\varphi_i)\}_{i\in I}$ such that

- $U_i \subset M$ is open for all $i \in I$ and $M = \bigcup_{i\in I} U_i$;
- $\varphi_i : U_i \to L^n$ is a homeomorphism onto an open subset of L^n for all $i \in I$; and

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- for all $i, j \in I$, the maps

$$\varphi_i(U_i \cap U_j) \underset{\varphi_i \circ \varphi_j^{-1}}{\overset{\varphi_j \circ \varphi_i^{-1}}{\rightleftarrows}} \varphi_j(U_i \cap U_j)$$

are locally analytic.

We say an atlas $\mathcal{A}$ is *maximal* if for any other atlas $\mathcal{B}$ such that $\mathcal{A} \cup \mathcal{B}$ is also an atlas, then $\mathcal{B} \subseteq \mathcal{A}$. We then say that M is a (locally L-analytic) *manifold of dimension* n if it is equipped with a maximal atlas, and the pairs (U_i, φ_i) are called *charts*.

Given a Hausdorff locally convex L-vector space V, we say that a map $f : M \to V$ is *locally analytic* if $f \circ \varphi^{-1} : \varphi(U) \to V$ is locally analytic for each chart (U, φ) of M. Similarly, a map between manifolds is locally analytic if it is locally analytic on the charts.

Finally we can talk about groups:

Definition 11.5 A manifold G is a *Lie group* if it is a group such that the multiplication $m : G \times G \to G$ is locally analytic.

Remark Given a Lie group G, the inversion map $g \mapsto g^{-1}$ is automatically a locally analytic isomorphism of manifolds (c.f. [33, Proposition 13.6]).

Example 11.6 The following are all examples of Lie groups:

(i) $(L^n, +)$ or $(\mathcal{O}_L^n, +)$.

(ii) $(L^\times, \cdot)$ or $(\mathcal{O}_L^\times, \cdot)$.

(iii) $(1 + p\mathbb{Z}_p, \cdot) \leqslant (\mathbb{Q}_p^\times, \cdot)$, i.e. elements of the form $1 + a_1 p + a_2 p^2 + \ldots$.

(iv) $\mathrm{GL}_n(L)$, $\mathrm{GL}_n(\mathcal{O}_L)$, $\mathrm{SL}_n(L)$, $\mathrm{SL}_n(\mathcal{O}_L)$.

(v) More generally, the L-valued points of any connected algebraic group over L. In particular, the Borel subgroup

$$B = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} \in \mathrm{GL}_n(L) \right\}$$

and the maximal torus

$$T = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \in \mathrm{GL}_n(L) \right\}$$

of $\mathrm{GL}_n(L)$ are Lie groups.

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(vi) The *Iwahori subgroup* of $\mathrm{GL}_2(\mathbb{Z}_p)$

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : c \in p\mathbb{Z}_p \right\}.$$

Most of these examples are algebraic in nature, but the point of the analytic setup is that we may study a class of representations larger than the algebraic ones.

11.2.3 Locally Analytic Representations

From now on, we fix complete non-archimedean fields $L \subseteq K$ such that the NAAV on K extends the one on L, and we fix G a locally L-analytic Lie group. Note that in this setup K is an L-Banach space. We will study representations of G on K-vector spaces. We assume that V is a Hausdorff locally convex K-vector space and we write

$$C^{\mathrm{an}}(G,V) := \{f : G \to V | f \text{ locally analytic}\}.$$

Definition 11.7 A representation $\rho : G \to \mathrm{GL}(V)$ is *locally analytic* if for each $v \in V$, the map $g \mapsto \rho(g)v$ belongs to $C^{\mathrm{an}}(G,V)$.

Remark This only depends on each vector $v \in V$, so given *any* representation on V, it makes sense to consider the *locally analytic vectors*,

$$V^{\mathrm{an}} := \{v \in V : (g \mapsto \rho(g)v) \in C^{\mathrm{an}}(G,V)\},$$

a locally analytic subrepresentation.

Example 11.8 (i) If G is algebraic (e.g. $\mathrm{GL}_n(L)$) then any algebraic representation of G is locally analytic, because the orbit maps are polynomial functions on G.

(ii) If $G = (\mathbb{Z}_p, +)$, we can define a character $\chi : G \to K^\times$ as follows. Pick $z \in K^\times$ such that $|z-1| < 1$. Then, for $a \in \mathbb{Z}_p$, set

$$\chi_z(a) = z^a := \sum_{n=0}^{\infty} (z-1)^n \binom{a}{n}.$$

Here the binomial coefficient is defined as $\binom{a}{n} = \frac{a(a-1)\dots(a-n+1)}{n!} \in \mathbb{Z}_p$. It was shown by Amice [1, 16.1.15] that χ_z is locally analytic.

(iii) Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, B the Borel subgroup, T the maximal torus. Let $\chi : T \to K^\times$ be a locally analytic character. As T can be identified with a quotient of B, we may lift χ to B. Then we have the locally analytic induction

$$\mathrm{Ind}_B^G(\chi) := \{f \in C^{\mathrm{an}}(G,K) : f(gb) = \chi(b^{-1})f(g)\ \forall g \in G, b \in B\}.$$

This is a locally analytic representation of G when G acts by left translation, called a *principal series* representation.

In fact, more generally, given any locally L-analytic group G and any locally analytic subgroup P such that G/P is compact, and given any locally analytic representation V of P, the induction

$$\mathrm{Ind}_P^G(V) := \left\{f \in C^{\mathrm{an}}(G,V) : f(gb) = b^{-1} \cdot f(g)\ \forall g \in G, b \in P\right\}$$

is a locally analytic G-representation (see [22, Satz 4.1.5]).

(iv) When $\chi = \mathbf{1}$ in (iii), the corresponding principal series representations $\mathrm{Ind}_B^G(\mathbf{1})$ can be identified with the space of locally analytic functions on $\mathbb{P}^1$, because $G/B \cong \mathbb{P}^1$. Moreover, we have a natural injection $\mathbf{1}_G \to \mathrm{Ind}_B^G(\mathbf{1})$ with image the constant functions $G \to K$. The quotient $\mathrm{St} := \mathrm{Ind}_B^G(\mathbf{1})/\mathbf{1}_G$ is called the *Steinberg representation*. This also has a geometrical interpretation, namely by a theorem of Morita [26] the Steinberg representation is isomorphic to the strong dual of the space of 1-forms on the Drinfeld upper half plane $\mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$.

