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Auslander–Reiten Theory of Finite-Dimensional Algebras

Raquel Coelho Simões

Introduction

Representation theory is the study of algebraic structures such as groups and rings via their actions on simpler algebraic structures such as vector spaces. In this survey chapter, which served as a basis for a lecture series at the LMS Autumn Algebra School in October 2020, we consider representation theory of finite-dimensional algebras. Quivers provide a useful concrete way to visualise algebras. Given a quiver, i.e. a directed graph, one can associate an algebra, called a *path algebra*, generated by the paths of the quiver. From the point of view of representation theory, the study of finite-dimensional algebras reduces to the study of quotients of path algebras.

A central aim in representation theory of finite-dimensional algebras is to classify all their modules and the morphisms between them. Due to the Krull–Schmidt theorem, the classification of modules can be reduced to the classification of indecomposable modules. That is, in some sense, indecomposable modules are the building blocks of all modules. It is then natural to ask when is a finite-dimensional algebra of *finite representation type*, i.e. when does it have finitely many indecomposable modules up to isomorphism? Gabriel's theorem [22] gives an elegant answer for path algebras of quivers without oriented cycles, also called *hereditary algebras*. This theorem is an example of an ADE classification, i.e. in terms of simply laced Dynkin diagrams. There are many other examples of objects classified by these diagrams, including representation-finite selfinjective algebras [35], irreducible root systems and semisimple Lie algebras (see e.g. [28]) and cluster algebras of finite type [20].

Auslander–Reiten theory gives us a way to visualise the representation theory of a finite-dimensional algebra using a quiver, called the Auslander–Reiten quiver. The vertices of this quiver correspond to the indecomposable modules

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and the arrows correspond to *irreducible morphisms*, which are the corresponding building blocks for the morphisms.

The aim of this survey chapter is to give a brief introduction to Auslander–Reiten theory and to provide methods for constructing Auslander–Reiten quivers. We present two methods to construct these quivers for some special classes of algebras. The first method is the *knitting algorithm*, which works for instance for hereditary algebras of finite representation type. The second method is a *geometric model* associated to (partial) triangulations of surfaces. This method, which has its origins in cluster-tilting theory [15], encodes the representation theory of an important class of algebras, called *gentle algebras* and more generally *skew-gentle algebras*, which have been the subject of intensive study since the 1980s due to the fact that they remain one of the relatively few classes of algebras for which the representation theory is computationally tractable.

The prerequisites are a basic knowledge of linear algebra and rings and modules. Knowledge of the basic concepts of category theory is beneficial, but not essential. The list of references is not exhaustive, but it includes some of the main references for this subject. We refer the reader to [5, 6, 9, 38] for further study on quiver representations and Auslander–Reiten theory. The language of categories used in these theories is also nicely explained in [6, 38].

Conventions: Throughout this chapter, we consider vector spaces, linear maps and algebras over an algebraically closed field $\mathbf{k}$. Every algebra will be a finite-dimensional associative algebra with unit and every module is considered to be a finite-dimensional right module. For a treatment of infinite-dimensional modules, see Chapter 4.

1.1 Bound Path Algebras

In this section we will associate algebras to quivers, i.e. directed graphs. From a representation-theoretic point of view, we will see that it is enough to study algebras associated to quivers.

Definition 1.1 A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of the following data:

1 a set Q_0 of vertices,
2 a set Q_1 of arrows between vertices,
3 two maps $s, t : Q_1 \to Q_0$, called *source* and *target*, respectively, such that, for each arrow $\alpha : i \to j \in Q_1$, $i = s(\alpha)$ and $j = t(\alpha)$.

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A quiver is *finite* if Q_0 and Q_1 are finite sets. Throughout these notes, we will only consider finite and connected quivers.

Definition 1.2 Let Q be a quiver.

1 A *path in Q of length ℓ* is a sequence $p = \alpha_1\alpha_2\cdots\alpha_\ell$, with $\alpha_i \in Q_1$ such that $s(\alpha_i) = t(\alpha_{i-1})$ for each $i = 2,\dots,\ell$. In particular, p has length 1 if and only if $p \in Q_1$.
2 We associate a path ε_i of length 0 to each vertex i of Q, which is called the *stationary path at i*.
3 If $s(\alpha_1) = t(\alpha_\ell)$, then p is said to be an *oriented cycle*. An oriented cycle of length 1 is called a *loop*. An *acyclic* quiver is a quiver with no oriented cycles.

Sometimes we denote a path from i to j by $i \rightsquigarrow j$.
Throughout $\mathbf{k}$ denotes an algebraically closed field.

Definition 1.3 The *path algebra* $\mathbf{k}Q$ of Q is an algebra whose underlying vector space has all the paths of Q as basis and with multiplication defined on two basis elements given by concatenation of paths, i.e. given two paths $p = \alpha_1\cdots\alpha_\ell, p' = \alpha'_1\cdots\alpha'_m$,

$$pp' = \begin{cases} \alpha_1\cdots\alpha_\ell\alpha'_1\cdots\alpha'_m & \text{if } t(\alpha_\ell) = s(\alpha'_1) \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.4 1 Let Q be the quiver:

$$\begin{tikzcd} 1 \arrow[loop above, "\alpha"] \end{tikzcd}$$

$\mathbf{k}Q$ has basis given by $\{\alpha^t \mid t \geqslant 0\}$, where α^0 denotes the stationary path ε_1. The multiplication is given by $\alpha^s\alpha^t = \alpha^{s+t}$. The algebra $\mathbf{k}Q$ is isomorphic to the algebra $\mathbf{k}[x]$ of polynomials with one indeterminate.

2 Let Q be the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$$

$\mathbf{k}Q$ is generated by the paths $\varepsilon_i (1 \leqslant i \leqslant n), \alpha_i (1 \leqslant i \leqslant n), \alpha_i\cdots\alpha_j (1 \leqslant i < j \leqslant n)$, and it is isomorphic to the algebra of upper triangular $n\times n$ matrices.

Remark The path algebra $\mathbf{k}Q$ satisfies the following properties:

1 $\mathbf{k}Q$ has an identity $1 = \sum_{i\in Q_0}\varepsilon_i$ if and only if Q_0 is finite.

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2 $\mathbf{k}Q$ is an associative algebra.

3 $\mathbf{k}Q$ is finite dimensional if and only if Q is finite and acyclic.

Definition 1.5 Let Q be a finite quiver.

1 The *arrow ideal* R_Q is the two-sided ideal of $\mathbf{k}Q$ generated by all arrows in Q.
2 An *admissible ideal* I is a two-sided ideal of $\mathbf{k}Q$ such that there is $m \geqslant 2$ for which $R_Q^m \subseteq I \subseteq R_Q^2$.
3 Given an admissible ideal I, the quotient algebra $\mathbf{k}Q/I$ is said to be a *bound path algebra*.

The bound path algebra $\mathbf{k}Q/I$ is finite dimensional, since $R_Q^m \subseteq I$ and it is connected (i.e. it is not the direct product of two algebras) because Q is connected and $I \subseteq R_Q^2$.

A *relation* ρ is a linear combination $\rho = \sum_p \lambda_p p$ of paths, all with length at least two, and with same start and same endpoints. It is easy to check that any admissible ideal can be generated by a set of relations.

Example 1.6 Let Q be the quiver

$$\begin{tikzcd}
& 2 \arrow[dl, "\alpha_2"'] & \\
1 \arrow[loop left, "\alpha_3"] & & 3 \arrow[ul, "\alpha_1"'] \arrow[ll, "\alpha_6"] \arrow[dl, "\alpha_5"] \\
& 4 \arrow[ul, "\alpha_4"] &
\end{tikzcd}$$

The ideal $I_1 = \langle \alpha_1\alpha_2 - \alpha_5\alpha_4, \alpha_6\alpha_3, \alpha_2\alpha_3, \alpha_3^4 \rangle$ is admissible since $R_Q^5 \subseteq I \subseteq R_Q^2$.
The ideal $I_2 = \langle \alpha_1\alpha_2 - \alpha_5\alpha_4, \alpha_6\alpha_3, \alpha_2\alpha_3 \rangle$ is not admissible because $\alpha_3^m \notin I_2$ for all $m \geqslant 2$.
The ideal $I_3 = \langle \alpha_1\alpha_2 - \alpha_6 \rangle$ is not admissible as $\alpha_1\alpha_2 - \alpha_6 \notin R_Q^2$.

The following theorem is due to [22].

Theorem 1.7 *Any finite-dimensional algebra A is Morita equivalent to a bound path algebra* $\mathrm{k}Q/I$*, i.e.* $\mathsf{mod}(A) \simeq \mathsf{mod}(\mathrm{k}Q/I)$.

For a proof, see [6, I.6.10, II.3.7].

1.2 Representations of a Bound Path Algebra

In the previous section we saw that quivers provide a nice way to visualise finite-dimensional algebras. Now, we will explain how quivers can be used to visualise also modules and morphisms between modules.

Throughout this section Q denotes a finite, connected quiver and I an admissible ideal. Note that if $I = 0$ is admissible, then Q must be acyclic.

Definition 1.8 A *representation* $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ *of* Q is given by:

- $\mathbf{k}$-vector spaces M_i for all $i \in Q_0$, and
- linear maps $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ for all $\alpha \in Q_1$.

Let $p = \alpha_1 \cdots \alpha_\ell$ be a path in Q and $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be a representation of Q. We denote by φ_p the composition of linear maps $\varphi_p = \varphi_{\alpha_\ell} \cdots \varphi_{\alpha_1}$. Given a relation $\rho = \sum_p \lambda_p p$ in I, we have $\varphi_\rho = \sum_p \lambda_p \varphi_p$.

Definition 1.9 A representation $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q is said to be *bound by* I, or to be a *representation of* (Q, I), if $\varphi_\rho = 0$ for all $\rho \in I$.

A representation M is *finite dimensional* if M_i is finite dimensional for all $i \in Q_0$. The *dimension vector* of M is the vector $\underline{\dim} M = (\dim M_i)_{i \in Q_0}$.

Example 1.10 Consider the quiver Q:

$$\begin{tikzcd}
& 2 \arrow[dl, "\alpha_2"'] & \\
1 \arrow[rr, "\alpha_3"'] & & 3 \arrow[ul, "\alpha_1"']
\end{tikzcd}$$

bound by $I = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1 \rangle$. The representation:

$$\begin{tikzcd}
& \mathbf{k}^2 \arrow[dl, "\left[\begin{smallmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{smallmatrix}\right]"'] & \\
\mathbf{k}^3 \arrow[rr, "\left[\begin{smallmatrix}0 & 1 & 0\\ 0 & 0 & 0\end{smallmatrix}\right]"'] & & \mathbf{k}^2 \arrow[ul, "\left[\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right]"']
\end{tikzcd}$$

is bound by I. However, the representation given by

$$\begin{tikzcd}
& \mathbf{k} \arrow[dl, "1"'] & \\
\mathbf{k} \arrow[rr, "\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]"'] & & \mathbf{k}^2 \arrow[ul, "\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]"']
\end{tikzcd}$$

is not bound by I.

