12

Purity

Contents

We study the notion of purity for additive categories that are locally finitely presented. A typical example is the category of modules over a ring. We are

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

mostly interested in pure-injective objects; they enjoy decomposition properties that are analogous to those of injective objects in Grothendieck categories.

A basic idea is to assign to a locally finitely presented category $\mathcal{A}$ an essentially small abelian category $\mathrm{Ab}(\mathcal{A})$ such that the objects in $\mathcal{A}$ identify with exact functors $\mathrm{Ab}(\mathcal{A})^{\mathrm{op}} \to \mathrm{Ab}$. For instance, when $\mathcal{A} = \operatorname{Mod}\Lambda$ is the category of modules over a ring Λ, then $\mathrm{Ab}(\mathcal{A})$ equals the free abelian category $\mathrm{Ab}(\Lambda)$ over Λ.

Viewing objects of $\mathcal{A}$ as exact functors leads naturally to the notion of a definable subcategory of $\mathcal{A}$ if we consider all exact functors which vanish on a specific Serre subcategory of $\mathrm{Ab}(\mathcal{A})$. In particular, we see that any such definable subcategory is determined by its indecomposable pure-injective objects.

12.1 Purity

In this section we introduce for locally finitely presented categories the notion of purity. This is based on the concept of a pure-exact sequence, and there are several ways to define this. For example, a sequence is pure-exact if it is a filtered colimit of split exact sequences. We can embed any locally finitely presented category $\mathcal{A}$ into a Grothendieck category $\mathbf{P}(\mathcal{A})$ such that pure-exactness identifies with the usual notion of exactness in abelian categories. We call this the purity category of $\mathcal{A}$. From this embedding we deduce that every object admits a pure-injective envelope.

From Left Exact to Exact Functors

Let $\mathcal{A}$ be a locally finitely presented category and set $\mathcal{C} = \operatorname{fp}\mathcal{A}$. We introduce the embedding $\mathcal{A} \hookrightarrow \mathbf{P}(\mathcal{A})$ into a Grothendieck category, which is our main tool.

A functor $F\colon \mathcal{C} \to \mathrm{Ab}$ is *finitely presented* if it admits a presentation

$$\operatorname{Hom}_{\mathcal{C}}(D,-) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(C,-) \longrightarrow F \longrightarrow 0. \tag{12.1.1}$$

We denote by $\operatorname{Fp}(\mathcal{C},\mathrm{Ab})$ the category of finitely presented functors and observe that $\operatorname{Fp}(\mathcal{C},\mathrm{Ab})$ is abelian since $\mathcal{C}$ admits cokernels.

A functor F in $\operatorname{Fp}(\mathcal{C},\mathrm{Ab})$ induces the functor

$$\bar{F}\colon \mathcal{A} \longrightarrow \mathrm{Ab}, \quad X \mapsto \operatorname*{colim}_{(C,\phi)\in\mathcal{C}/X} F(C)$$

using the presentation (11.1.17) of X as a filtered colimit of finitely presented objects. A presentation (12.1.1) of F then yields the presentation

$$\operatorname{Hom}_{\mathcal{A}}(D,-) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(C,-) \longrightarrow \bar{F} \longrightarrow 0. \tag{12.1.2}$$

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Remark 12.1.3. A functor of the form $\bar{F}\colon \mathcal{A} \to \mathrm{Ab}$ with $F \in \mathrm{Fp}(\mathcal{C}, \mathrm{Ab})$ preserves filtered colimits and products. This is clear when $\bar{F} = \mathrm{Hom}_{\mathcal{A}}(C, -)$ for some $C \in \mathrm{fp}\,\mathcal{A}$, and the general case follows from the presentation (12.1.2); for a converse see Corollary 12.2.11.

For $X \in \mathcal{A}$ we consider the *evaluation*

$$\bar{X}\colon \mathrm{Fp}(\mathcal{C}, \mathrm{Ab}) \longrightarrow \mathrm{Ab}, \quad F \mapsto \bar{F}(X).$$

Clearly, the functor $\bar{X}$ is exact when $X \in \mathcal{C}$, and $\bar{X} = \operatorname{colim} \bar{X}_i$ is exact when $X = \operatorname{colim} X_i$, since taking filtered colimits is exact. This yields the functor

$$\mathrm{ev}\colon \mathcal{A} \longrightarrow \mathbf{P}(\mathcal{A}) := \mathrm{Lex}(\mathrm{Fp}(\mathcal{C}, \mathrm{Ab}), \mathrm{Ab}), \quad X \mapsto \bar{X}.$$

The category $\mathbf{P}(\mathcal{A})$ is by definition the *purity category* of $\mathcal{A}$. It is a locally finitely presented Grothendieck category and the finitely presented objects form an abelian category that is equivalent to $\mathrm{Fp}(\mathcal{C}, \mathrm{Ab})^{\mathrm{op}}$.

Let us collect some basic properties of this evaluation functor. We write $\mathrm{Ex}(\mathrm{Fp}(\mathcal{C}, \mathrm{Ab}), \mathrm{Ab})$ for the category of exact functors $\mathrm{Fp}(\mathcal{C}, \mathrm{Ab}) \to \mathrm{Ab}$.