Remark Even if $G = (\mathbb{Z}_p, +)$, we can construct infinitely many irreducible, infinite dimensional, locally analytic representations. If $z \in K^\times$ as in example (ii) and z is transcendental over $\mathbb{Q}_p$, and assuming that K is the smallest complete field containing z, then Diarra [16, Théorème 5] showed that K is then an irreducible $\mathbb{Q}_p$-representation V_z of G via

$$\rho(a)v = \sum_{n=0}^{\infty} (z-1)^n \binom{a}{n} v,$$

and moreover he showed that if we instead choose $z' \in K^\times$ with $|z'| \neq |z|$, then V_z is not isomorphic to $V_{z'}$. Hence locally analytic representations are too wild to study in general. We need a nicer subclass of representations within it.

11.3 The Distribution Algebra

The analytic nature of both the groups and representations makes it hard to work with them directly. In order to study these representations more

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algebraically, we define an algebra $D(G,K)$ such that

$$\left\{\begin{array}{l}\text{sufficiently nice}\\ \text{loc. an. representations}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{sufficiently nice}\\ D(G,K)\text{-modules}\end{array}\right\}.$$

Here, 'sufficiently nice' will have to be some topological properties. Later, we will see how to replace some of these topological properties with more algebraic ones.

We fix fields $\mathbb{Q}_p \subseteq L \subseteq K$ with $L/\mathbb{Q}_p$ finite, and assume further that K is *spherically complete* with respect to a NAAV extending the one on L. We refer the reader to the Appendix for the precise meaning of this, but we recall that this is automatically satisfied if K is a finite extension of $\mathbb{Q}_p$.

11.3.1 The Space of Distributions

Recall that given a locally L-analytic manifold M, we have $C^{\text{an}}(M,K) = \{f : M \to K | f \text{ is locally analytic}\}$. Assume that M is strictly paracompact, i.e. any open cover can be refined to one where the opens are disjoint, and of dimension d. We point out that this is not a very restrictive condition, for instance it is always satisfied when M is a Lie group, see [33, Corollary 18.8]. In such a setting, Féaux de Lacroix [22, 2.1.10] defined a K-locally convex structure on $C^{\text{an}}(M,K)$, which we recall now.

First, note that if $(U_i,\varphi_i)_{i\in I}$ are the charts, by strict paracompactness we can refine it to assume that the U_i are disjoint. Then since $f : M \to K$ is locally analytic if and only if $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to K$ is locally analytic by definition, it now follows that $C^{\text{an}}(M,K) = \prod_{i\in I} C^{\text{an}}(U_i,K)$. Moreover, by definition of what it means for $f \circ \varphi_i^{-1}$ to be locally analytic and by strict paracompactness, for each f we can cover $\varphi_i(U_i)$ by disjoint open balls B_{ij} (which depend on f) so that f is holomorphic on each B_{ij}.

Summarising the above, given $f \in C^{\text{an}}(M,K)$ there is a family of charts $(V_i,\varphi_i)_{i\in J}$ of M and real numbers $\varepsilon_i > 0$ ($i \in J$) such that:

(i) $M = \coprod_{i\in J} V_i$;
(ii) $\varphi_i(V_i) = B(x_i,\varepsilon_i)$ for some $x_i \in L^d$; and
(iii) $f \circ \varphi_i^{-1}$ is holomorphic on $B(x_i,\varepsilon_i)$.

We then define an *index* $\mathcal{J}$ to be a family of charts $(V_i,\varphi_i)_{i\in J}$ of M satisfying (i)–(ii) above. For each index $\mathcal{J}$, we may form the product

$$\mathcal{F}_{\mathcal{J}}(M,K) := \prod_{i\in J} \mathcal{F}(x_i,\varepsilon_i,K).$$

Since each $\mathcal{F}(x_i,\varepsilon_i,K)$ is Banach by Definition 11.3, this can be given the structure of a locally convex K-vector space (see Appendix). From the above we can

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see that

$$C^{\mathrm{an}}(M,K) = \varinjlim_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(M,K),$$

where the limit is over all indices $\mathcal{I}$, and it therefore has the structure of a locally convex K-vector space as an inductive limit of locally convex spaces (see Appendix).

Definition 11.9 With M as above, the *space of distributions* on M is the dual $D(M,K) := C^{\mathrm{an}}(M,K)'$.

Lemma 11.10 *(i)* ([36, Lemma 2.1]) *If M is compact then $D(M,K)$ is a nuclear Fréchet space, and so in particular reflexive.*
(ii) ([22, 2.2.4]) *If $M = \coprod_{i\in I} M_i$, where the M_i are pairwise disjoint compact open subsets, then $D(M,K) = \bigoplus_{i\in I} D(M_i,K)$ topologically.*[1]

At first sight, it isn't a priori clear what the elements of $D(M,K)$ look like. The only easy examples are the *Dirac distributions* which are defined as follows: let $m \in M$, then we have an element $\delta_m \in D(M,K)$ given by $\delta_m(f) := f(m)$ for $f \in C^{\mathrm{an}}(M,K)$. This gives a map $M \to D(M,K)$, $m \mapsto \delta_m$.

11.3.2 The Convolution Product

From now on, $M = G$ is a Lie group. We now sketch the construction of the product on $D(G,K)$. It can be motivated as follows:

Note that, given a finite group H and a field k, one can identify the group algebra kH with the dual vector space of the space $C(H,k)$ of all functions $H \to k$. One can define Dirac distributions δ_h for $h \in H$ as above. The group operation $H \times H \to H$ gives rise to a map $C(H,k) \to C(H\times H,k)$. Passing to duals we get a map $k(H\times H) \cong kH \otimes_k kH \to kH$, which gives a k-algebra structure to kH and which agrees with the group operation on the Dirac deltas, i.e. $\delta_h \otimes \delta_{h'} \mapsto \delta_{hh'}$.