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Definition 1.11 Let $M = (M_i, \varphi_\alpha), N = (N_i, \psi_\alpha)$ be representations of (Q,I).

1 A *morphism of representations* $f : M \to N$ is a collection $(f_i)_{i\in Q_0}$ of linear maps, $f_i : M_i \to N_i$, such that for each $\alpha : i \to j \in Q_1$, the following diagram commutes:

$$\begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\ N_i \arrow[r, "\psi_\alpha"'] & N_j. \end{tikzcd}$$

2 The morphism $f = (f_i)_{i\in Q_0}$ is an *isomorphism* if each f_i is bijective.

Example 1.12 Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\alpha"] \end{tikzcd}$$

The following represents a morphism of representations:

$$\begin{tikzcd} \mathbf{k}^2 \arrow[loop left, "\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}"] & \mathbf{k}^2 \arrow[loop above, "\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}"] \end{tikzcd}$$

This morphism is bijective with inverse given by

$$\begin{tikzcd} \mathbf{k}^2 \arrow[loop left, "\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}"] & \mathbf{k}^2 \arrow[loop above, "\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}"] \arrow[l, "\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}"] \end{tikzcd}$$

We obtain the category $\mathsf{rep}(Q,I)$ of finite-dimensional bound quiver representations of (Q,I), whose objects are finite-dimensional bound quiver representations and maps are given by morphisms of bound quiver representations.

Given a finite-dimensional algebra A, we denote by $\mathsf{mod}(A)$ the category of finite-dimensional right A-modules. Note that we are adopting the same convention as [6] of taking right A-modules and reading paths in a quiver from left to right. Other sources may have the opposite convention.

Theorem 1.13 *There is an equivalence of categories* $\mathrm{mod}(\mathbf{k}Q/I) \simeq \mathsf{rep}(Q,I)$.

Proof Denote the algebra $\mathbf{k}Q/I$ by A and write $e_i = \varepsilon_i + I$. We begin by constructing a functor $F : \mathsf{mod}(A) \to \mathsf{rep}(Q,I)$.

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Given $M \in \mathrm{mod}(A)$, we define $F(M)$ to be the representation (M_i, φ_α), where $M_i = Me_i$, and $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is the map $me_{s(\alpha)} \mapsto m\bar{\alpha} := m(\alpha + I)$. Note that each φ_α is a $\mathbf{k}$-linear map since M is an A-module.

In order to check that $F(M) \in \mathrm{rep}(Q,I)$, we need to show that $F(M)$ is bound by I. Given a relation $\rho = \sum_{p:i\rightsquigarrow j} \lambda_p p$ in I, we have

$$\begin{aligned}
\varphi_\rho(me_i) &= \sum_{p:i\rightsquigarrow j} \lambda_p \varphi_p(me_i) \\
&= \sum_{p:i\rightsquigarrow j} \lambda_p m(p+I) \\
&= m \sum_{p:i\rightsquigarrow j} \lambda_p (p+I) \\
&= m(\rho + I) = m0 = 0.
\end{aligned}$$

This defines F on the objects. Now, let $f : M \to N$ be a morphism in $\mathrm{mod}(A)$, and let $F(M) = (M_i, \varphi_\alpha), F(N) = (N_i, \psi_\alpha)$. We define $F(f) = (f_i)_{i\in Q_0}$ by $f_i(me_i) := f(m)e_i$.

We need to check that $f_j\varphi_\alpha = \psi_\alpha f_i$ for each $\alpha : i \to j \in Q_1$. Indeed, given $me_i \in M_i$, we have

$$\begin{aligned}
f_j\varphi_\alpha(me_i) &= f_j(m\bar{\alpha}) = f(m\bar{\alpha})e_j \\
&= f(m)\bar{\alpha}e_j = \psi_\alpha(f(m)e_i) \\
&= \psi_\alpha(f_i(me_i)) = \psi_\alpha f_i(me_i).
\end{aligned}$$

Therefore, $F(f)$ is a morphism of representations.

It is easy to check that $F(f)$ is indeed a (covariant) functor, i.e. that $F(1_M) = 1_{F(M)}$ for any A-module M, and $F(gf) = F(g)F(f)$ for $f : L \to M, g : M \to N \in \mathrm{mod}(A)$.

The next step is to construct a functor $G : \mathrm{rep}(Q,I) \to \mathrm{mod}(A)$. Given $(M_i, \varphi_\alpha) \in \mathrm{rep}(Q,I)$, we define $G(M_i, \varphi_\alpha) = M$ as follows. The underlying vector space of M is $\oplus_{i\in Q_0} M_i$. It is enough to define the right A-action on paths in Q. Let p be a path in Q and $m = (m_i)_{i\in Q_0}$ be an element of M. If $p = \varepsilon_i$ for some i, let $mp := m_i$, and if p has length $\geqslant 1$, we define mp to be the following element in M:

$$(mp)_k := \begin{cases} 0 & \text{if } k \neq t(p) \\ \varphi_p(m_{s(p)}) & \text{if } k = t(p). \end{cases}$$

In order to check that the A-action is well defined, we need to show that if $\rho = \sum_{p:i\rightsquigarrow j} \lambda_p p \in I$, then $m\rho = 0$. Indeed, we have that $m\rho$ is the element

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in M whose only possible non-zero coordinate is $(m\rho)_j = \sum \lambda_p \varphi_p(m_i)$. But $\sum \lambda_p \varphi_p(m_i) = 0$ since (M_i, φ_α) is bound by I.

The definition of G on morphisms is as follows: given $f = (f_i) : (M_i, \varphi_\alpha) \to (N_i \psi_\alpha)$, we have $G(f) : M \to N$ defined by $G(f)(m) := (f_i(m_i))_{i \in Q_0}$.

Clearly $G(f)$ is linear as each f_i is linear. In order to show that $G(f)$ is a module homomorphism, it is enough to check $G(f)(ma) = G(f)(m)a$ for all $m = m_i \in M_i$ and $a = p + I \in A$, where p is a path from i to j.

On the one hand, we have $(ma)_k = 0$ for $k \neq j$ and $(ma)_j = \varphi_p(m_i)$, and so

$$(G(f)(ma))_k = \begin{cases} 0 & \text{if } k \neq j \\ f_j \varphi_p(m_i) = \psi_p f_i(m_i) & \text{if } k = j. \end{cases}$$

On the other hand, $(G(f)(m))_k = 0$ for $k \neq i$, and $(G(f)(m))_i = f_i(m_i)$, and so according to the definition of A-action,

$$(G(f)(m)a)_k = \begin{cases} 0 & \text{if } k \neq j \\ \psi_p(f_i(m_i)) & \text{if } k = j. \end{cases}$$

It is easy to check that G is indeed a functor and that $FG \simeq 1_{\mathsf{rep}(Q,I)}$ and $GF \simeq 1_{\mathsf{mod}(A)}$, thus giving the required equivalence of categories. $\square$

1.3 Representation Finite Hereditary Algebras

The Krull–Schmidt theorem states that every module over an algebra can be written as a direct sum of indecomposable modules in a unique way (up to isomorphism and changing the order). Therefore, in order to classify all the modules over an algebra, it is sufficient to classify the indecomposable ones.

In this section we discuss representation types of algebras, and discuss the simplest case one can hope for, which is when there are finitely many indecomposable modules.

Definition 1.14

1 Given two representations $M = (M_i, \varphi_\alpha)$, $N = (N_i, \psi_\alpha)$ of Q, we can construct a new representation

$$M \oplus N := \left(M_i \oplus N_i, \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \psi_\alpha \end{bmatrix} \right),$$

called the *direct sum of M and N*.

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2 A representation M is *indecomposable* if $M \neq 0$ and it cannot be written as a direct sum of two non-zero representations.

Example 1.15

1 Let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3$. The representation

$$M = \mathbf{k} \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbf{k}^2 \xrightarrow{\begin{bmatrix}0 & 1\end{bmatrix}} \mathbf{k}$$

is not indecomposable since $M \cong (\, \mathbf{k} \xrightarrow{1} \mathbf{k} \longrightarrow 0 \,)$ $\oplus (\, 0 \longrightarrow \mathbf{k} \xrightarrow{1} \mathbf{k} \,)$.

2 Let Q be the quiver $1 \rightrightarrows 2$. We have

$$\mathbf{k} \underset{\begin{bmatrix}2\\4\end{bmatrix}}{\overset{\begin{bmatrix}1\\2\end{bmatrix}}{\rightrightarrows}} \mathbf{k}^2 \;\cong (\, \mathbf{k} \underset{2}{\overset{1}{\rightrightarrows}} \mathbf{k} \,) \oplus (\, 0 \rightrightarrows \mathbf{k} \,).$$

Definition 1.16 A connected algebra A is:

1 of *finite representation type* if, up to isomorphism, there are only finitely many indecomposable objects in $\mathsf{mod}(A)$.

2 *hereditary* if $A \cong \mathbf{k}Q$ for some finite, connected and acyclic quiver Q.

Representation finite hereditary algebras have been classified by Gabriel.

Theorem 1.17 (Gabriel's theorem) *An hereditary algebra* $\mathbf{k}Q$ *is of finite representation type if and only if Q is an orientation of an ADE diagram, i.e. the underlying graph of Q is of one of the following forms:*

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$$A_n : \quad \begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & n-1 \arrow[r, dash] & n \end{tikzcd} ,n \geqslant 1$$

$$D_n : \quad \begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & n-2 \arrow[r, dash] \arrow[d, dash] & n-1 \\ & & & & n & \end{tikzcd} ,n \geqslant 4$$

$$E_6 : \quad \begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 4 \arrow[r, dash] & 5 \\ & & 6 & & \end{tikzcd}$$

$$E_7 : \quad \begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \\ & & 7 & & & \end{tikzcd}$$

$$E_8 : \quad \begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] & 7 \\ & & 8 & & & & \end{tikzcd}$$

There are two different proofs of this theorem in [6, 38] worth studying. The proof in [6] uses *reflection functors*, which are at the origin of *tilting theory*, where one studies an algebra by comparing its representation theory with that of a simpler algebra. The proof in [38] uses algebraic geometry, namely by studying the space of representations of a quiver with a given dimension vector, which is an algebraic variety.

There are two subtypes of infinite-representation algebras (Drozd's tame-wild dichotomy, 1977):

- **tame type**: infinitely many indecomposable finite-dimensional representations (up to isomorphism), but which are *possible to parametrise*.
- **wild type**: infinitely many indecomposable finite-dimensional representations (up to isomorphism) which *cannot be parametrised*.