Lemma 12.1.4. *The functor* $\mathrm{ev}\colon \mathcal{A} \to \mathbf{P}(\mathcal{A})$ *is fully faithful and induces an equivalence*

$$\mathcal{A} \xrightarrow{\sim} \mathrm{Ex}(\mathrm{Fp}(\mathcal{C}, \mathrm{Ab}), \mathrm{Ab}).$$

Moreover, the functor preserves filtered colimits, products, and cokernels.

Proof First observe that $\bar{X}$ is an exact functor for any $X \in \mathcal{A}$, since evaluation is exact. When we identify $\mathcal{A} = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$, then the quasi-inverse functor

$$\mathrm{Ex}(\mathrm{Fp}(\mathcal{C}, \mathrm{Ab}), \mathrm{Ab}) \longrightarrow \mathcal{A}$$

sends F to $F \circ h$, where

$$h\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Fp}(\mathcal{C}, \mathrm{Ab}), \quad C \mapsto \mathrm{Hom}_{\mathcal{C}}(C, -)$$

denotes the Yoneda functor.

For $F \in \mathrm{Fp}(\mathcal{C}, \mathrm{Ab})$ the corresponding functor $\bar{F}\colon \mathcal{A} \to \mathrm{Ab}$ preserves filtered colimits and products. Thus ev preserves filtered colimits and products, since in $\mathbf{P}(\mathcal{A})$ these are computed pointwise. It remains to consider cokernels. For $C \in \mathcal{C}$ we have $\bar{C} = \mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(C, -), -)$. Thus the restriction $\mathrm{ev}|_{\mathcal{C}}$ preserves cokernels. It follows that ev preserves cokernels, since any cokernel sequence in $\mathcal{A}$ can be written as a filtered colimit of cokernel sequences in $\mathcal{C}$. □

Remark 12.1.5. The category $\mathcal{A}$ viewed as a subcategory of $\mathbf{P}(\mathcal{A})$ is covariantly finite; this follows from Proposition 11.1.27.

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Pure-Exactness and Pure-Injectives

A sequence of morphisms $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ is called *pure-exact* if the induced sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{A}}(C, X) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(C, Y) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(C, Z) \longrightarrow 0$$

of abelian groups is exact for all finitely presented $C \in \mathcal{A}$. In that case the morphism $X \to Y$ is called a *pure monomorphism*. An object $Q \in \mathcal{A}$ is *pure-injective* if every pure monomorphism $X \to Y$ induces a surjective map $\operatorname{Hom}_{\mathcal{A}}(Y, Q) \to \operatorname{Hom}_{\mathcal{A}}(X, Q)$.

Lemma 12.1.6. *For a sequence of morphisms $\eta\colon 0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ the following are equivalent.*

(1) *The sequence η is pure-exact.*
(2) *The sequence η is a filtered colimit of split exact sequences.*
(3) *The sequence $\bar{\eta}\colon 0 \to \bar{X} \to \bar{Y} \to \bar{Z} \to 0$ is exact in $\mathbf{P}(\mathcal{A})$.*

Proof (1) $\Rightarrow$ (2): Write $Z = \operatorname{colim} Z_i$ as a filtered colimit of finitely presented objects. Composing η with $Z_i \to Z$ yields a split exact sequence $\eta_i\colon 0 \to X \to Y_i \to Z_i \to 0$, and $\eta = \operatorname{colim} \eta_i$.

(2) $\Rightarrow$ (3): The assignment $X \mapsto \bar{X}$ preserves filtered colimits, and in $\mathbf{P}(\mathcal{A})$ a filtered colimit of exact sequences is exact.

(3) $\Rightarrow$ (1): For $C \in \operatorname{fp} \mathcal{A}$ the sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(\bar{C}, \bar{X}) \longrightarrow \operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(\bar{C}, \bar{Y}) \longrightarrow \operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(\bar{C}, \bar{Z}) \longrightarrow 0$$

is exact by Lemma 11.1.26. Thus η is pure-exact. □

Lemma 12.1.7. *A morphism $X \to Y$ in $\mathcal{A}$ is a pure monomorphism if and only if $\bar{X} \to \bar{Y}$ is a monomorphism in $\mathbf{P}(\mathcal{A})$.*

Proof Complete the morphism $\alpha\colon X \to Y$ to an exact sequence $X \xrightarrow{\alpha} Y \to Z \to 0$ in $\mathcal{A}$. If α is a pure monomorphism, then $\bar{\alpha}$ is a monomorphism by Lemma 12.1.6. Conversely, if $\bar{\alpha}$ is a monomorphism, then the sequence $0 \to \bar{X} \xrightarrow{\bar{\alpha}} \bar{Y} \to \bar{Z} \to 0$ in $\mathbf{P}(\mathcal{A})$ is exact, since ev is right exact by Lemma 12.1.4. Thus α is a pure monomorphism by Lemma 12.1.6. □

Lemma 12.1.8. *The functor* ev: $\mathcal{A} \to \mathbf{P}(\mathcal{A})$ *identifies the pure-injective objects in $\mathcal{A}$ with the injective objects in $\mathbf{P}(\mathcal{A})$.*

Proof An injective object in $\mathbf{P}(\mathcal{A})$ is of the form $\bar{X}$ for some $X \in \mathcal{A}$ by Lemma 11.1.26 and Lemma 12.1.4. Clearly, X is pure-injective, since ev sends any pure monomorphism in $\mathcal{A}$ to a monomorphism in $\mathbf{P}(\mathcal{A})$ by Lemma 12.1.7.