We now replicate this construction for the distributions on G. The key fact is that there is a canonical isomorphism

$$D(G\times G,K) \cong D(G,K)\widehat{\otimes}_K D(G,K),$$

where $\widehat{\otimes}_K$ denotes the completion of the usual algebraic tensor product with respect to the so-called inductive topology – see [39, Section 12]. Also, the group multiplication $m : G\times G \to G$ induces a map $C^{\mathrm{an}}(G,K) \to C^{\mathrm{an}}(G\times G,K)$, $f \mapsto f\circ m$. Dually this gives a map $D(G\times G,K) \to D(G,K)$.

[1] This is useful when $M = G$ is a Lie group and the M_i are left cosets of some compact open subgroup G_0 (e.g. $G = \mathrm{GL}_n(\mathbb{Q}_p)$ and $G_0 = \mathrm{GL}_n(\mathbb{Z}_p)$).

Given $u, v \in D(G,K)$, we define their *convolution* $u * v$ to be the image of $u \otimes v$ under the composite

$$D(G,K)\widehat{\otimes}_K D(G,K) \xrightarrow{\cong} D(G \times G, K) \to D(G,K).$$

The main result is then:

Theorem 11.11 ([21, 4.4.1 & 4.4.4]) *Convolution defines a separately continuous product on $D(G,K)$ with unit δ_1. When G is compact, this makes $D(G,K)$ into a Fréchet algebra i.e. $* : D(G,K) \times D(G,K) \to D(G,K)$ is continuous.*

The main moral of the story to come is that we gain more control by working with $D(G,K)$-modules rather than locally analytic G-representations directly.

We now briefly describe more concretely the above definition of the distribution algebra when $G = \mathbb{Z}_p$. By the compact topology on $\mathbb{Z}_p$, any index is finite and consists of disjoint balls which may be shrunk so that they all have the same radius. This radius is a negative power of p, and moreover the set of open balls of radius p^{-j} in $\mathbb{Z}_p$, for a fixed $j \geqslant 1$, are in bijection with $\mathbb{Z}/p^j\mathbb{Z}$. Thus we see that

$$C^{\mathrm{an}}(\mathbb{Z}_p, K) = \varinjlim_{j \geqslant 1} \prod_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{F}(b, p^{-j}, K)$$

with the obvious restriction maps in the directed system. Passing to the dual, we have

$$D(\mathbb{Z}_p, K) = \varprojlim_{j \geqslant 1} \prod_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{F}(b, p^{-j}, K)'.$$

As an algebra, $D(\mathbb{Z}_p, K)$ also has a geometric description. Indeed, recall the locally analytic characters from Example 11.8(ii): given any $z \in K$ with $|1 - z| < 1$, there is a locally analytic character χ_z of $\mathbb{Z}_p$. Thus, given any distribution $\lambda \in D(\mathbb{Z}_p, K)$, we may 'evaluate' λ at z by evaluating λ at $\chi_z \in C^{\mathrm{an}}(\mathbb{Z}_p, K)$. This associates to λ a function called its 'Fourier transform'. Using Fourier theory, Amice showed that this explicitly realises the distribution algebra as analytic functions on an open ball:

Theorem 11.12 ([2, 1.3 & 2.3.4]) *Let $X = \{a \in K : |a| < 1\}$ be the open unit ball. Then $D(\mathbb{Z}_p, K)$ is canonically isomorphic to the ring of rigid analytic functions on X.*

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11.3.3 A More Explicit Description of the Distribution Algebra

So far the only elements of $D(G,K)$ that we have come across are the Dirac delta distributions δ_g for $g \in G$,

$$\begin{aligned}\delta_g : C^{\mathrm{an}}(G,K) &\to K,\\ f &\mapsto f(g).\end{aligned}$$

The definition of the convolution product yields immediately the following lemma.

Lemma 11.13 *The map $g \mapsto \delta_g$ is a continuous map of monoids $G \to D(G,K)$, i.e. $\delta_{gh} = \delta_g \cdot \delta_h$ for any $g,h \in G$.*

In particular, there is a natural algebra morphism $K[G] \to D(G,K)$. If G is a compact, we can even go further: define the completed group algebra (or Iwasawa algebra) by

$$K[[G]] := (\varprojlim \mathbb{Z}_p[G/N]) \otimes_{\mathbb{Z}_p} K,$$

where the inverse limit is taken over all open normal subgroups N of G (note that G/N is then a finite group by compactness of G).

In fact (compare e.g. the introduction of [38]), $K[[G]]$ is the dual of all *continuous* functions $G \to K$, so our morphism actually extends naturally to an algebra morphism $\theta : K[[G]] \to D(G,K)$.

Theorem 11.14 ([38, Theorem 5.2]) *Let G_0 denote G, but regarded as a Lie group over $\mathbb{Q}_p$ rather than L. The morphism $L[[G]] \to K[[G]] \to D(G_0,K)$ is faithfully flat.*

In other words, we can study the ('more classical') $L[[G]]$-modules by passing to $D(G_0,K)$-modules (applying $D(G_0,K) \otimes_{L[[G]]} -$) without losing any information.

Proposition 11.15 ([36, Lemma 3.1]) *The Dirac distributions δ_g span a dense subspace of $D(G,K)$.*

Proof We only sketch the idea here. Let $H \leqslant G$ be a compact open subgroup. Then the coset decomposition of Lemma 11.10(ii) yields

$$D(G,K) = \bigoplus_{g \in G/H} \delta_g * D(H,K).$$

Using Lemma 11.13, we can thus reduce to the case where G itself is compact. In particular, $D(G,K)$ is a reflexive Fréchet space (Lemma 11.10(i)), so that $D(G,K)' \cong C^{\mathrm{an}}(G,K)$. Since an element f of $C^{\mathrm{an}}(G,K)$ is zero if and only if

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$f(g) = \delta_g(f)$ is zero for all g, the result then follows from the Hahn–Banach theorem as given in Theorem 11.35 in the Appendix (as there is no non-zero functional on $D(G,K)$ vanishing on the closure of the K-span of all Dirac distributions). □

We thus often think of $D(G,K)$ as an 'analytic' group algebra. A special property of $D(G,K)$ is that apart from the distributions induced by the group G, it also contains distributions induced by the Lie algebra:

Let $\mathfrak{g}$ be the Lie algebra of G, e.g. $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{Q}_p)$. Write $\mathfrak{g}_K = \mathfrak{g} \otimes K$. If $x \in \mathfrak{g}$, we can form the distribution $\mathrm{dist}(x)$ by

$$\mathrm{dist}(x)(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f(\exp(tx)))|_{t=0}.$$

This gives a linear map $\mathrm{dist} : \mathfrak{g} \to D(G,K)$, sending $[x,y]$ to the commutator $\mathrm{dist}(x)\mathrm{dist}(y) - \mathrm{dist}(y)\mathrm{dist}(x)$ – so we obtain an algebra morphism $d : U(\mathfrak{g}_K) \to D(G,K)$.