Precise definitions of tame and wild algebras can be found for example in [40, Definition XIX.1.3, Definition XIX.3.3]

Hereditary algebras of tame type correspond to acyclic orientations of the Euclidean quivers:

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$\widetilde{A}_n$:
$$\begin{tikzcd}
 & & & n+1 \arrow[dlll, dash] \arrow[drr, dash] & & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & n-1 \arrow[r, dash] & n, n \geqslant 1
\end{tikzcd}$$

$\widetilde{D}_n$:
$$\begin{tikzcd}
1 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 4 \arrow[r, dash] & \cdots \arrow[r, dash] & n-1 \arrow[r, dash] \arrow[d, dash] & n, n \geqslant 4 \\
 & 2 & & & n+1 &
\end{tikzcd}$$

$\widetilde{E}_6$:
$$\begin{tikzcd}
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 6 \arrow[r, dash] & 7 \\
 & & 5 \arrow[d, dash] & & \\
 & & 4 & &
\end{tikzcd}$$

$\widetilde{E}_7$:
$$\begin{tikzcd}
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] \arrow[d, dash] & 6 \arrow[r, dash] & 7 \arrow[r, dash] & 8 \\
 & & & 5 & & &
\end{tikzcd}$$

$\widetilde{E}_8$:
$$\begin{tikzcd}
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] & 7 \arrow[r, dash] & 8 \arrow[r, dash] & 9 \\
 & & 4 & & & & &
\end{tikzcd}$$

Example 1.18 Let Q be the quiver $1 \rightrightarrows 2$. The indecomposable representations over $\mathbf{k}Q$ are of the following form:

$$\mathbf{k}^n \underset{J_{n,\lambda}}{\overset{1}{\rightrightarrows}} \mathbf{k}^n \ , \ \mathbf{k}^n \underset{1}{\overset{J_{n,0}}{\rightrightarrows}} \mathbf{k}^n \ , \ \mathbf{k}^{n+1} \underset{[0 \quad 1]}{\overset{[1 \quad 0]}{\rightrightarrows}} \mathbf{k}^n \ , \ \mathbf{k}^n \underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightrightarrows}} \mathbf{k}^{n+1} \ ,$$

where $n > 0$, and $J_{n,\lambda}$ denotes the nilpotent $n \times n$ Jordan block corresponding to the eigenvalue $\lambda \in \mathbf{k}$.

Example 1.19 The path algebra $\mathbf{k}Q$ associated to the quiver:

- $1 \longrightarrow 2$ is of finite type.
- $1 \rightrightarrows 2$ is of tame type.
- $1 \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} 2$ is of wild type.

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1.4 Auslander–Reiten Theory

In this section we give a brief overview of Auslander–Reiten (AR) theory, giving the basic concepts and main results in order to define the AR-quiver and describe the knitting algorithm, which provides a method to construct the AR-quiver of the representation-finite hereditary algebras.

1.4.1 (Short) Exact Sequences and Extensions

Definition 1.20 Let A be a finite-dimensional algebra.
A sequence of objects and morphisms in $\mathsf{mod}(A)$ of the form

$$\cdots \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \cdots$$

is *exact* if $\mathrm{im} f_i = \mathsf{ker} f_{i+1}$ for all i.
A *short exact sequence* (s.e.s. for short) is an exact sequence of the form

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 .$$

In other words, f is injective, g is surjective and $\mathrm{im} f = \mathsf{ker}\, g$. This is also called an *extension of N by L.*

Note that, in an exact sequence, we have $f_{i+1} f_i = 0$ for all i.

Example 1.21 1 Given a morphism $f : M \to N$ of A-modules, the sequence

$$0 \longrightarrow \mathsf{ker} f \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} \mathsf{coker} f \longrightarrow 0 ,$$

where i is the inclusion and p is the projection, is exact, and

$$0 \longrightarrow \mathsf{ker} f \xrightarrow{i} M \xrightarrow{q} M/\mathsf{ker} f \longrightarrow 0$$

is short exact.

2 Let Q be the quiver $1 \longrightarrow 2$, and consider the representations $S(2) := 0 \longrightarrow \mathbf{k}$, $M := \mathbf{k} \xrightarrow{1} \mathbf{k}$ and $S(1) := \mathbf{k} \longrightarrow 0$. Then

$$0 \longrightarrow S(2) \xrightarrow{(0,1)} M \xrightarrow{(1,0)} S(1) \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow S(2) \xrightarrow{(0,1)} S(1) \oplus S(2) \xrightarrow{(1,0)} S(1) \longrightarrow 0$$

are short exact sequences, where each component of the pairs $(0,1)$ and $(1,0)$ denotes a linear map between the vector spaces at the corresponding vertex of Q.

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The following lemma, known as the splitting lemma, holds for any abelian category (see [6, Definition A.1.5] for the definition of abelian category).

Lemma 1.22 *Given a s.e.s.* $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *in* $\mathsf{mod}(A)$, *the following statements are equivalent:*

1 *f is a split monomorphism (also called a section), i.e. there exists $h : M \to L$ such that $hf = 1_L$.*
2 *g is a split epimorphism (also called a retraction), i.e. there exists $h' : N \to M$ such that $gh' = 1_N$.*
3 *The sequence is equivalent to the s.e.s.* $0 \longrightarrow L \xrightarrow{i} L \oplus N \xrightarrow{p} N \longrightarrow 0$ *, i.e. there is a commutative diagram:*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "g"] \arrow[d, "\cong"] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "i"'] & L \oplus N \arrow[r, "p"'] & N \arrow[r] & 0. \end{tikzcd}$$

In this case, the s.e.s. is said to split.

The set of equivalence classes $\mathrm{Ext}^1(N,L)$ of extensions of N by L, with the equivalence relation defined in Lemma 1.22 (3), is an abelian group, whose zero element is the class of the split extension. For more details, see for instance [37, Section 7.2].

Example 1.23 Let Q be the quiver $1 \rightrightarrows 2$, which is known as the Kronecker quiver. The sequences

$$0 \longrightarrow S(2) \longrightarrow E \longrightarrow S(1) \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow S(2) \longrightarrow E' \longrightarrow S(1) \longrightarrow 0 \ ,$$

where $S(1) = \mathbf{k} \rightrightarrows 0$, $S(2) = 0 \rightrightarrows \mathbf{k}$, $E = \mathbf{k} \underset{0}{\overset{1}{\rightrightarrows}} \mathbf{k}$ and $E' = \mathbf{k} \underset{1}{\overset{0}{\rightrightarrows}} \mathbf{k}$ are non-equivalent short exact sequences.

1.4.2 Simple, Projective and Injective Representations

Let $A = \mathbf{k}Q/I$, with Q a finite, connected quiver, and I an admissible ideal.

A *simple A-module* is a non-zero module that has no proper submodules. A *simple representation of* (Q,I) is a representation that corresponds to a simple A-module under the equivalence in Theorem 1.13.

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Proposition 1.24 *The simple representations of (Q,I) are, up to isomorphism, of the form $S(i) = (S(i)_j, \varphi_\alpha)$ for each $i \in Q_0$, where $\varphi_\alpha = 0$ for all $\alpha \in Q_1$ and*

$$S(i)_j = \begin{cases} \boldsymbol{k} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

An A-module P is *projective* if any s.e.s. ending at P splits, i.e. $\mathrm{Ext}^1(P,-) = 0$. An A-module I is *injective* if any s.e.s. starting at I splits, i.e. $\mathrm{Ext}^1(-,I) = 0$. The reader can find the definition and basic results on Hom and Ext functors in both [6] and [38].

Remark

1 P is projective if and only if for every epimorphism $f : M \to N$ and every morphism $g : P \to N$, there is $g' : P \to M$ such that $g = fg'$. In other words, $\mathrm{Hom}(P,-)$ maps surjective morphisms to surjective morphisms.
2 I is injective if and only if for every monomorphism $u : L \to M$ and every morphism $g : L \to I$, there is $g' : M \to I$ such that $g = g'u$. In other words, $\mathrm{Hom}(-,I)$ maps injective morphisms to surjective morphisms.

Proposition 1.25 *The projective representations of (Q,I) are, up to isomorphism, of the form $P(i) = (P(i)_j, \varphi_\alpha)$ for each $i \in Q_0$, where:*

- *$P(i)_j$ is the vector space generated by $\{p + I \mid p$ path from i to $j\}$.*
- *Given an arrow $\alpha : j \to \ell$, $\varphi_\alpha : P(i)_j \to P(i)_\ell$ is the linear map defined on the basis by composing the paths from i to j with the arrow α.*

Similarly, the injective representations of (Q,I) are, up to isomorphism, of the form $I(i) = (I(i)_j, \varphi_\alpha)$ for each $i \in Q_0$, where:

- *$I(i)_j$ is the vector space generated by $\{p + I \mid p$ path from j to $i\}$.*
- *Given an arrow $\alpha : j \to \ell$, $\varphi_\alpha : P(i)_j \to P(i)_\ell$ is the linear map defined on the basis by deleting the arrow α from the paths from j to i that start with α and sending to zero the remaining paths.*

Example 1.26 Consider the algebra given by the quiver

$$\begin{tikzcd}
 & 2 \arrow[dl, "a"'] \arrow[dr, "d"] & \\
1 \arrow[dr, "b"'] & & 3 \\
 & 4 \arrow[uu, "c"'] &
\end{tikzcd}$$

subject to the relations $ca = 0 = ab$.

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The projective and injective representations are as follows:

$$P(1) = I(3) = \begin{tikzcd} & \mathbf{k} \arrow[dl, "0"'] \arrow[dr, "1"] & \\ \mathbf{k} \arrow[dr, "1"'] & & \mathbf{k} \\ & \mathbf{k} \arrow[uu, "1"'] & \end{tikzcd} \qquad P(2) = \begin{tikzcd} & \mathbf{k} \arrow[dl, "1"'] \arrow[dr, "1"] & \\ \mathbf{k} \arrow[dr, "0"'] & & \mathbf{k} \\ & 0 \arrow[uu, "0"'] & \end{tikzcd}$$

$$P(3) = S(3) = \begin{tikzcd} & 0 \arrow[dl, "0"'] \arrow[dr, "0"] & \\ 0 \arrow[dr, "0"'] & & \mathbf{k} \\ & 0 \arrow[uu, "0"'] & \end{tikzcd} \qquad P(4) = \begin{tikzcd} & \mathbf{k} \arrow[dl, "0"'] \arrow[dr, "1"] & \\ 0 \arrow[dr, "0"'] & & \mathbf{k} \\ & \mathbf{k} \arrow[uu, "1"'] & \end{tikzcd}$$

$$I(1) = \begin{tikzcd} & \mathbf{k} \arrow[dl, "1"'] \arrow[dr, "0"] & \\ \mathbf{k} \arrow[dr, "0"'] & & 0 \\ & 0 \arrow[uu, "0"'] & \end{tikzcd} \qquad I(2) = \begin{tikzcd} & \mathbf{k} \arrow[dl, "0"'] \arrow[dr, "0"] & \\ \mathbf{k} \arrow[dr, "1"'] & & 0 \\ & \mathbf{k} \arrow[uu, "1"'] & \end{tikzcd}$$

$$I(4) = \begin{tikzcd} & 0 \arrow[dl, "0"'] \arrow[dr, "0"] & \\ \mathbf{k} \arrow[dr, "1"'] & & 0 \\ & \mathbf{k} \arrow[uu, "0"'] & \end{tikzcd}$$