Now suppose that $X \in \mathcal{A}$ is pure-injective and choose an injective envelope

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

$\bar{\alpha}\colon \bar{X} \to \bar{Y}$ in $\mathbf{P}(\mathcal{A})$. Then α is a pure monomorphism by Lemma 12.1.7, and therefore a split monomorphism. It follows that $\bar{X}$ is an injective object. □

A pure monomorphism $\phi\colon X \to Y$ in $\mathcal{A}$ is called a *pure-injective envelope* of X, if Y is pure-injective and if every endomorphism $\alpha\colon Y \to Y$ satisfying $\phi = \alpha\phi$ is invertible.

Theorem 12.1.9. *Every object $X \in \mathcal{A}$ admits a pure-injective envelope. Moreover, a morphism $X \to Y$ is a pure-injective envelope if and only if the induced morphism $\bar{X} \to \bar{Y}$ is an injective envelope in* $\mathbf{P}(\mathcal{A})$.

Proof Choose a morphism $\phi\colon X \to Y$ such that $\bar{\phi}\colon \bar{X} \to \bar{Y}$ is an injective envelope in $\mathbf{P}(\mathcal{A})$ (Corollary 2.5.4). Then ϕ is a pure monomorphism by Lemma 12.1.7 and Y is pure-injective by Lemma 12.1.8. The additional minimality property for every endomorphism $Y \to Y$ follows from the corresponding characterisation of injective envelopes (Lemma 2.1.19). Clearly, a pure-injective envelope is essentially unique, and this yields the second part of the assertion. □

The pure-exact sequences provide an exact structure on $\mathcal{A}$. We give an application which is a variation of Example 11.1.25.

Example 12.1.10. Let $(\mathcal{T}, \mathcal{F})$ be a split torsion pair for $\operatorname{fp}\mathcal{A}$. Then $(\vec{\mathcal{T}}, \vec{\mathcal{F}})$ is a torsion pair for $\mathcal{A}$ and each object $X \in \mathcal{A}$ fits into a pure-exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \vec{\mathcal{T}}$ and $X'' \in \vec{\mathcal{F}}$.

The Spectrum of Indecomposable Injectives

Let $\mathcal{A}$ be a Grothendieck category. We denote by $\operatorname{Sp}\mathcal{A}$ a representative set of the isomorphism classes of indecomposable injective objects in $\mathcal{A}$ (the *spectrum* of indecomposable injectives). Note that $\operatorname{Sp}\mathcal{A}$ is a set, because $\mathcal{A}$ has a set of generators and each object in $\operatorname{Sp}\mathcal{A}$ is the injective envelope of X/U for some generating object X and some subobject $U \subseteq X$.

Lemma 12.1.11. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the objects in* $\operatorname{Sp}\mathcal{A}$ *form a set of cogenerators for $\mathcal{A}$.*

Proof Let $X \in \mathcal{A}$ be a non-zero object. Thus we find $C \in \operatorname{fp}\mathcal{A}$ and a non-zero monomorphism $C/U \to X$ for some subobject $U \subseteq C$. Using Zorn's lemma, we choose a maximal subobject $V \subseteq C$ containing U and an injective envelope $C/V \to Q$. This yields a non-zero morphism $X \to Q$. Clearly, Q is indecomposable. □

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Our next goal is the definition of a topology on the spectrum of $\mathcal{A}$. We fix a locally finitely presented Grothendieck category $\mathcal{A}$ such that $\operatorname{fp}\mathcal{A}$ is abelian. For classes $\mathcal{C} \subseteq \operatorname{fp}\mathcal{A}$ and $\mathcal{U} \subseteq \operatorname{Sp}\mathcal{A}$ set

$$\mathcal{C}^{\perp} = \{X \in \operatorname{Sp}\mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(C, X) = 0 \text{ for all } C \in \mathcal{C}\} \subseteq \operatorname{Sp}\mathcal{A}$$
$${}^{\perp}\mathcal{U} = \{C \in \operatorname{fp}\mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(C, X) = 0 \text{ for all } X \in \mathcal{U}\} \subseteq \operatorname{fp}\mathcal{A}.$$

Lemma 12.1.12. *The assignment* $\mathcal{U} \mapsto \overline{\mathcal{U}} := ({}^{\perp}\mathcal{U})^{\perp}$ *defines a closure operator on* $\operatorname{Sp}\mathcal{A}$*. Thus the subsets* $\mathcal{U} \subseteq \operatorname{Sp}\mathcal{A}$ *satisfying* $\mathcal{U} = \overline{\mathcal{U}}$ *form the closed subsets of a topology on* $\operatorname{Sp}\mathcal{A}$*.*