Lemma 11.16 ([22, Korollar 4.7.4]) *The map d is injective. The closure of $U(\mathfrak{g}_K)$ in $D(G,K)$ is a Fréchet algebra which we denote by $\widehat{U(\mathfrak{g}_K)}$.*

At first, the object $\widehat{U(\mathfrak{g}_K)}$ might seem strange, but its elements are actually very concrete. If $x_1,\dots,x_d$ is an ordered K-basis of $\mathfrak{g}_K$, then the Poincaré–Birkhoff–Witt theorem states that $U(\mathfrak{g}_K)$ admits a K-basis of the form

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d},$$

where $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}_0^d$. Now [36, Lemma 2.4] shows that an arbitrary element of $\widehat{U(\mathfrak{g}_K)}$ can be written uniquely as

$$\sum_{\alpha \in \mathbb{N}_0^d} \lambda_\alpha x^\alpha,\ \lambda_\alpha \in K,\ \pi^{-|\alpha| n} \lambda_\alpha \to 0 \text{ as } |\alpha| \to \infty\ \forall n,$$

where $\pi \in K$ is any non-zero element with $|\pi| < 1$.

It is worth contemplating this particular convergence condition for a while, as it appears quite naturally in several places in p-adic representation theory and rigid analytic geometry.

Recall from subsection 11.2.1 that a power series $\sum \lambda_\alpha X^\alpha$ converges on a ball of radius ε if and only if $\varepsilon^{|\alpha|}|\lambda_\alpha| \to 0$. We should therefore think of the convergence condition in the description of $\widehat{U(\mathfrak{g}_K)}$ as requiring an infinite radius of convergence – i.e. elements of $\widehat{U(\mathfrak{g}_K)}$ 'look like' analytic functions on the dual vector space $(\mathfrak{g}_K)^*$. Just as $U(\mathfrak{g}_K)$ is a non-commutative deformation of the ring of polynomial functions on $(\mathfrak{g}_K)^*$, as expressed in the Poincaré–Birkhoff–Witt isomorphism $\operatorname{gr} U(\mathfrak{g}_K) \cong \operatorname{Sym}(\mathfrak{g}_K)$, we should think of $\widehat{U(\mathfrak{g}_K)}$ as a non-commutative version of the ring of (globally) analytic functions on $(\mathfrak{g}_K)^*$.

In summary, $D(G,K)$ is a topological group algebra which is sufficiently thickened to also incorporate the infinitesimal information, present in the form of the Lie algebra.

We now turn to the study of $D(G,K)$-modules and their role in locally analytic representation theory. Just as with the usual group algebra, a locally analytic G-representation carries a natural $D(G,K)$-module structure (this is actually a bit subtle to show, see [36, section 3] for details). It turns out, however, that it is more useful to dualize this operation to get a better handle on the topology.

Theorem 11.17 ([36, Corollary 3.3]) *There is an anti-equivalence of categories*

$$\{\textit{locally analytic } G\textit{-representations on spaces of compact type}\}$$
$$\downarrow$$
$$\{\textit{sep. continuous } D(G,K)\textit{-modules in nuclear Fréchet spaces}\}$$

given by sending V to its strong dual V'.

Remark We refer to the appendix for the definitions of 'compact type' and 'nuclear', and only comment on their function in this result: compact type is a property that ensures that V is reflexive, i.e. $(V')' \cong V$, and nuclear Fréchet spaces are those spaces which are dual to spaces of compact type. The conditions are therefore necessary in the theorem above to ensure that the duality functor is an anti-equivalence on the underlying topological vector spaces.

11.4 Fréchet–Stein Algebras

We saw above how we can think of locally analytic G-representations (of compact type) as certain topological modules over the distribution algebra $D(G,K)$. The problem persists however that these are topological modules, and doing algebra with topological objects is quite difficult. For instance, the category of topological $D(G,K)$-modules knows no 'isomorphism theorem': if $f: M \to N$ is a morphism, then $M/\ker f$ need not be isomorphic to the image of f, as the quotient topology on $M/\ker f$ need not agree with the subspace topology on $\operatorname{im} f$. In particular, we are dealing with categories that are not abelian.

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11.4.1 Toy Model: Noetherian Banach Algebras and Finitely Generated Modules

Let A be a Noetherian Banach K-algebra, i.e. it is a Noetherian K-algebra which is complete with respect to some (submultiplicative) norm. The category of normed A-modules (or of Banach modules if we insist on completeness) displays the same problems alluded to above. But here there is an excellent remedy.

Theorem 11.18 (see [7, sections 3.7.2, 3.7.3]) *Any (abstract) finitely generated A-module can be endowed with a canonical Banach norm such that any A-module map between finitely generated modules is continuous. These norms are compatible with the formation of submodules, quotients and direct sums.*

More abstractly: there is a fully faithful functor from (abstract!) finitely generated A-modules to the category of Banach A-modules, exhibiting the former as an (abelian!) subcategory of the latter.

Proof (Rough sketch.) If M is a finitely generated A-module, there exists some surjection $A^r \to M$. We can check that this endows M with a Banach norm by checking that every submodule of A^r is closed. This quotient norm then has the property that $M \to N$ is continuous if and only if the composition $A^r \to M \to N$ is. But if N is another finitely generated A-module endowed with such a norm, then any A-linear map $A^r \to N$ is a sum of action maps and hence continuous. This shows that any A-module map $M \to N$ is automatically continuous. In particular (taking $M = N$), Banach norms arising from a different generating set give rise to an equivalent norm. Now check that any submodule of a finitely generated A-module is a closed subspace with respect to this norm, by reducing to the case of A^r. □

Remark This applies e.g. to the Tate algebra

$$K\langle x\rangle = \left\{ \sum_{i\in\mathbb{N}_0} a_i x^i : |a_i| \to 0 \right\}$$

of analytic functions on the closed unit disk, ensuring that p-adic analytic geometry (and its theory of coherent modules) is well behaved.