Notation: We will simplify the notation of an indecomposable representation, by encoding their composition series whenever possible. For instance, the projective module $P(1)$ in the example above can be denoted by

$$P(1) = {\scriptsize\begin{matrix}1\\4\\2\\3\end{matrix}},$$

meaning $P(1)_i = \mathbf{k}$ for all $i \in Q_0$, and there is an identity map from top to bottom, i.e.

$$(P(1))_1 \xrightarrow{1} (P(1))_4 \xrightarrow{1} (P(1))_2 \xrightarrow{1} (P(1))_3 \,.$$

This module has a unique composition series given by:

$$\begin{tikzcd}
0 \arrow[r, hook] & 3 \arrow[r, hook] \arrow[d, two heads] & {\begin{smallmatrix}2\\3\end{smallmatrix}} \arrow[r, hook] \arrow[d, two heads] & {\begin{smallmatrix}4\\2\\3\end{smallmatrix}} \arrow[r, hook] \arrow[d, two heads] & {\begin{smallmatrix}1\\4\\2\\3\end{smallmatrix}} \arrow[d, two heads] \\
& 3 & 2 & 4 & 1
\end{tikzcd}$$

Theorem 1.27 *Given a representation $M = (M_i, \varphi_\alpha)$ in $\mathsf{rep}(Q,I)$, we have for all $i \in Q_0$,*

$$\mathrm{Hom}(P(i),M) \simeq M_i \simeq \mathrm{Hom}(M,I(i)).$$

For a proof, see for instance [6, Lemma III.2.11]. Recall the definition of dimension vector in Definition 1.9.

Corollary 1.28 *If* $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *is a s.e.s. in* $\mathsf{rep}(Q,I)$*, then*

$$\underline{\dim} M = \underline{\dim} L + \underline{\dim} N.$$

1.4.3 Irreducible Morphisms and AR-sequences

We now introduce the definition of irreducible morphisms, which are in some sense the building blocks for the morphisms between modules, and we define an important class of short exact sequences, called AR-sequences, which can be defined in terms of irreducible morphisms and indecomposable modules.

Definition 1.29 A morphism $f : M \to N$ is *irreducible* if:

- f is not a split monomorphism,
- f is not a split epimorphism and
- if $f = gh$, then h is a split monomorphism or g is a split epimorphism.

We note that an irreducible morphism is either injective or surjective, but not both. Moreover, the third condition says that an irreducible morphism admits no nontrivial factorisation.

Example 1.30 Let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3$. The map $S(3) \xrightarrow{(0,0,1)} P(2)$ is irreducible. But the map $S(3) \xrightarrow{(0,0,1)} P(1)$ is not irreducible as it factors nontrivially through $P(2)$.

Given two indecomposable A-modules M and N, the set $\mathsf{Irr}(M,N)$ of irreducible morphisms from M to N is a vector space. In fact, $\mathsf{Irr}(M,N)$ is given by the quotient $\mathsf{rad}_A(M,N)/\mathsf{rad}_A^2(M,N)$. For a definition of the (mth power of the)

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radical of $\mathsf{mod}(A)$, and a proof of this fact, we refer the reader to [6, Section IV.1, Appendix A.3].

Definition 1.31 An s.e.s. $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is an *AR-sequence* if the following conditions hold:

1 L, N are indecomposable;
2 f, g are irreducible morphisms.

Remark An AR-sequence is also known as an *almost-split sequence*, in the sense that any map $u : L \to U$ that is not a split monomorphism (resp. any map $v : V \to N$ that is not split epimorphism) factors through f (resp. g).

Remark

1 An AR-sequence never splits. Therefore, no AR-sequence starts with an injective module or ends with a projective module.
2 An AR-sequence is uniquely determined, up to isomorphism, by each of its end terms.

Theorem 1.32 (Auslander–Reiten theorem) *Let M be an indecomposable A-module.*

1 If M is non-projective, there is an AR-sequence
$0 \longrightarrow \tau M \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$ *ending at M.*
2 If M is non-injective, there is an AR-sequence
$0 \longrightarrow M \xrightarrow{f} E' \xrightarrow{g} \tau^{-1} M \longrightarrow 0$ *starting at M.*

The module τM is called the *AR-translate of M*, and $\tau^{-1}M$ is the *inverse AR-translate of M*. We note that if M is non-projective indecomposable (resp. non-injective indecomposable) then τM (resp. $\tau^{-1}M$) is non-injective indecomposable (resp. non-projective indecomposable).

We recommend [6, Section IV] for a proof of Theorem 1.32. Key tools in this proof are the *AR-formulas*, which describe the relationship between morphisms and extensions. Namely, for any pair of modules $M, N \in \mathsf{mod}(A)$, we have:

$$\mathrm{Ext}^1(M,N) \cong D\underline{\mathrm{Hom}}(\tau^{-1}N, M) \cong D\overline{\mathrm{Hom}}(N, \tau M).$$

Here, D is the standard $\mathbf{k}$-duality $\mathrm{Hom}_{\mathbf{k}}(-,\mathbf{k})$, $\tau^{-1}I = 0$ for all injective module I, $\tau P = 0$ for all projective module P, and the underlining (resp. overlining) means we are considering morphisms that do not factor through projective (resp. injective) modules.

When A is an hereditary algebra, the AR-formulas can be simplified to

$$\mathrm{Ext}^1(M,N) \cong D\mathrm{Hom}(\tau^{-1}N,M) \cong D\mathrm{Hom}(N,\tau M).$$

1.4.4 The AR-quiver and the Knitting Algorithm

Given a finite-dimensional algebra A, we can record the information about $\mathsf{mod}(A)$ in a quiver, called the AR-quiver. In the case when A is of finite representation type, this quiver gives a complete picture of the representation theory of A.

Definition 1.33 The *AR-quiver* $\Gamma(\mathsf{mod}(A))$ of $\mathsf{mod}(A)$ is defined by:

- the vertices of $\Gamma(\mathsf{mod}(A))$ are the isomorphism classes of indecomposable A-modules,
- the arrows correspond to basis elements of the vector space of irreducible morphisms between indecomposable modules.

Note that there are no loops in $\Gamma(\mathsf{mod}(A))$. This follows from the fact that we are dealing with finite-dimensional modules and that any irreducible morphism is either a monomorphism or an epimorphism, but not both. Moreover, in the case when A is representation-finite, the AR-quiver has no multiple arrows, i.e. all the vector spaces of irreducible morphisms between two indecomposable modules have dimension $\leqslant 1$ (cf. [6, Proposition IV.4.9]).

Each AR-sequence $0 \longrightarrow \tau M \longrightarrow L_1 \oplus \cdots \oplus L_r \longrightarrow M \longrightarrow 0$ is represented in the AR-quiver by a *mesh*:

$$\begin{tikzcd}
 & L_1 \arrow[ddr] & \\
 & L_2 \arrow[dr] \arrow[dd, dotted, no head] & \\
\tau M \arrow[uur] \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & M \\
 & L_r \arrow[ur] &
\end{tikzcd}$$

The AR-quiver is a *translation quiver*, i.e. for each arrow $M \to L$, for which $\tau^{-1}M \neq 0$ (resp. $\tau L \neq 0$), there is an arrow $L \to \tau^{-1}M$ (resp. $\tau L \to M$).

Theorem 1.34 (Auslander's Theorem) *[8] If the AR-quiver Γ of a connected finite-dimensional algebra A has a connected component $\mathcal{C}$ such that the lengths*

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of the modules in $\mathcal{C}$ are bounded, then A is of finite representation type, and $\Gamma = \mathcal{C}$.

In particular, the AR-quiver of a representation-finite algebra consists of one finite component. Moreover, in this case the AR-quiver completely describes $\mathsf{mod}(A)$, in the sense that every module is a direct sum of finite-dimensional indecomposable modules and every nonzero non-isomorphism between indecomposable modules is a sum of compositions of irreducible morphisms.

The *knitting algorithm* is an algorithm that allows us to construct, in some special cases, the AR-quiver (or part thereof). One of these special cases is when $A = \mathbf{k}Q$, where the underlying graph of Q is ADE. It owes its name to the fact that it recursively constructs one mesh after the other, from left to right.

What follows is a description of this algorithm. We start by computing all the projective modules and their radicals.

The radical $\mathsf{rad}(M)$ of a module M is the intersection of all maximal submodules of M. The representation $(P(i)'_j, \varphi'_\alpha)$ corresponding to the radical $\mathsf{rad}(P(i))$ of the projective $P(i) = (P(i)_j, \varphi_\alpha)$ at i is such that $P(i)'_j = P(i)_j$ if $i \neq j, P(i)_i$ is the vector space spanned by all nonconstant paths from i to i, and φ'_α is the restriction of φ_α to $P(i)_{s(\alpha)}$.

Proposition 1.35 *Every direct predecessor of $P(i)$ in $\Gamma(\mathsf{mod}(A))$, i.e. every indecomposable module X for which there is an irreducible morphism $X \to P(i)$ is a direct summand of $\mathsf{rad}(P(i))$. In the case when A is hereditary, all predecessors of projective modules are projective modules.*

Base step:

1 Draw a vertex for each simple projective $P(i)$.
2 If $P(i)$ is a summand of $\mathsf{rad}(P)$ for some projective P, then add a vertex corresponding to P and arrows from $P(i)$ to P (the number of arrows equals the multiplicity of $P(i)$ in $\mathsf{rad}(P)$).
3 Add vertices associated to remaining summands R of $\mathsf{rad}(P)$ and arrows $R \to P$ (the number of arrows equals the multiplicity of R in $\mathsf{rad}(P)$).
4 Repeat previous steps for each simple projective.

At this point we get a quiver Δ_0.
Induction Δ_n from Δ_{n-1}:
If $X \in \Delta_{n-1}$ and all its direct predecessors are in Δ_{n-1}, then:

1 if X is a direct summand of $\mathsf{rad}(Q)$ for some projective Q not in Δ_{n-1}, add a vertex associated to Q and arrows $X \to Q$ (the number of arrows equals the multiplicity of X in $\mathsf{rad}(Q)$).

2 if X is not injective, add a vertex corresponding to $\tau^{-1}X$ and for each arrow $X \to Y$, add $Y \to \tau^{-1}X$.

If A is hereditary of finite representation type, it is known that each indecomposable A-module is uniquely determined by its dimension vector. Therefore, in order to calculate $\tau^{-1}X$ in the knitting algorithm, one can simply use the formula $\underline{\dim}\tau^{-1}X = \sum_{X\to Y}\underline{\dim}Y - \underline{\dim}X$, by Corollary 1.28.