Proof Following Kuratowski's axiomatisation of a topological space we need to verify that

(1) $\overline{\varnothing} = \varnothing$,
(2) $\mathcal{U} \subseteq \overline{\mathcal{U}}$ for every subset $\mathcal{U}$,
(3) $\overline{\mathcal{U}} = \overline{\overline{\mathcal{U}}}$ for every subset $\mathcal{U}$,
(4) $\overline{\mathcal{U}_1 \cup \mathcal{U}_2} = \overline{\mathcal{U}_1} \cup \overline{\mathcal{U}_2}$ for every pair of subsets $\mathcal{U}_1$ and $\mathcal{U}_2$.

The conditions (1)–(3) are easily checked; so it remains to show (4). From ${}^{\perp}(\mathcal{U}_1 \cup \mathcal{U}_2) \subseteq {}^{\perp}\mathcal{U}_1 \cap {}^{\perp}\mathcal{U}_2$ it follows that $\overline{\mathcal{U}_1} \cup \overline{\mathcal{U}_2} \subseteq \overline{\mathcal{U}_1 \cup \mathcal{U}_2}$. Now choose $X \notin \overline{\mathcal{U}_1} \cup \overline{\mathcal{U}_2}$, and we claim this implies $X \notin \overline{\mathcal{U}_1 \cup \mathcal{U}_2}$. Choose non-zero morphisms $\phi_i \colon C_i \to X$ with $C_i \in {}^{\perp}\mathcal{U}_i$. We have $\operatorname{Im}\phi_1 \cap \operatorname{Im}\phi_2 \neq 0$ since X is indecomposable. Choosing a finitely generated subobject $0 \neq U \subseteq \operatorname{Im}\phi_1 \cap \operatorname{Im}\phi_2$, there are finitely generated subobjects $U_i \subseteq C_i$ such that $\phi_i(U_i) = U$. We obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
0 & \longrightarrow & V & \xrightarrow{\alpha} & U_1 \oplus U_2 & \longrightarrow & U \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & W & \longrightarrow & C_1 \oplus C_2 & \longrightarrow & X
\end{array}$$

The morphisms $\alpha_i \colon V \to U_i$ are epimorphisms. Thus there are finitely generated subobjects $V_i \subseteq V$ such that $\alpha_i(V_i) = U_i$. Now set $C = (U_1 \oplus U_2)/\alpha(V_1 + V_2)$. We have $C \in \operatorname{fp}\mathcal{A}$ since $\operatorname{fp}\mathcal{A}$ is abelian, and one checks that $\operatorname{Hom}_{\mathcal{A}}(C, X) \neq 0$. On the other hand, $C \in {}^{\perp}(\mathcal{U}_1 \cup \mathcal{U}_2)$ since C is a quotient of each U_i. Therefore $X \notin \overline{\mathcal{U}_1 \cup \mathcal{U}_2}$ and the proof is complete. □

Proposition 12.1.13. *Let* $\mathcal{A}$ *be a locally finitely presented Grothendieck category and suppose that* $\operatorname{fp}\mathcal{A}$ *is abelian. Then the assignments* $\mathcal{C} \mapsto \mathcal{C}^{\perp}$ *and* $\mathcal{U} \mapsto {}^{\perp}\mathcal{U}$ *provide mutually inverse and inclusion reversing bijections between the Serre subcategories of* $\operatorname{fp}\mathcal{A}$ *and the closed subsets of* $\operatorname{Sp}\mathcal{A}$*.*

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Proof Clearly, both maps are well defined. Let $\mathcal{U} \subseteq \operatorname{Sp} \mathcal{A}$ be closed. Then $({}^{\perp}\mathcal{U})^{\perp} = \mathcal{U}$ by definition. Now let $\mathcal{C} \subseteq \operatorname{fp} \mathcal{A}$ be a Serre subcategory. The inclusion $\mathcal{C} \subseteq {}^{\perp}(\mathcal{C}^{\perp})$ is clear. For the other inclusion we apply Corollary 11.1.33. Thus $\vec{\mathcal{C}}$ is a localising subcategory satisfying $\vec{\mathcal{C}} \cap \operatorname{fp} \mathcal{A} = \mathcal{C}$. Furthermore, $\mathcal{C}^{\perp} = \vec{\mathcal{C}}^{\perp}$ and $\mathcal{C}^{\perp}$ identifies with $\operatorname{Sp}(\mathcal{A}/\vec{\mathcal{C}})$. The category $\mathcal{A}/\vec{\mathcal{C}}$ is locally finitely presented. Thus $\mathcal{C} = {}^{\perp}(\mathcal{C}^{\perp})$ by Lemma 12.1.11, since $\vec{\mathcal{C}} = {}^{\perp}(\vec{\mathcal{C}}^{\perp})$. □