11.4.2 Fréchet–Stein Algebras and Coadmissible Modules

It turns out that $D(G,K)$ is hardly ever Noetherian Banach – recall for example from Theorem 11.12 that $D(\mathbb{Z}_p,K)$ is the ring of analytic functions on the *open*

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unit disk. As the open disk is the union of countably many closed disks (with radius approaching 1), $D(\mathbb{Z}_p, K)$ is not Noetherian Banach, but rather an inverse limit of Noetherian Banach algebras.

Definition 11.19 ([38, section 3]) Let A be a Fréchet K-algebra. We say that A is a *Fréchet–Stein algebra* if A can be written as a countable inverse limit $A = \varprojlim A_n$, where each A_n is a Noetherian Banach K-algebra such that $A_{n+1} \to A_n$ has dense image and turns A_n into a flat A_{n+1}-module on both sides.

An A-module M is called *coadmissible* if $M = \varprojlim M_n$, where M_n is a finitely generated A_n-module, and the natural morphism $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ is an isomorphism.

Example 11.20 As before, let $\pi \in K$ be non-zero with $|\pi| < 1$. Let

$$A_n = K\langle \pi^n x\rangle = \left\{\sum a_i x^i : \ \pi^{-in} a_i \to 0\right\}$$

be the ring of analytic function on a closed disk of radius $|\pi|^{-n}$. Then $A = \varprojlim A_n$ is the ring of analytic functions on $X = \cup \mathrm{Sp} A_n$, the ‘affine line’. Then A is a Fréchet–Stein algebra, and coadmissible A-modules are precisely the global sections of coherent $\mathcal{O}_X$-modules.

By exactly the same argument, $D(\mathbb{Z}_p, K)$ is a Fréchet–Stein algebra, and coadmissible $D(\mathbb{Z}_p, K)$-modules are given as global sections of coherent modules on the open unit disk.

Just as in these examples, the M_n can in general be recovered from M, which allows us to go back and forth between M and its ‘Noetherian levels’.

Lemma 11.21 ([38, Corollary 3.1]) *If M is a coadmissible A-module, then the natural morphism $A_n \otimes_A M \to M_n$ is an isomorphism.*

Proof (Sketch.) By our density assumption, $A_n \otimes_A M \to M_n$ has dense image. It follows from our toy model that any (automatically finitely generated) A_n-submodule of M_n is closed, so the morphism is in fact surjective. For injectivity, suppose that $\sum_{i=1}^k b_i \otimes x_i \in A_n \otimes_A M$ is in the kernel. We now consider the morphisms $\varphi_m : A_m^k \to M_m$ sending $(a_1, \dots, a_k)$ to $\sum a_i \overline{x_i}$, where $\overline{x_i}$ denotes the image of x_i in M_m. By construction, $\varprojlim \ker \varphi_m$ is a coadmissible A-module, and applying the same surjectivity argument as above, this time to $\varprojlim \ker \varphi_m$, we can compute that $\sum b_i \otimes x_i = 0$. This proves injectivity. □

We now extend the ideas from our toy model to the Fréchet–Stein setting.

Lemma 11.22 ([38, paragraph after Lemma 3.6]) *Any coadmissible A-module can be endowed with a canonical Fréchet topology such that any A-module*

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morphism between coadmissible A-modules is continuous. Kernel and cokernel of any morphism between coadmissible A-modules are also coadmissible.

Proof Equip each M_n with its canonical Banach norm and take the limit. Any morphism $M \to N$ then gives rise to A_n-module morphisms $M_n \to N_n$ by Lemma 11.21, and these are continuous by our toy model. Thus $M \to N$ is continuous by definition of the inverse limit topology. It is easy to verify that kernels and cokernels are given as the inverse limit of the corresponding kernels and cokernels on the Noetherian level. □

Proposition 11.23 ([38, Lemma 3.6]) *Let A be a Fréchet–Stein algebra and let M be a coadmissible A-module. Let $N \leqslant M$ be a submodule. The following are equivalent:*

(i) N is coadmissible.
(ii) N is closed with respect to the canonical topology on M.
(iii) M/N is coadmissible.

Corollary 11.24 *The category of coadmissible A-modules is an abelian category and contains all finitely presented A-modules.*

In other words, the theory of coadmissible modules over Fréchet–Stein algebras follows the same philosophy as finitely generated modules over Noetherian Banach algebras did in our toy model: they provide us with an abelian category of 'topological' modules which can be manipulated purely algebraically.

11.4.3 Distribution Algebras of Compact Groups Are Fréchet–Stein

It thus remains to show that distribution algebras are indeed Fréchet–Stein algebras (provided that G is compact). The proof of this result, given in [38], is quite involved, but we can at least indicate the main ideas. We have already discussed the example of $G = \mathbb{Z}_p$ above, which is not difficult to generalize to $G = \mathbb{Z}_p^d$. We then generalize further to the case where G is still structurally similar to $\mathbb{Z}_p^d$ – namely, that G is uniform pro-p. Since any compact p-adic Lie group contains such a group as an open subgroup, we can then conclude with the usual coset decomposition from Lemma 11.10(ii).

Instead of giving a formal definition, let us treat uniform pro-p groups as a black box and discuss which properties might help us to generalize from $\mathbb{Z}_p^d$ to a more general class of groups.

Definition 11.25 Let G be a locally $\mathbb{Q}_p$-analytic group which is pro-p. We say that G admits an *ordered basis* if there exists an ordered set of topological

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generators $h_1, \dots, h_d$ such that the map

$$\mathbb{Z}_p^d \to G,$$
$$(x_1, \dots, x_d) \mapsto h_1^{x_1} \dots h_d^{x_d}$$

is a homeomorphism.

In particular, an ordered basis yields a global chart for the group G, giving us isomorphisms of topological vector spaces $C^{\mathrm{an}}(\mathbb{Z}_p^d, K) \cong C^{\mathrm{an}}(G, K)$ and hence $D(\mathbb{Z}_p^d, K) \cong D(G, K)$. Note, however, that this tells us nothing about the algebra structure on $D(G, K)$: if the h_i do not commute, then we obtain a non-commutative convolution product which cannot be read off from the chart. In other words, we can use this description to bring elements of $D(G, K)$ into a standard form, but the question remains how to multiply two such expressions.