Note that there is a dual version of the knitting algorithm where one starts by computing injective modules, and considering the dual of Proposition 1.35 which states that every direct successor of $I(i)$ in $\Gamma(\mathsf{mod}(A))$ is a direct summand of $I(i)/S(i)$, and if A is hereditary then all successors of injective modules are injective modules.

Let A be a hereditary algebra. If A is of finite representation type, then the algorithm terminates when we have reached all the injective modules. If A is not of finite type, then the algorithm does not terminate, and what the algorithm produces is the postprojective component of the AR-quiver (cf. [6, Corollary VIII.2.3]). For the definition of postprojective component, see e.g. [6, Definition VIII.2.2]. Note that some authors refer to postprojective components as preprojective components.

Example 1.36 Let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5$ of type A_5. The AR-quiver of $\mathbf{k}Q$ is given by:

$$\begin{tikzcd}
& & \substack{1\\2\\3} \arrow[dr] \arrow[rr, dash, dashed] & & \substack{4\\5} \arrow[dr] & & \\
& \substack{2\\3} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \substack{1\\2\,4\\3\,5} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & 4 & \\
3 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \substack{2\,4\\3\,5} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \substack{1\\2\,4\\3} \arrow[ur] \arrow[dr] & & \\
& \substack{4\\3\,5} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \substack{2\,4\\3} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \substack{1\\2} \arrow[dr] & \\
5 \arrow[ur] \arrow[rr, dash, dashed] & & \substack{4\\3} \arrow[ur] \arrow[rr, dash, dashed] & & 2 \arrow[ur] \arrow[rr, dash, dashed] & & 1
\end{tikzcd}$$

Example 1.37 Let Q be the following quiver of type D_4:

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] \arrow[d] & 3 \\
& 4 &
\end{tikzcd}$$

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The AR-quiver of $\mathbf{k}Q$ is given by:

$$
\begin{tikzcd}
 & & \begin{smallmatrix}&1&\\&2&\\3&&4\end{smallmatrix} \arrow[dr] \arrow[rr, dashed, no head] & & 2 \arrow[dr] \arrow[rr, dashed, no head] & & 1 \\
 & \begin{smallmatrix}&2&\\3&&4\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[rr, dashed, no head] & & \begin{smallmatrix}&1&\\2&&2\\3&&4\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[rr, dashed, no head] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] & \\
3 \arrow[ur] \arrow[rr, dashed, no head] & & \begin{smallmatrix}2\\4\end{smallmatrix} \arrow[ur] \arrow[rr, dashed, no head] & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[ur] & & \\
4 \arrow[uur] \arrow[rr, dashed, no head] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[uur] \arrow[rr, dashed, no head] & & \begin{smallmatrix}1\\2\\4\end{smallmatrix} \arrow[uur] & &
\end{tikzcd}
$$

Here we have:
$$
\begin{smallmatrix}&1&\\2&&2\\3&&4\end{smallmatrix} \quad = \quad
\begin{tikzcd}
\mathbf{k} \arrow[r, "\begin{bmatrix}1\\1\end{bmatrix}"] & \mathbf{k}^2 \arrow[r, "\begin{bmatrix}1 & 0\end{bmatrix}"] \arrow[d, "\begin{bmatrix}0 & 1\end{bmatrix}"] & \mathbf{k} \\
 & \mathbf{k} &
\end{tikzcd}
$$

1.5 Geometric Models

The knitting algorithm might not work when we start with a non-simple projective module.

For instance, consider the quiver Q

$$
\begin{tikzcd}
1 \arrow[rr, "\alpha_1"] & & 4 \arrow[dl, "\alpha_2"] \\
 & 3 \arrow[ul, "\alpha_3"] \arrow[dl, "\alpha_4"'] & \\
2 \arrow[rr, "\alpha_5"'] & & 5 \arrow[ul, "\alpha_6"']
\end{tikzcd}
$$

together with the admissible ideal $I = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1, \alpha_4\alpha_5, \alpha_5\alpha_6, \alpha_6\alpha_4 \rangle$, and let $\mathsf{A} = \mathbf{k}Q/I$.

Suppose we start the knitting algorithm with $P(5)$, whose radical is $\mathfrak{m}(P(5)) = \begin{smallmatrix}3\\1\end{smallmatrix}$. This module is not the summand of the radical of any other projective module, and so according to the algorithm, we would knit the following mesh:

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$$\begin{tikzcd}
 & \begin{smallmatrix}5\\3\\1\end{smallmatrix} \arrow[dr] & \\
\begin{smallmatrix}3\\1\end{smallmatrix} \arrow[ur] \arrow[rr, dashed, no head] & & 5.
\end{tikzcd}$$

However, this mesh is not correct; the algorithm did not compute the irreducible morphism $\begin{smallmatrix}5\\3\\1\end{smallmatrix} \to 3$.

This section is devoted to a different way of computing the AR-quiver of certain classes of algebras, using the geometry of Riemann surfaces with boundary.

1.5.1 Geometric Model of Type A_n

We start by illustrating how to construct the AR-quiver of a hereditary algebra of type A_n, with the example

$$Q = \; 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5 \; .$$

Consider a disc with $8(= n+3)$ marked points on its boundary, together with the triangulation T, i.e. maximal set of non-crossing diagonals, given in Figure 1.1.

Before associating an algebra to these data, we need to introduce some terminology and notation, which follows that of [21]. For further study on the background of combinatorial topology of surfaces we refer the reader to [29].

A *boundary segment* in the marked disc S (or any marked surface) is a segment of a boundary component between two marked points. A *curve* is a continuous map $\gamma\colon [0,1] \to \mathsf{S}$. We always consider curves up to homotopy relative to their endpoints. A curve γ is said to be an *arc* if it satisfies the following properties:

- The endpoints of γ are marked points on the boundary.
- γ intersects the boundary of the surface only in its endpoints.

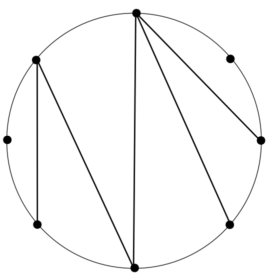

Figure 1.1 A triangulation of an octagon.

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- γ is not homotopic to a point or a boundary segment.

Figure 1.2 illustrates all these concepts.

Given a marked point p, let m', m'' be two points in the same boundary component of p such that m', m'' are not marked points and p is the only marked point lying in the boundary segment δ between m' and m''. Draw a curve c homotopic to δ but lying in the interior of the disc except for its endpoints m' and m''. The *complete fan at p* is the sequence of diagonals in T that c crosses in the clockwise order.

We can now associate a quiver Q_{T} to this triangulation, in the following way:

- Vertices of Q_{T} are in one-to-one correspondence with diagonals of T. We will use the same notation for both.
- Given two vertices i and j, there is an arrow $i \to j$ if and only if i and j share a marked point p and j is the immediate successor of i in the complete fan at p.

Note that we can associate a marked point to each arrow of Q_{T}. Namely, using the notation above, the marked point associated to the arrow $i \to j$ is p.

The quiver Q_{T} in Figure 1.3 is indeed Q, and in fact one can obtain any orientation of a Dynkin graph of type A_n from a triangulation of a disc with

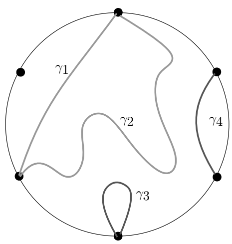

Figure 1.2 The arcs γ_1 and γ_2 are homotopic to each other. The curve γ_3 is homotopic to a point. The curve γ_4 is homotopic to a boundary segment.

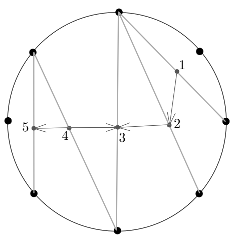

Figure 1.3 The quiver Q_{T} of the triangulation.

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$n+3$ marked points on the boundary whose triangles are *outer-triangles*, i.e. triangles with at least one side on the boundary of the disc.

We will now describe how to obtain the AR-quiver of $\mathbf{k}Q$ from this triangulation.

We will always consider arcs up to homotopy relative to their endpoints. Given an arc γ distinct from any diagonal of T, we define a representation $M_\gamma = (M_i, \varphi_\alpha)$ of $\mathbf{k}Q_\mathsf{T}$, as follows:

$$M_i = \begin{cases} \mathbf{k} & \text{if } \gamma \text{ crosses diagonal } i \\ 0 & \text{otherwise,} \end{cases} \qquad \varphi_\alpha = \begin{cases} 1 & \text{if } M_{s(\alpha)} = M_{t(\alpha)} = \mathbf{k} \\ 0 & \text{otherwise.} \end{cases}$$

Irreducible morphisms correspond to pivoting one of the endpoints of an arc to its counterclockwise neighbour (*pivoting elementary move*). Given an arc γ, we define its translate $\tau(\gamma)$ to be the arc obtained from γ by rotating both endpoints to their counterclockwise neighbour. In particular, $M_\gamma = P(i)$ (resp. $M_\gamma = I(j)$) if and only if $\tau\gamma = i$ (resp. $\tau^{-1} = j$).

A presentation of the AR-quiver of $\mathsf{mod}(\mathbf{k}Q_\mathsf{T})$ in terms of these combinatorics is presented in Figure 1.4.

Extensions have a nice description in terms of arcs. Indeed, there is an extension from N to M if and only if the corresponding arcs γ_N and γ_M cross each other as in Figure 1.5.

The summands of the middle term of the extension correspond to the dashed arcs in Figure 1.5.

1.5.2 Geometric Model for Cluster-tilted Algebras of Type A_n

Cluster-tilted algebras arise in the context of cluster-tilting theory. We refer the reader to [4] for a nice survey on this class of algebras.

Cluster-tilted algebras of type A_n are precisely the algebras associated to an arbitrary triangulation of the $(n+3)$-gon.

An arbitrary triangulation T may include *inner triangles*, i.e. triangles whose three boundaries are all diagonals of T. The quiver Q_T is defined as above, but now we include relations $\alpha\beta$, if $s(\alpha), t(\alpha) = s(\beta), t(\beta)$ are the boundaries of an inner triangle.

The algebra A at the start of this section is a cluster-tilted algebra of type A, which can be obtained from the triangulation in Figure 1.6.

Using the same rule for arcs, pivot elementary moves and translates, we are now able to compute the AR-quiver of $\mathsf{mod}(\mathsf{A})$ in terms of the geometric model (see Figure 1.7).

Note that when we have inner triangles, we can get a new type of crossing, see Figure 1.8.