We discuss briefly an alternative closure operation. Let $\mathcal{A}$ be a Grothendieck category and fix $\mathcal{U} \subseteq \operatorname{Sp} \mathcal{A}$. We denote by $\widehat{\mathcal{U}}$ the set of objects $X \in \operatorname{Sp} \mathcal{A}$ such that $X \subseteq \prod_{i \in I} Y_i$ for some set of objects $Y_i \in \mathcal{U}$. Now consider the localising subcategory $\mathcal{A}_{\mathcal{U}} = \{X \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(X, \mathcal{U}) = 0\}$. Then we have

$$\widehat{\mathcal{U}} = \{X \in \operatorname{Sp} \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(\mathcal{A}_{\mathcal{U}}, X) = 0\}$$

by Corollary 2.2.18.

Lemma 12.1.14. *Let $\mathcal{A}$ be a locally noetherian Grothendieck category and $\mathcal{U} \subseteq \operatorname{Sp} \mathcal{A}$. Then $\widehat{\mathcal{U}} = \overline{\mathcal{U}}$.*

Proof The inclusion $\widehat{\mathcal{U}} \subseteq \overline{\mathcal{U}}$ is automatic since ${}^{\perp}\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$. On the other hand, $\mathcal{A}_{\mathcal{U}}$ is generated by ${}^{\perp}\mathcal{U}$ since $\mathcal{A}$ is locally noetherian. Thus we have equality. □

Compactness

Let $\mathcal{C}$ be an abelian category and $\mathcal{X}$ a class of objects in $\mathcal{C}$. We write $S\langle \mathcal{X} \rangle$ for the smallest Serre subcategory containing $\mathcal{X}$.

Lemma 12.1.15. *If an object $X \in \mathcal{C}$ belongs to $S\langle \mathcal{X} \rangle$, then $X \in S\langle \mathcal{X}_0 \rangle$ for some finite set of objects $\mathcal{X}_0 \subseteq \mathcal{X}$.*

Proof The objects in $S\langle \mathcal{X} \rangle$ are obtained by closing the objects in $\mathcal{X}$ under forming subobjects, quotients, and extensions. For each $X \in S\langle \mathcal{X} \rangle$, finitely many such operations suffice. □

Let us call $\mathcal{C}$ *finitely generated* if $\mathcal{C} = S\langle X \rangle$ for some object $X \in \mathcal{C}$. An equivalent condition is the following. For any family $(\mathcal{C}_i)_{i \in I}$ of Serre subcategories $\bigvee_{i \in I} \mathcal{C}_i = \mathcal{C}$ implies $\bigvee_{i \in J} \mathcal{C}_i = \mathcal{C}$ for some finite subset $J \subseteq I$.

Recall that a topological space T is *quasi-compact* if for any family $(U_i)_{i \in I}$ of open subsets $\bigcup_{i \in I} U_i = T$ implies $\bigcup_{i \in J} U_i = T$ for some finite subset $J \subseteq I$.

Lemma 12.1.16. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category and suppose that $\operatorname{fp} \mathcal{A}$ is abelian. For a closed subset $\mathcal{V} \subseteq \operatorname{Sp} \mathcal{A}$ and $\mathcal{U} = \operatorname{Sp} \mathcal{A} \setminus \mathcal{V}$, we have*

(1) *$\mathcal{U}$ is quasi-compact if and only if ${}^{\perp}\mathcal{V}$ is finitely generated, and*

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

(2) *$\mathcal{V}$ is quasi-compact if and only if* $(\operatorname{fp}\mathcal{A})/({}^{\perp}\mathcal{V})$ *is finitely generated.*

Proof This is an immediate consequence of the correspondence in Proposition 12.1.13, since for any family $(\mathcal{C}_i)_{i\in I}$ of Serre subcategories of $\operatorname{fp}\mathcal{A}$

$$\Big(\bigvee_{i\in I}\mathcal{C}_i\Big)^{\perp}=\bigcap_{i\in I}(\mathcal{C}_i^{\perp}).\qquad\square$$

The Spectrum of Indecomposable Pure-Injectives

Let $\mathcal{A}$ be a locally finitely presented category and denote by $\operatorname{Ind}\mathcal{A}$ a representative set of the isomorphism classes of indecomposable pure-injective objects in $\mathcal{A}$ (the *spectrum* of indecomposable pure-injectives or *Ziegler spectrum*).