On the other hand, if the h_i do commute, then the above chart is a group isomorphism and we have indeed reduced to the case of $G = \mathbb{Z}_p^d$. The crucial idea is therefore to consider groups admitting an ordered basis which is commutative 'up to higher order terms'.

Lemma 11.26 ([17, Theorem 8.18, Theorem 4.9, Lemma 4.10]) *Let G be a uniform pro-p group. Then G is locally* $\mathbb{Q}_p$*-analytic and admits an ordered basis* $\mathrm{h}_1, \dots, \mathrm{h}_d$ *such that each commutator* $h_i h_j h_i^{-1} h_j^{-1}$ *is a pth power, i.e. there exists* $g \in G$ *such that* $h_i h_j h_i^{-1} h_j^{-1} = g^p$*.*

With this result in place, one can employ graded methods to show (after a significant amount of work) the following.

Proposition 11.27 ([38, Theorem 4.10]) *If G is a uniform pro-p group, viewed as a locally* $\mathbb{Q}_p$*-analytic group, then* $D(G, K)$ *is a Fréchet–Stein algebra.*

Lemma 11.28 ([17, Corollary 8.34]) *Every compact locally* $\mathbb{Q}_p$*-analytic group contains an open normal subgroup which is uniform pro-p.*

Theorem 11.29 ([38, Theorem 5.1], [30, Theorem 2.3]) *Let G be a compact locally L-analytic group and let* $\mathfrak{g}$ *be its Lie algebra. Then* $D(G, K)$ *and* $\widehat{U(\mathfrak{g}_K)}$ *are Fréchet–Stein algebras.*

Proof (Sketch.) For $\widehat{U(\mathfrak{g}_K)}$, note the similarity to the example of analytic functions discussed before. See [30, Theorem 2.3], which reformulates [25, Theorem 1.4.2]. A more general argument can be found in [5, Theorem 6.7].

Let G_0 be G, viewed as a locally $\mathbb{Q}_p$-analytic group, and let H be an open, normal, uniform pro-p subgroup. By Proposition 11.27, $D(H, K)$ is Fréchet–Stein, so we can write $D(H, K) \cong \varprojlim D(H, K)_n$ for some suitable Noetherian

Banach algebras. By Lemma 11.10.(ii),

$$D(G_0,K) \cong \oplus_{g\in G_0/H}\delta_g * D(H,K),$$

i.e. $D(G_0,K)$ is a free $D(H,K)$-module of finite rank (by compactness). It is then not too difficult to check that $D(G_0,K)$ is also Fréchet–Stein – it is trivially a coadmissible $D(H,K)$-module, and the only thing one needs to verify is that the algebra structure on $D(G_0,K)$ extends to an algebra structure on

$$D(G_0,K)_n := D(H,K)_n \otimes_{D(H,K)} D(G_0,K) \cong \oplus\delta_g * D(H,K)_n,$$

as $D(G_0,K)_n$ then inherits all desired properties from $D(H,K)_n$. Thus $D(G_0,K)$ is Fréchet–Stein.

Finally, embedding locally $\mathbb{Q}_p$-analytic functions $G \to K$ into the space of locally L-analytic functions $G \to K$ yields dually a surjection $D(G_0,K) \to D(G,K)$, exhibiting $D(G,K)$ as a topological quotient algebra of $D(G_0,K)$. We can thus deduce that $D(G,K)$ is Fréchet–Stein. □

It is worth pointing out a feature which is quite common in this theory: the general case is often much less accessible than the case '$L = \mathbb{Q}_p$', so that we are forced to take a detour via G_0 and descend along $D(G_0,K) \to D(G,K)$. This also explains the restrictions in the faithful flatness result Theorem 11.14: while some properties, like being Fréchet–Stein, are preserved when taking the (Fréchet) quotient, this certainly need not be the case for the property of being faithfully flat. We can prove the result for G_0 (again making use of uniform pro-p subgroups), but a priori this does not tell us anything about the general case.

Let G be a compact locally L-analytic group. By the above, it now makes sense to talk about coadmissible $D(G,K)$-modules. We note that coadmissible modules are indeed contained in the category appearing in the equivalence of Theorem 11.17.

Lemma 11.30 ([38, Lemma 6.1]) *Let G be compact. Then any coadmissible $D(G,K)$-module, endowed with its canonical Fréchet structure, is a nuclear Fréchet space.*

This justifies the following definition.

Definition 11.31 Let G be a locally L-analytic group, and let V be a locally analytic G-representation of compact type. We say that V is *admissible* if V' is a coadmissible $D(H,K)$-module for some (equivalently, for any) compact subgroup $H \leqslant G$.

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We have thus reached our goal. We have the following commutative diagram (for G compact), where the horizontal arrows are (anti-)equivalences of categories and the vertical arrows are fully faithful embeddings.

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{admissible}\\ G\text{-representations}\end{array}\right\} & \xrightarrow{\cong} & \left\{\begin{array}{c}\text{coadmissible}\\ D(G,K)\text{-modules}\end{array}\right\} \\
\big\downarrow & & \big\downarrow \\
\left\{\begin{array}{c}\text{locally analytic}\\ G\text{-representations on}\\ \text{spaces of compact type}\end{array}\right\} & \xrightarrow{\cong} & \left\{\begin{array}{c}\text{sep. continuous}\\ D(G,K)\text{-modules in}\\ \text{nuclear Fréchet spaces}\end{array}\right\}
\end{array}$$

As mentioned before, the crucial feature is that the category of coadmissible $D(G,K)$-modules is an abelian category which embeds naturally into the category of abstract $D(G,K)$-modules. In this way, admissible representations can now be studied purely algebraically.

11.4.4 Some Further Directions

We give here a (non-exhaustive) list of further developments in the field:

(i) As mentioned in the introduction, the p-adic local Langlands conjecture relates n-dimensional Galois representations to certain unitary Banach representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. Often it is helpful to restrict a unitary representation to its locally analytic vectors (analogously to the smooth vectors for complex representations). From this viewpoint, the faithful flatness result Theorem 11.14 ensures that the functor of taking locally analytic vectors is exact and does not annihilate any non-zero representations (see [38, Theorem 7.1]).