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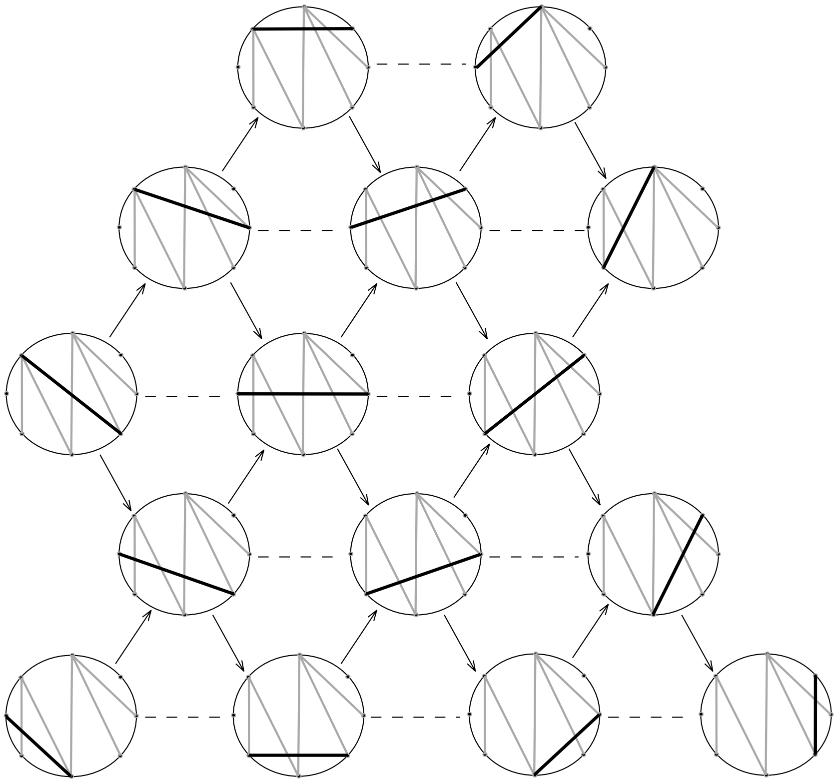

Figure 1.4 The geometric model of the AR-quiver of $\mathsf{mod}(\mathbf{k}Q_{\mathsf{T}})$.

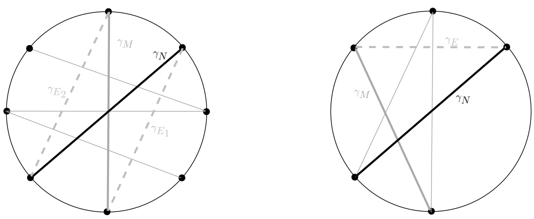

Figure 1.5 Extensions of N by M as crossings of γ_N and γ_M.

However, this type of crossing does not give rise to an extension, and so all extensions are described in the same way as we have seen above. For more details see [16].

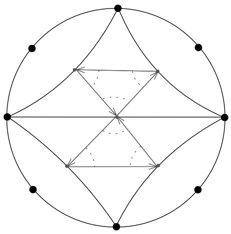

Figure 1.6 The triangulation associated to A.

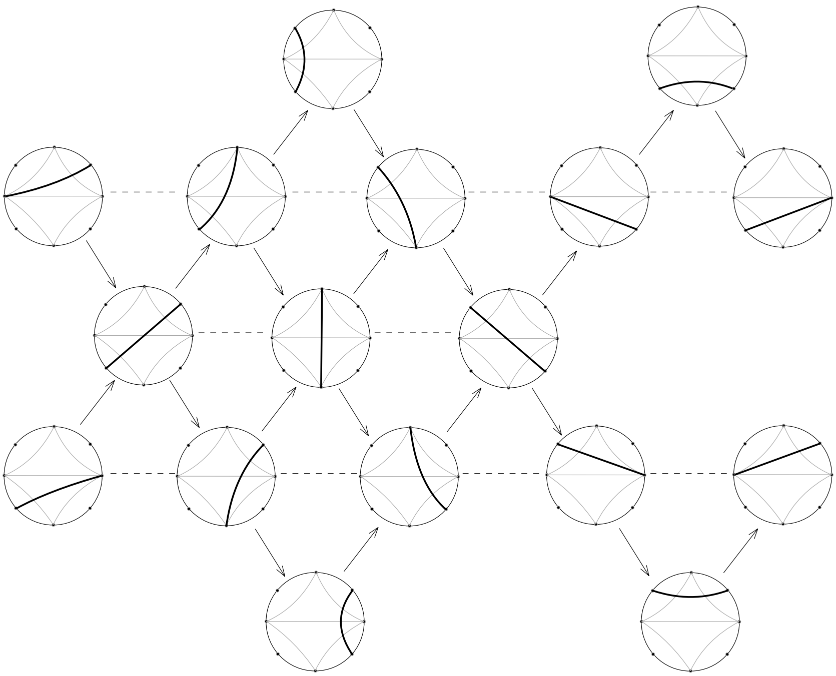

Figure 1.7 The geometric model of the AR-quiver of $\mathsf{mod}(\mathsf{A})$.

1.5.3 Geometric Model for Gentle Algebras

We will now consider two possible generalisations of this combinatorial construction: on the one hand we can consider partial triangulations instead (i.e. any set of non-crossing diagonals), and on the other hand we can consider other surfaces.

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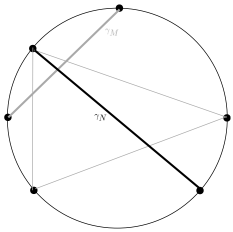

Figure 1.8 Crossing associated to an inner triangle.

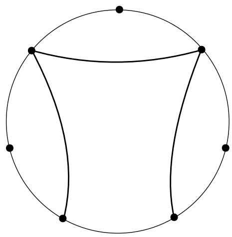

Figure 1.9 The partial triangulation of the algebra C.

Let C be the bound path algebra given by $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ bound by $\alpha\beta$. This algebra can be obtained from the partial triangulation of a disc in Figure 1.9.

The quiver is obtained in the same way as before. The relations are given by composition of two arrows in the same region. Note that this rule applied to an arbitrary triangulation of the disc gives rise to the same rule described in the previous subsection.

For partial triangulations, not every arc gives rise to an indecomposable module and two different arcs may give rise to the same indecomposable module. Therefore, we need to define *permissible arcs* and *equivalence of arcs* (this is not the same as homotopy).

An arc is permissible if each consecutive crossing corresponds to an arrow in the quiver. See Figure 1.10 for a counter-example.

Two arcs are isomorphic if they intersect the same diagonals of the partial triangulation (see Figure 1.11).

Indecomposable modules are therefore in bijection with equivalence classes of permissible arcs.

If we perform a pivot elementary move as described in the previous subsections, we may get an isomorphic arc. Hence, an irreducible morphism corresponds to a sequence of pivot elementary moves until one gets a non-equivalent arc.

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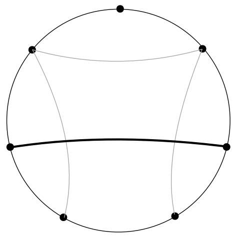

Figure 1.10 An arc that is not permissible.

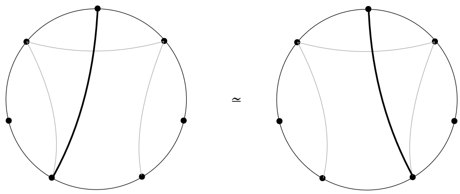

Figure 1.11 Isomorphic permissible arcs.

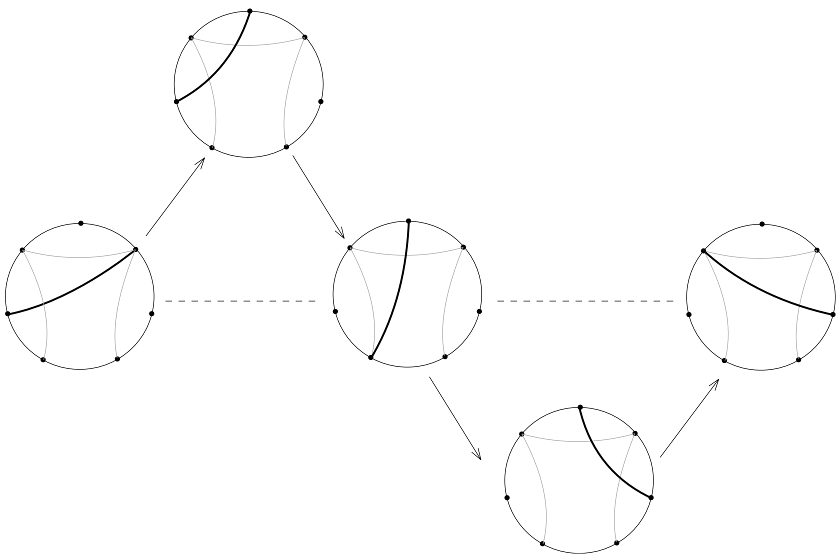

Figure 1.12 The geometric model of the AR-quiver of $\mathsf{mod}(\mathsf{C})$.

The AR-quiver of $\mathsf{mod}(\mathsf{C})$ is given in Figure 1.12.

Now, let us consider an example coming from an annulus (see Figure 1.13). The quiver of the algebra D associated to this partial triangulation is defined

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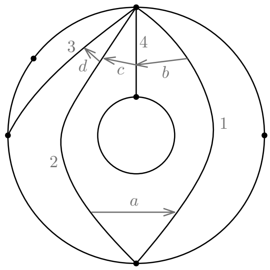

Figure 1.13 A partial triangulation in an annulus and corresponding quiver.

as previously. But we refine the definition of relations as follows: the composition of two arrows with different marked points is zero and if α is a loop, i.e. its start and endpoints correspond to a loop arc of the partial triangulation, then $\alpha^2 = 0$.

The algebra D is then the algebra considered in Example 1.26, given by the quiver

$$\begin{tikzcd}
& 2 \arrow[dl, "a"'] \arrow[dr, "d"] & \\
1 \arrow[dr, "b"'] & & 3 \\
& 4 \arrow[uu, "c"'] &
\end{tikzcd}$$

bound by the relations $ca = 0 = ab$. By refining the notions of permissible arcs, equivalence of arcs and pivot elementary moves, we get the AR-quiver of D as in Figure 1.14.

An algebra associated to an unpunctured surface with a finite set of marked points on the boundary is called a *tilting algebra*. It turns out that these algebras are precisely gentle algebras.

Definition 1.38 A finite-dimensional algebra A is *gentle* if it admits a presentation $A = \mathbf{k}Q/I$ satisfying the following conditions:

1 Each vertex of Q is the source of at most two arrows and the target of at most two arrows.
2 For each arrow α in Q, there is at most one arrow β in Q such that $\alpha\beta \notin I$, and there is at most one arrow γ such that $\gamma\alpha \notin I$.
3 For each arrow α in Q, there is at most one arrow δ in Q such that $\alpha\delta \in I$, and there is at most one arrow μ such that $\mu\alpha \in I$.
4 I is generated by paths of length 2.