Lemma 12.1.17. *Let $\mathcal{A}$ be a locally finitely presented category. The functor* $\operatorname{ev}\colon\mathcal{A}\to\mathbf{P}(\mathcal{A})$ *induces a bijection* $\operatorname{Ind}\mathcal{A}\xrightarrow{\sim}\operatorname{Sp}\mathbf{P}(\mathcal{A})$*. Therefore every object admits a pure monomorphism into a pure-injective object which is a product of indecomposable objects.*

Proof The first assertion is clear from Lemma 12.1.8. Let $X\in\mathcal{A}$ and consider the canonical morphism

$$X\longrightarrow\prod_{\substack{Q\in\operatorname{Ind}\mathcal{A}\\ \phi\in\operatorname{Hom}_{\mathcal{A}}(X,Q)}}Q.$$

It follows from Lemma 12.1.7 and Lemma 12.1.11 that this is a pure monomorphism. $\square$

We use the identification $\operatorname{Ind}\mathcal{A}\xrightarrow{\sim}\operatorname{Sp}\mathbf{P}(\mathcal{A})$ and obtain a topology on $\operatorname{Ind}\mathcal{A}$. For classes $\mathcal{C}\subseteq\operatorname{fp}\mathbf{P}(\mathcal{A})$ and $\mathcal{U}\subseteq\operatorname{Ind}\mathcal{A}$ we set

$$\mathcal{C}^{\perp}=\{X\in\operatorname{Ind}\mathcal{A}\mid\operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(C,\bar{X})=0\text{ for all }C\in\mathcal{C}\}\subseteq\operatorname{Ind}\mathcal{A}$$
$${}^{\perp}\mathcal{U}=\{C\in\operatorname{fp}\mathbf{P}(\mathcal{A})\mid\operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(C,\bar{X})=0\text{ for all }X\in\mathcal{U}\}\subseteq\operatorname{fp}\mathbf{P}(\mathcal{A}).$$

Lemma 12.1.18. *The assignment* $\mathcal{U}\mapsto\overline{\mathcal{U}}:=({}^{\perp}\mathcal{U})^{\perp}$ *defines a closure operator on* $\operatorname{Ind}\mathcal{A}$*. Thus the subsets* $\mathcal{U}\subseteq\operatorname{Ind}\mathcal{A}$ *satisfying* $\mathcal{U}=\overline{\mathcal{U}}$ *form the closed subsets of a topology on* $\operatorname{Ind}\mathcal{A}$.

Proof Apply Lemma 12.1.12. $\square$

12.2 Definable Subcategories

Let $\mathcal{A}$ be a locally finitely presented category. We set

$$\mathcal{C}:=\operatorname{fp}\mathcal{A}\qquad\text{and}\qquad\operatorname{Ab}(\mathcal{A}):=\operatorname{Fp}(\mathcal{C},\operatorname{Ab})^{\mathrm{op}}.$$

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Note that $\mathrm{Ab}(\mathcal{A}) \xrightarrow{\sim} \mathrm{fp}\,\mathbf{P}(\mathcal{A})$ via $F \mapsto \mathrm{Hom}(-, F)$; see Theorem 11.1.15. The category $\mathrm{Ab}(\mathcal{A})$ is abelian and $\mathcal{A}$ identifies with the category of exact functors $\mathrm{Ab}(\mathcal{A})^{\mathrm{op}} \to \mathrm{Ab}$ via the assignment $X \mapsto \bar{X}$; see Lemma 12.1.4. The kernel of any exact functor $\bar{X}$ is a Serre subcategory of $\mathrm{Ab}(\mathcal{A})$ and therefore a natural invariant of X.

In this section we study the class of definable subcategories of $\mathcal{A}$. The terminology is justified by the fact that any definable subcategory is given by a family of morphisms in $\mathcal{C}$. Observe that a morphism ϕ in $\mathcal{C}$ yields a functor $F = \mathrm{Coker}\,\mathrm{Hom}_{\mathcal{C}}(\phi, -)$ in $\mathrm{Fp}(\mathcal{C}, \mathrm{Ab})$, and for $X \in \mathcal{A}$ we have

$$\bar{F}(X) = 0 \quad \Longleftrightarrow \quad \mathrm{Hom}_{\mathcal{A}}(\phi, X) \text{ is surjective} \quad \Longleftrightarrow \quad \bar{X}(F) = 0.$$

Definable Subcategories

A full subcategory $\mathcal{B} \subseteq \mathcal{A}$ is called *definable* if it is of the form

$$\mathcal{B} = \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(\phi_i, X) \text{ is surjective for all } i \in I\}$$

for a family of morphisms $(\phi_i)_{i \in I}$ in $\mathrm{fp}\,\mathcal{A}$; thus it is 'defined' by the ϕ_i. Similarly, a subset $\mathcal{U} \subseteq \mathrm{Ind}\,\mathcal{A}$ is *Ziegler closed* if there is a family $(\phi_i)_{i \in I}$ of morphisms in $\mathrm{fp}\,\mathcal{A}$ such that

$$\mathcal{U} = \{X \in \mathrm{Ind}\,\mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(\phi_i, X) \text{ is surjective for all } i \in I\}.$$

Let us consider the pairing

$$\mathrm{Fp}(\mathcal{C}, \mathrm{Ab}) \times \mathcal{A} \longrightarrow \mathrm{Ab}, \quad (F, X) \mapsto \bar{F}(X) = \bar{X}(F).$$