(ii) These lectures should already have illustrated that some ideas from p-adic geometry arise quite naturally in the study of locally analytic representations. It is therefore not surprising that a number of distinctly geometric tools have been developed, often influenced by complex geometric representation theory. For instance, if $G = \mathbb{G}(L)$ for some split reductive algebraic group $\mathbb{G}$, one can establish an equivalence of categories between coadmissible $D(G,K)$-modules with trivial infinitesimal central character and a certain class of G-equivariant p-adic $\mathcal{D}$-modules on the p-adic analytic flag variety, analogously to Beilinson–Bernstein theory over the complex numbers – see e.g. [4], [3], [29].

There is also a growing body of literature on particular representations coming from geometry as certain cohomology groups (e.g. the cohomology of Drinfeld's upper half plane and its tower of coverings), often motivated by the Langlands programme – see e.g. [27], [20], [18].

(iii) One phenomenon which we have exploited regularly is the description of $D(\mathbb{Z}_p, K)$ as the ring of functions on the open unit disk, where we could view the open unit disk as the set of all locally analytic characters of $\mathbb{Z}_p$. This observation is also quite helpful in the theory of (φ, Γ)-modules (and hence in Colmez's work on the pLLC for $\mathrm{GL}_2(\mathbb{Q}_p)$). Schneider and others have generalised this approach to consider 'character varieties' for the ring of integers of more general finite field extensions of $\mathbb{Q}_p$ (see [34], [6]).

(iv) The study of coadmissible $\widehat{U(\mathfrak{g}_K)}$-modules can be developed along similar lines as the representation theory of complex Lie algebras. For instance, the centre of $\widehat{U(\mathfrak{g}_K)}$ can be described via an extension of the usual Harish-Chandra isomorphism, c.f. [25, Theorem 2.1.6]. In fact, if G is the group of L-rational points of a connected, split reductive algebraic group over L and if the field K is discretely valued, the Harish-Chandra isomorphism extends to the distribution algebra $D(G, K)$, c.f. [25, Theorem 2.4.2]. There is also an analogue of the BGG category $\mathcal{O}$ for $\widehat{U(\mathfrak{g}_K)}$, see [31].

(v) For more details on locally analytic induction, see the discussion of principal series representations of GL_2 in [36]. We also mention that Orlik–Strauch [28] give a functor which constructs a locally analytic G-representation out of a P-equivariant $\widehat{U(\mathfrak{g}_K)}$-module in the BGG category $\mathcal{O}$ and a smooth P-representation (P being a parabolic subgroup of a split reductive group G).

(vi) As one might imagine, there are many links between the p-adic theory and the mod p theory: for example, if V is a unitary Banach representation of G over $\mathbb{Q}_p$, then its unit ball is a representation over $\mathbb{Z}_p$ and we can consider its reduction mod p. There is a mod p Langlands conjecture which is being developed in tandem with the p-adic one, see e.g. [11], [10].

11.5 Appendix: Some Non-archimedean Functional Analysis

Throughout we made use of various notions appearing in non-archimedean functional analysis. Here we collect the various facts and definitions required. A main reference for this material is e.g. [32]. Throughout this appendix, K is a field equipped with a NAAV and is complete.

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11.5.1 Locally Convex Spaces

Definition 11.32 A (non-archimedean) *semi-norm* on a K-vector space V is a function $q : V \to \mathbb{R}_{\geqslant 0}$ satisfying:

(i) $q(av) = |a|\, q(v)$ for all $v \in V$ and $a \in K$; and
(ii) $q(v+w) \leqslant \max(q(v), q(w))$ for all $v, w \in V$.

Condition (ii) is called the *strong triangle inequality*. Note that (i) implies in particular that $q(0) = 0$ and (i) and (ii) imply that the set $\{v \in V : q(v) = 0\}$ is a vector subspace of V. If furthermore the condition

(iii) $q(v) = 0$ if and only if $v = 0$

is satisfied then we say that q is a *norm*. When q is a norm, it is conventional to denote it by $||\cdot||$.

Given a family of semi-norms $(q_i)_{i \in I}$ on V, we may assign to V the coarsest topology such that $q_i : V \to \mathbb{R}$ is continuous for all $i \in I$ and all translation maps $v + \cdot : V \to V$, for $v \in V$, are continuous. In such a case we call V a *locally convex space*. If the topology is defined by a single (semi-)norm q, we call V a *(semi-)normed space*.

We note here that when a locally convex space is finite dimensional, it has nice properties:

Proposition 11.33 ([32, Proposition 4.13]) *Every Hausdorff locally convex topology on a finite dimensional vector space $V = K^n$ ($n \geqslant 1$) is equivalent to the one defined by the norm $||(a_1, \ldots, a_n)|| = \max_{1 \leqslant i \leqslant n} |a_i|$.*

There is another equivalent definition of a locally convex topology (see [32, Proposition 4.4]). Given a K-vector space V, a *lattice* in V is an $\mathcal{O}_K$-submodule M of V such that the canonical map $M \otimes_{\mathcal{O}_K} K \to V$ is an isomorphism. Then V is locally convex if and only if there exists a non-empty family of lattices $\{M_i\}_{i \in I}$ of V with the properties

(i) for any $i \in I$ and $a \in K^\times$ there exists $j \in I$ such that $M_j \subseteq aM_i$; and
(ii) for any two $i, j \in I$, there exists $k \in I$ such that $M_k \subseteq M_i \cap M_j$.

A basis for the topology on V is then given by the subsets of the form $v + M_i$ for $v \in V$ and $i \in I$.

We now describe explicitly various constructions with locally convex spaces that we use in this chapter.