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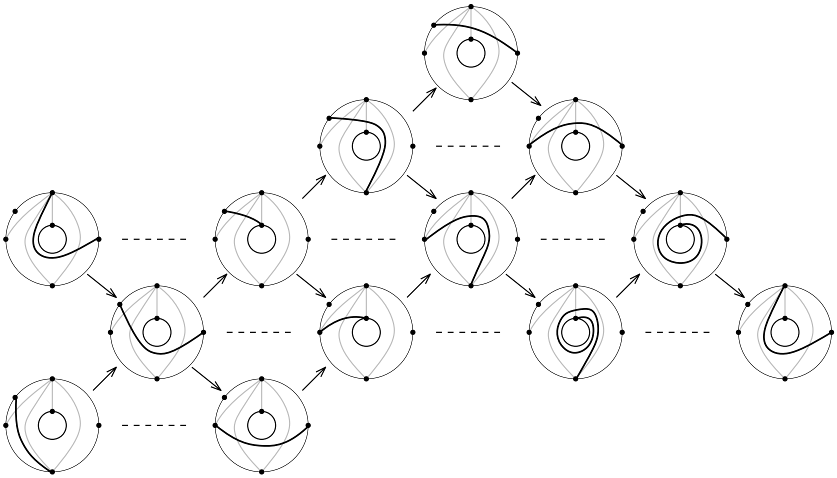

Figure 1.14 The geometric model of the AR-quiver of $\mathsf{mod}(\mathsf{D})$.

Gentle algebras first appeared in the context of tilting theory [5] (see also [6, Section IX]), where iterated tilted algebras of types A and $\widetilde{A}$ were observed to satisfy the properties above. Gentle algebras, which are tame, remain one of the relatively few classes of algebras for which the representation theory is computationally tractable. Partly due to this reason, there has been widespread interest in this class of algebras in many different contexts, such as Fukaya categories [25], dimer models [12], enveloping algebras of Lie algebras [27] and cluster theory [7, 23, 30]. The geometric model for the module category of gentle algebras presented above is given in [10]. Derived categories of gentle algebras have also been described geometrically [33] and an important application is a geometric description of derived equivalences of gentle algebras [3]. We refer the reader to [13, 16, 17, 34] for further examples of recent developments in this area.

Ribbon graphs are the bridge between gentle algebras and unpunctured surfaces (cf. [39]). A *ribbon graph* is an undirected finite graph with a cyclic ordering of the half edges at each vertex. Let $A = \mathbf{k}Q/I$ be a gentle algebra, and $\mathcal{M}$ the set of maximal paths in Q avoiding relations together with the stationary paths e_v, for each vertex v of valency 1 or valency 2 such that v is the middle vertex of path of length 2 not in I. Note that each vertex in Q_0 appears twice in the paths in $\mathcal{M}$.

The vertices of the ribbon graph Γ corresponding to A are in one-to-one correspondence with the elements in $\mathcal{M}$. The edges in Γ are in one-to-one correspondence with the vertices of Q. More precisely, given $v \in Q_0$, the corresponding edge connects the two elements in $\mathcal{M}$ containing v. The cyclic ordering of the half edges at each vertex in Γ is determined by the paths in $\mathcal{M}$.

Example 1.39 Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 2 \arrow[r, "c"] & 3 \arrow[loop right, "d"] \end{tikzcd}$$

Let $I = \langle ac, d^2 \rangle$, and $A = \mathbf{k}Q/I$. Then $\mathcal{M} = \{bcd, a\}$ and the corresponding ribbon graph is given in Figure 1.15. Here, the arrows correspond to the path of the corresponding vertex of the ribbon graph.

By replacing edges with oriented strips, vertices with oriented discs and gluing these according to the orientation along the faces of Γ, we get an oriented surface S in such a way that the faces of Γ correspond to the boundary components of S (cf. [32]). The embedding of Γ in the surface defines the partial triangulation. Note that we may have to add marked points to avoid having arcs homotopic to boundary segments. Figure 1.16 shows the surface associated to the gentle algebra in Example 1.39.

1.5.4 Geometric Model for Skew-gentle Algebras

The geometric model given in Subsection 1.5.3 has been recently extended to a wider class of algebras, called skew-gentle algebras, by considering punctured surfaces (see [26]). A skew-gentle algebra can be obtained from a gentle algebra by replacing some relations of the form ε^2, where ε is a loop, by $\varepsilon^2 - \varepsilon$.

Definition 1.40 Let Q be a quiver, I a set of paths in Q, and Sp a subset of Q_0 such that $\mathbf{k}Q^{sp}/I^{sp}$ is a gentle algebra, where Q^{sp} is obtained from Q by adding a loop ε_i at each vertex i in Sp, and $I^{sp} = I \cup \{\varepsilon_i^2 \mid i \in Sp\}$. The vertices in Sp are called *special vertices* and the ε_i are the *special loops*.

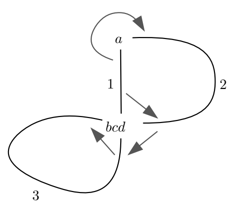

Figure 1.15 The ribbon graph of A.

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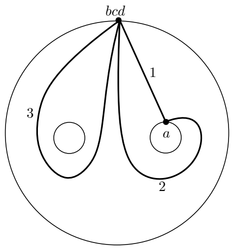

Figure 1.16 The surface and the ribbon graph of A as a partial triangulation.

A finite-dimensional algebra is *skew-gentle* if it admits a presentation of the form $\mathbf{k}Q^{sp}/I^{sg}$, where Q^{sp} comes from a triple (Q,I,Sp) as above and $I^{sg} = I \cup \{\varepsilon_i^2 - \varepsilon \mid i \in Sp\}$.

Note that the presentation I^{sg} is not admissible. However there is an isomorphism $\mathbf{k}Q^{sp}/I^{sg} \simeq \mathbf{k}\hat{Q}/\hat{I}$, with $\hat{I}$ admissible, where Q^{sp} and I^{sg} are defined as follows.

- $\hat{Q}_0$ is obtained from Q_0 by splitting each special vertex i into two vertices i^+ and i^-;
- $\hat{Q}_1$ is obtained from Q_1 by splitting an arrow a for which $s(a) \notin Sp$ and $t(a) \in Sp$ (resp. $s(a) \in Sp$ and $t(a) \notin Sp$) into two arrows a_+ with $t(a_+) = t(a)^+$ and a_- with $t(a_-) = t(a)^-$ (resp. a^+ with $s(a^+) = s(a)^+$ and a^- with $s(a^-) = s(a)^-$);
- given $ab \in I$, if $t(a) \notin Sp$, then all resulting paths in Q_1 of length 2 lie in $\hat{I}$, and if $t(a) \in Sp$, then $a_+b^+ - a_-b^- \in \hat{I}$. All relations in $\hat{I}$ are obtained this way.

Remark 1 Any gentle algebra is skew-gentle; take $Sp = \varnothing$.

2 The linearly oriented quivers of type D and $\widetilde{D}$ are hereditary skew-gentle algebras. Indeed, let $Q = \; 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$ and $I = \varnothing$. If we set $Sp = \{n\}$ (resp. $Sp = \{1,n\}$), then the corresponding skew-gentle algebra is isomorphic to the hereditary algebra of type D_{n+1} (resp. $\widetilde{D}_{n+1}$) with linear orientation.

Consider the following triangulation of a punctured disc:

We can define a quiver Q associated to this triangulation in the same manner as above, giving us

$$Q = \; 1 \longrightarrow 2 \longrightarrow 3 \circlearrowright \varepsilon_3 .$$

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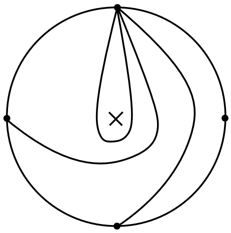

Figure 1.17 A triangulation of a punctured disc.

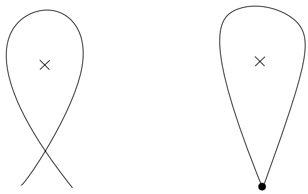

Figure 1.18 Permissible arcs in punctured surfaces do not satisfy these local configurations.

The vertex 3 associated to the loop arc delimiting a monogon with a puncture in its interior is considered to be a special vertex and so the algebra E corresponding to the triangulation in Figure 1.17 is the bound path algebra defined by Q and the relation $\varepsilon_3^2 - \varepsilon_3$. By Remark 1.5.4 (2), this algebra is isomorphic to the hereditary path algebra of the quiver:

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] \arrow[d] & 3 \\ & 4 & \end{tikzcd}$$

We consider *tagged permissible arcs* in the punctured disc, i.e. pairs (γ, σ), where γ is an arc whose endpoints are marked points in the boundary or the puncture, γ is not an arc in the triangulation and it does not cut out a once-punctured monogon by its self-intersection (see Figure 1.18), and

$$\sigma : \{t \mid \gamma(t) \text{ is a puncture}\} \to \{0, 1\}$$

is a map. If $\sigma(t) = 1$, we put a tag on the arc γ near the puncture.

In what follows, we describe E-modules via representations of the quiver of type D_4. There is a one-to-one correspondence between permissible tagged arcs and the indecomposable E-modules, which is again described via crossings. Given a permissible tagged arc (γ, σ), the corresponding indecomposable E-module $M(\gamma, \sigma)$ is uniquely determined by its support. The support at vertex i

is given by the number of times γ crosses the diagonal indexed by i, if i is not a special vertex. If i is a special vertex, and γ crosses i but is not incident with the puncture, then $(M(\gamma,\sigma))_j = \mathbf{k}$, where $j = 3,4$. If i is a special vertex, γ crosses i and it is incident with the puncture, then

$$(M(\gamma,\sigma))_j = \begin{cases} 0 & \text{if } \sigma = 0, j = 4 \text{ or } \sigma = 1, j = 3 \\ \mathbf{k} & \text{if } \sigma = 0, j = 3 \text{ or } \sigma = 1, j = 4. \end{cases}$$

The pivot elementary moves are described in the same manner as in the gentle case, except for the following cases.

Case 1: If (γ,σ) is an arc incident with the puncture, then the pivot elementary move consists of replacing γ by a loop arc around the puncture and ending at the other endpoint of γ and pivoting one step in the anticlockwise direction the endpoint which does not create a self-crossing.

Case 2: Given a permissible arc γ which is not incident with the puncture, if one obtains a loop arc around the puncture after performing a pivot elementary move, then this move corresponds to two irreducible maps, whose targets correspond to the tagged and untagged arcs incident with the puncture and the endpoint of the loop arc.

The AR-translate is described by clockwise rotation of the endpoints which are marked points in the boundary, and by changing the tag at the puncture.

A geometric model of the AR-quiver of $\mathsf{mod}(\mathsf{E})$ is thus described in Figure 1.19.

One can associate an algebra to a partial triangulation of a punctured surface containing a loop arc around each puncture. These algebras are called *skew-tilting algebras*, and they coincide with the skew-gentle algebras. The definition of permissible arcs and equivalence classes of arcs passes across to the punctured case, permissible arcs incident with punctures can be tagged or untagged and we also allow permissible arcs whose both endpoints are punctures; these will correspond to four non-isomorphic indecomposable modules, determined by their tags at each endpoint. The case where there are tagged permissible arcs whose both endpoints are punctures only show up in the case when the algebra is not of finite representation type.