For classes $\mathcal{F} \subseteq \mathrm{Fp}(\mathcal{C}, \mathrm{Ab})$ and $\mathcal{X} \subseteq \mathcal{A}$ we set

$$\mathcal{F}^{\perp} = \{X \in \mathcal{A} \mid \bar{F}(X) = 0 \text{ for all } F \in \mathcal{F}\} \subseteq \mathcal{A}$$
$${}^{\perp}\mathcal{X} = \{F \in \mathrm{Fp}(\mathcal{C}, \mathrm{Ab}) \mid \bar{F}(X) = 0 \text{ for all } X \in \mathcal{X}\} \subseteq \mathrm{Fp}(\mathcal{C}, \mathrm{Ab}).$$

The pairing admits another interpretation. To this end identify $\mathrm{Fp}(\mathcal{C}, \mathrm{Ab})$ with the full subcategory of finitely presented objects in the purity category $\mathbf{P}(\mathcal{A})$ via the Yoneda embedding $F \mapsto \mathrm{Hom}(F, -)$. Then we have for all $X \in \mathcal{A}$

$$\bar{F}(X) \cong \mathrm{Hom}_{\mathbf{P}(\mathcal{A})}(F, \bar{X}).$$

Lemma 12.2.1. *The following holds.*

(1) $\mathcal{F}^{\perp}$ *is a definable subcategory of* $\mathcal{A}$.
(2) ${}^{\perp}\mathcal{X}$ *is a Serre subcategory of* $\mathrm{Fp}(\mathcal{C}, \mathrm{Ab})$.

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Proof (1) For $F = \operatorname{Coker}\operatorname{Hom}_{\mathcal{C}}(\phi, -)$ in $\mathcal{F}$, we have $\bar{F}(X) = 0$ if and only if $\operatorname{Hom}_{\mathcal{A}}(\phi, X)$ is surjective. Thus $\mathcal{F}^{\perp}$ is a definable subcategory for any choice of $\mathcal{F}$.

(2) The assignment $F \mapsto \bar{F}(X)$ is exact for fixed $X \in \mathcal{A}$. Thus ${}^{\perp}\mathcal{X}$ is a Serre subcategory for any choice of $\mathcal{X}$. □

We obtain for $\mathcal{X} \subseteq \mathcal{A}$ an abelian category by forming the quotient

$$\operatorname{Ab}(\mathcal{A}) \twoheadrightarrow \operatorname{Ab}(\mathcal{X}) := (\operatorname{Fp}(\mathcal{C}, \operatorname{Ab})/{}^{\perp}\mathcal{X})^{\mathrm{op}}.$$

Note that any inclusion $\mathcal{X}' \subseteq \mathcal{X}$ induces an exact functor $\operatorname{Ab}(\mathcal{X}) \twoheadrightarrow \operatorname{Ab}(\mathcal{X}')$.

There is the following fundamental correspondence for definable subcategories.

Theorem 12.2.2. *Let $\mathcal{A}$ be a locally finitely presented category.*

(1) *The assignments $\mathcal{F} \mapsto \mathcal{F}^{\perp}$ and $\mathcal{X} \mapsto {}^{\perp}\mathcal{X}$ provide mutually inverse and inclusion reversing bijections between the Serre subcategories of* $\operatorname{Ab}(\mathcal{A})$ *and the definable subcategories of $\mathcal{A}$.*
(2) *The assignment*

$$\mathcal{A} \supseteq \mathcal{B} \longmapsto \mathcal{B} \cap \operatorname{Ind}\mathcal{A} \subseteq \operatorname{Ind}\mathcal{A}$$

provides an inclusion preserving bijection between the definable subcategories of $\mathcal{A}$ and the Ziegler closed subsets of $\operatorname{Ind}\mathcal{A}$.

The first part of Theorem 12.2.2 has an immediate consequence.

Corollary 12.2.3. *For a definable subcategory $\mathcal{B} \subseteq \mathcal{A}$ the assignment $X \mapsto \bar{X}$ (Lemma 12.1.4) induces the following commutative square*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\sim} & \operatorname{Ex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}}, \operatorname{Ab}) \\ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{\sim} & \operatorname{Ex}(\operatorname{Ab}(\mathcal{A})^{\mathrm{op}}, \operatorname{Ab}) \end{array}$$

where the inclusion on the right is induced by composing with the canonical functor $\operatorname{Ab}(\mathcal{A}) \twoheadrightarrow \operatorname{Ab}(\mathcal{B})$.

Proof Let $\mathcal{S} \subseteq \operatorname{Fp}(\mathcal{C}, \operatorname{Ab})$ be a Serre subcategory. Then the exact functors $\operatorname{Fp}(\mathcal{C}, \operatorname{Ab}) \to \operatorname{Ab}$ that vanish on $\mathcal{S}$ identify with the exact functors $\frac{\operatorname{Fp}(\mathcal{C},\operatorname{Ab})}{\mathcal{S}} \to \operatorname{Ab}$; see Proposition 11.1.31. □

Proof of Theorem 12.2.2 (1) It is convenient to work in the purity category $\mathbf{P}(\mathcal{A})$ and we identify $\operatorname{Fp}(\mathcal{C}, \operatorname{Ab})$ with the full subcategory of finitely presented