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- Suppose $\{V_i\}_{i\in I}$ is an inductive system of locally convex K-vector spaces. Then we can equip $\varinjlim_{i\in I} V_i$ with the final locally convex topology with respect to the maps $\varphi_j : V_j \to \varinjlim_{i\in I} V_i$. Explicitly, it is defined by the family of all lattices $M \subseteq \varinjlim_{i\in I} V_i$ such that $\varphi_j^{-1}(M) \subseteq V_j$ is open for all $j \in I$.
- Suppose $\{V_i\}_{i\in I}$ is a family of locally convex K-vector spaces. Then we may equip $\bigoplus_{i\in I} V_i$ with the final locally convex topology with respect to the inclusions $V_j \to \bigoplus_{i\in I} V_i$ as above.
- Suppose again that $\{V_i\}_{i\in I}$ is a family of locally convex K-vector spaces. Then we may define a locally convex topology on the direct product $\prod_{i\in I} V_i$ this time using seminorms as follows: if for each $j \in I, (q_{jk})_{k\in J}$ denotes the family of seminorms defining the topology on V_j, then we give $\prod_{i\in I} V_i$ the locally convex topology defined by the family of seminorms

$$\prod_{i\in I} V_i \to V_j \overset{q_{jk}}{\to} \mathbb{R}_{\geqslant 0}$$

for $j \in I$ and $k \in J$.
- Given a locally convex K-vector space V, we let

$$V' = \{\varphi \in V^* : \varphi \text{ is continuous}\}$$

be the continuous dual of V. We define a subset $B \subseteq V$ to be *bounded* if for any open lattice $M \subseteq V$, there exists $a \in K$ such that $B \subseteq aM$.[2] Given a bounded $B \subseteq V$, we may define a seminorm p_B on V' by

$$p_B(\varphi) := \sup_{b\in B} |\varphi(b)|.$$

The seminorms $\{p_B : B \text{ bounded in } V\}$ define a locally convex topology on V' called the *strong topology*. When V is normed, this topology is just the normed topology on V' given by the operator norm

$$||\varphi|| := \sup_{0\neq v\in V} \frac{||\varphi(v)||}{||v||}.$$

We will always assume in this chapter that our duals are equipped with the strong topology. We next turn to the Hahn–Banach theorem. For this we need the notion of a spherically complete base field.

Definition 11.34 We say that our field K is *spherically complete* if for any decreasing sequence $B_1 \supset B_2 \supset \cdots$ of balls in K, the intersection $\bigcap_{i\in\mathbb{N}} B_i$ is non-empty.

[2] When V is normed this is equivalent to saying $||B||$ is bounded in $\mathbb{R}$.

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As an example, any finite extension of $\mathbb{Q}_p$ is spherically complete because these fields are locally compact. But not all complete non-archimedean fields are spherically complete. For instance, one may consider the algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. There is a unique extension of the p-adic absolute value to it, but it is no longer complete. Its completion, denoted by $\mathbb{C}_p$, is a complete non-archimedean field and is not spherically complete. See [32, Section I.1] for more details.

This condition is required in order to get that the dual V' of a locally convex vector space is non-zero, which is a consequence of the following Hahn–Banach theorem:

Theorem 11.35 ([32, Proposition 9.2]) *Suppose that K is spherically complete. Let V be a K-vector space, q a seminorm on V and $V_0 \leqslant V$ a vector subspace. Then for any linear form $f_0 : V_0 \to K$ such that $|f_0(v)| \leqslant q(v)$ for all $v \in V_0$, there exists a linear form $f : V \to K$ such that $f|_{V_0} = f_0$ and $|f(v)| \leqslant q(v)$ for all $v \in V$.*

It follows as a corollary of this theorem that any continuous seminorm on a vector subspace V_0 of a locally convex space V extends to a continuous seminorm on V.

11.5.2 Fréchet and Banach spaces

If $||\cdot||$ is a norm on V, the normed space topology on V as we defined it is the usual metric topology coming from $||\cdot||$.

Definition 11.36 A normed space that is complete, meaning that all Cauchy sequences converge, is called a *Banach space*. More generally, a locally convex space that is metrizable and complete is called a *Fréchet space*.

Of course, any Banach space is in particular a Fréchet space. Also, K is itself a Banach space with the absolute value as a norm, and more generally K^n is Banach with the topology given in Proposition 11.33.

In general, a Fréchet space V can be described as follows. Because V is metrizable, there is an increasing countable family of seminorms $q_1 \leqslant q_2 \leqslant \cdots$ defining the topology on V (see [32, Proposition 8.1]). Then for each $i \geqslant 1$, we denote by V_i the metric space completion of $V/\{v \in V : q_i(v) = 0\}$, which is a Banach space with the norm induced by q_i. There is moreover a continuous linear map $V_{i+1} \to V_i$. Then V is canonically isomorphic to the projective limit $\varprojlim V_i$. So one may describe Fréchet spaces as being the countable projective limits of Banach spaces.

We now turn to the notion of a compact type space:

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Definition 11.37 Let V and W be Hausdorff locally convex K-vector spaces.

(i) A subset $B \subseteq V$ is called *compactoid* if for any open lattice $M \subseteq V$, there are finitely many vectors $v_1, \dots, v_n \in V$ such that $B \subseteq M + \mathcal{O}_K v_1 + \dots + \mathcal{O}_K v_n$.
(ii) A bounded $\mathcal{O}_K$-submodule $B \subseteq V$ is called *c-compact* if it is compactoid and complete.
(iii) A continuous linear map $f : V \to W$ is called *compact* if there is an open lattice $M \subseteq V$ such that the closure of $f(M)$ in W is bounded and c-compact.
(iv) V is said to be of *compact type* if it is the locally convex inductive limit of a sequence

$$V_1 \xrightarrow{\iota_1} V_2 \xrightarrow{\iota_2} V_3 \xrightarrow{\iota_3} \cdots$$

where each V_i is a Banach space and each map ι_i is an injective compact map.

The key property of these spaces is the following:

Theorem 11.38 ([32, Proposition 16.10]) *Any locally convex K-vector space of compact type $V = \varinjlim_{n\geqslant 1} V_n$ is reflexive, i.e. the evaluation map $V \to (V')'$ is an isomorphism, its strong dual V' is Fréchet and there is a canonical isomorphism $V' \cong \varprojlim_{n\geqslant 1} V_i'$.*

There is a dual notion:

Definition 11.39 A Fréchet space V is called *nuclear* if $V \cong \varprojlim_{n\geqslant 1} V_n$ is a countable projective limit of Banach spaces, where the transition maps $V_{n+1} \to V_n$ are all compact.

There is more generally a notion of nuclear locally convex space, see [32, Section 19]. The key fact we'll need is the following:

Theorem 11.40 ([36, Theorem 1.3]) *A Fréchet space V is the strong dual of a locally convex K-vector space of compact type if and only if V is nuclear.*

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