The following gives an example of a skew-gentle algebra of finite-representation type coming from a partial triangulation of a punctured disc. Consider the path algebra F of the quiver

$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[loop above, "\varepsilon_2"] & 3 \end{tikzcd}$$

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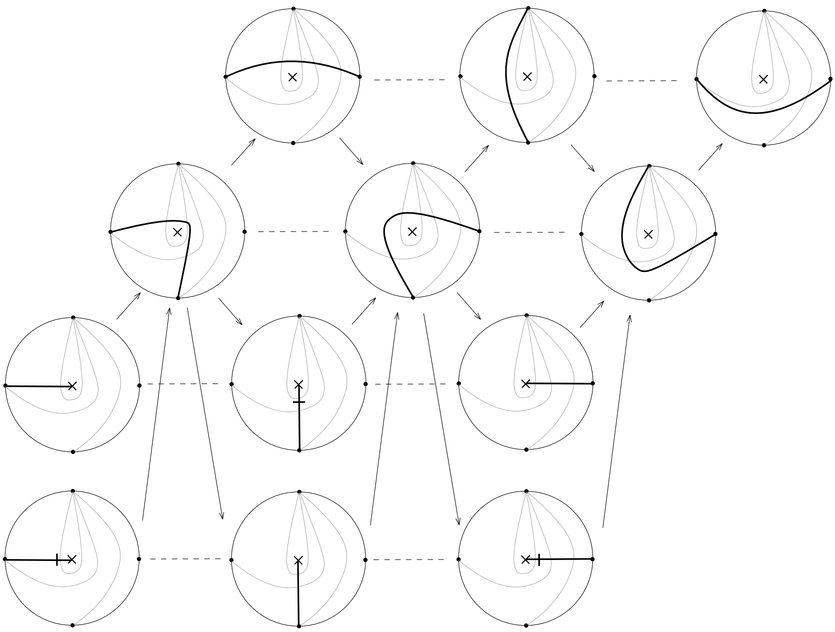

Figure 1.19 The geometric model of the AR quiver of $\mathsf{mod}(\mathsf{E})$.

bound by the relations $\varepsilon_2^2 - \varepsilon_2 = 0$ and $ab = 0$. This algebra, which is isomorphic to the bound path algebra of the quiver

$$\begin{tikzcd}
 & 2 \arrow[dr, "b"] & \\
1 \arrow[ur, "a"] \arrow[dr, "c"'] & & 4 \\
 & 3 \arrow[ur, "d"'] &
\end{tikzcd}$$

bound by the relation $ab = cd$, is associated to the partial triangulation of the punctured disc in Figure 1.20.

The AR-quiver of F is given in Figure 1.21.

For recent developments of the study of derived categories of skew-gentle algebras via geometric models, see [2, 31].

We note that the geometric models of (skew-)gentle algebras described above are based on the description of the AR theory coming from deep results classifying indecomposable representations of classes of algebras of tame represen-

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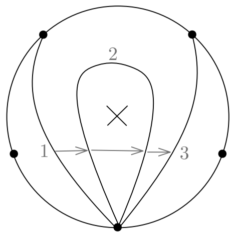

Figure 1.20 The partial triangulation associated to F.

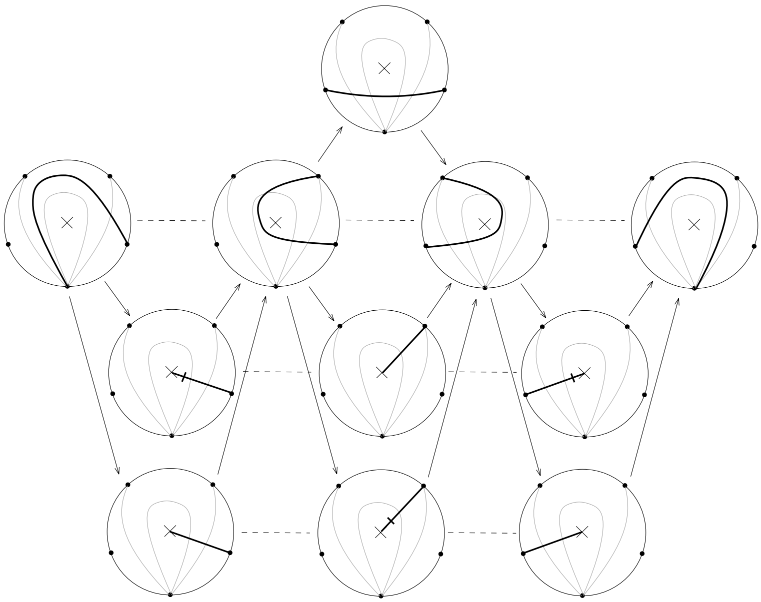

Figure 1.21 The geometric model of the AR-quiver of $\mathsf{mod}(\mathsf{F})$.

tation type (cf. [11, 14, 18, 19, 24]). The indecomposable modules are split into two classes: string modules and band modules. All the examples we considered in Section 1.5 are representation-finite, in which case we only have string modules. These correspond to the (tagged) permissible arcs for which at least one endpoint is a marked point. Band modules lying in homogeneous tubes correspond to certain closed curves in the surface and band modules lying in the bottom of tubes of rank 2 correspond to tagged permissible arcs whose both endpoints are punctures. In any case, the set of (tagged) permissible arcs describe in particular τ-rigid modules, which leads us to the final subsection of this chapter.

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1.5.5 An Application: τ-tilting Theory

Classical tilting theory compares the representation theory of two algebras, one of which is the endomorphism algebra of a tilting module over the other algebra. In 2012, Adachi, Iyama and Reiten introduced τ-tilting theory, which can be seen as a "mutation closure" of tilting theory [1]. For more on τ-tilting theory, see Chapter 2. In this subsection we use the geometric models described above to give a classification of support τ-tilting modules, the main objects of study in τ-tilting theory. This classification was obtained in [26].

Definition 1.41 Let A be an algebra and M an A-module. Denote by $|A|$ the number of simple A-modules, and by $|M|$ the number of indecomposable summands of M.

1 M is *τ-rigid* if $\mathrm{Hom}_A(M, \tau M) = 0$.
2 M is *τ-tilting* if M is τ-rigid and $|M| = |A|$.
3 M is *support τ-tilting* if there is an idempotent $e \in A$ such that M is a τ-tilting $(A/\langle e\rangle)$-module.

Proposition 1.42 *[1, Proposition 2.3] M is support τ-tilting if and only if M is τ-rigid and there is a projective module P such that* $\mathrm{Hom}_A(P,M) = 0$ *and* $|M| + |P| = |A|$.

Let A be a skew-gentle algebra, S the associated punctured surface and P the associated partial triangulation of S. Given a tagged permissible arc γ, we represent by $[\gamma]$ the arc that is equivalent to γ such that the starting/ending segments have the form represented in Figure 1.22.

A *generalised permissible arc* in S is either $[\gamma]$, where γ is a permissible arc, or an arc whose completion is in P, where the *completion* $\tilde{\gamma}$ of an arc γ is described in Figure 1.23.

Besides the natural definition of crossing of arcs in the interior of the surface, we also need to consider crossings at a puncture. Two tagged generalised

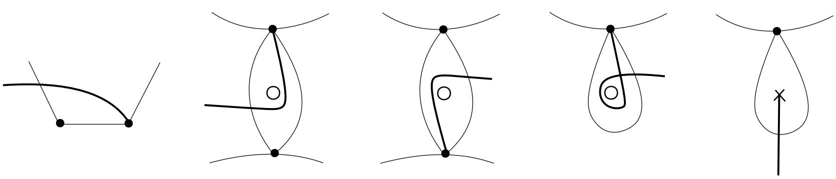

Figure 1.22 Starting/ending segment of $[\gamma]$.

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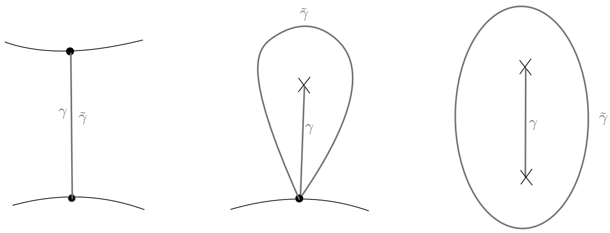

Figure 1.23 Completion of a diagonal in P.

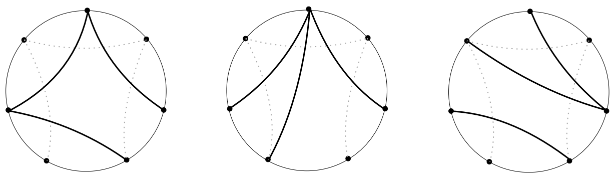

Figure 1.24 τ-tilting C-modules.

permissible arcs cross at a puncture p if p is an endpoint of both arcs, they have different tags at p and if the arcs are homotopic, then the other endpoint of both arcs is also a puncture p' and the tags also differ at p'.

The geometric description of support τ-tilting modules over a skew-gentle algebra is given as follows.

Theorem 1.43 *[26, Corollary 5.9] There is a one-to-one correspondence between the set of maximal collections of noncrossing tagged generalised permissible arcs in S and the set of support τ-tilting A-modules. Moreover, a collection of noncrossing tagged generalised permissible arcs is maximal if and only if its cardinality is $|A|$.*

Example 1.44 Recall the gentle algebra C from the start of Subsection 1.5.3. The τ-tilting modules over C are given by the collection of thick arcs in Figure 1.24. The support τ-tilting C-modules with two summands are given by the collection of thick arcs in Figure 1.25. Here, the thin arc represents the projective module associated to the support τ-tilting C-module. The remaining support τ-tilting C-modules and corresponding projective modules are given by the collection of thick and thin arcs respectively in Figure 1.26.

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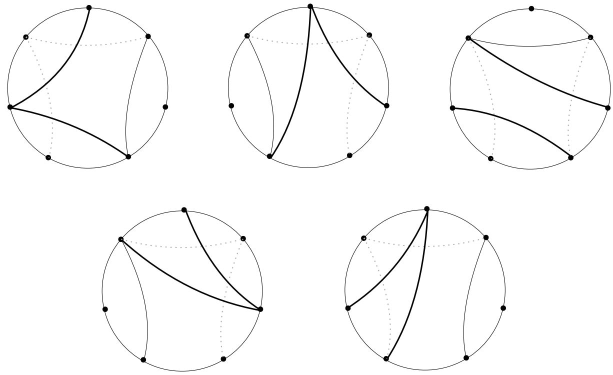

Figure 1.25 Support τ-tilting C-modules with two summands.

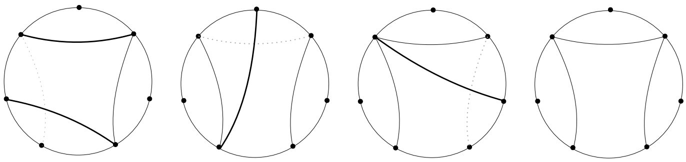

Figure 1.26 The remaining support τ-tilting C-modules.

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