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

objects in $\mathbf{P}(\mathcal{A})$ via the Yoneda embedding $F \mapsto \operatorname{Hom}(F,-)$. Then we have for all $X \in \mathcal{A}$

$$\bar{F}(X) \cong \operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(F, \bar{X}).$$

We use the bijection $\operatorname{Ind}\mathcal{A} \xrightarrow{\sim} \operatorname{Sp}\mathbf{P}(\mathcal{A})$ from Lemma 12.1.17 and combine this with the bijection from Proposition 12.1.13. Thus the assignments $\mathcal{F} \mapsto \mathcal{F}^{\perp} \cap \operatorname{Ind}\mathcal{A}$ and $\mathcal{U} \mapsto {}^{\perp}\mathcal{U}$ provide mutually inverse and inclusion reversing bijections between the Serre subcategories of $\operatorname{Fp}(\mathcal{C}, \mathrm{Ab})$ and the Ziegler closed subsets of $\operatorname{Ind}\mathcal{A}$.

Fix a Serre subcategory $\mathcal{S}$ of $\operatorname{Fp}(\mathcal{C}, \mathrm{Ab})$. Then the above bijections imply $\mathcal{S} = {}^{\perp}(\mathcal{S}^{\perp})$.

Now fix a definable subcategory $\mathcal{B} = \mathcal{F}^{\perp}$ of $\mathcal{A}$, which is given by some $\mathcal{F} \subseteq \operatorname{Fp}(\mathcal{C}, \mathrm{Ab})$. Let $\mathcal{S} \subseteq \operatorname{Fp}(\mathcal{C}, \mathrm{Ab})$ denote the smallest Serre subcategory containing $\mathcal{F}$. Clearly, $\mathcal{B} = \mathcal{S}^{\perp}$. Thus we have

$$({}^{\perp}\mathcal{B})^{\perp} = ({}^{\perp}(\mathcal{S}^{\perp}))^{\perp} = \mathcal{S}^{\perp} = \mathcal{B},$$

where one uses the equality $\mathcal{S} = {}^{\perp}(\mathcal{S}^{\perp})$ from the first part of the proof.

(2) The assertion claims that a definable subcategory $\mathcal{B}$ is determined by $\mathcal{B} \cap \operatorname{Ind}\mathcal{A}$. This follows from (1). In fact, $\mathcal{B}$ identifies with $\operatorname{Ex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}}, \mathrm{Ab})$ as in Corollary 12.2.3, and it remains to observe that $\mathcal{B} \cap \operatorname{Ind}\mathcal{A}$ identifies with the indecomposable injective objects in $\operatorname{Lex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}}, \mathrm{Ab})$, which form a set of cogenerators; see Lemma 12.1.11. □

Closure Properties of Definable Subcategories

Definable subcategories are characterised by some natural closure properties. The proof of this requires some preparations.

Lemma 12.2.4. *A filtered colimit* $\operatorname{colim}_{i \in \mathcal{I}} X_i$ *and a family of monomorphisms* $(X_i \to Y_i)_{i \in \mathcal{I}}$ *in a Grothendieck category induce a monomorphism*

$$\operatorname*{colim}_{i \in \mathcal{I}} X_i \longrightarrow \operatorname*{colim}_{i \in \mathcal{I}} \Big(\prod_{i \to j} Y_j \Big).$$

Proof For each $i \in \mathcal{I}$ we have a canonical monomorphisms $X_i \to \prod_{i \to j} Y_j$, where $i \to j$ runs through all morphisms in $\mathcal{I}$ starting at i and each component is given by the composite $X_i \to X_j \to Y_j$. A morphism $i \to i'$ in $\mathcal{I}$ yields a commuting square.

$$\begin{array}{ccc} X_i & \longrightarrow & \prod_{i \to j} Y_j \\ \downarrow & & \downarrow \\ X_{i'} & \longrightarrow & \prod_{i' \to j'} Y_{j'} \end{array}$$

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Taking colimits yields the desired morphism, which is a monomorphism since filtered colimits preserve monomorphisms. □

Theorem 12.2.5. *A full subcategory of a locally finitely presented category is definable if and only if it is closed under taking products, filtered colimits, and pure subobjects.*

Proof Let $\mathcal{A}$ be a locally finitely presented category. As before, it is convenient to work in $\mathbf{P}(\mathcal{A})$ and we identify $\operatorname{Fp}(\mathcal{C}, \mathrm{Ab})$ with the full subcategory of finitely presented objects in $\mathbf{P}(\mathcal{A})$.

One direction is clear, since for any $F \in \operatorname{Fp}(\mathcal{C}, \mathrm{Ab})$, the functor $\bar{F} \colon \mathcal{A} \to \mathrm{Ab}$ preserves filtered colimits, products, and sends pure monomorphisms to monomorphisms. This follows easily from the presentation (12.1.2).

Now suppose that $\mathcal{B} \subseteq \mathcal{A}$ is closed under taking products, filtered colimits, and pure subobjects. Set

$$\mathcal{F} = \{X \in \mathbf{P}(\mathcal{A}) \mid X \subseteq \bar{Y} \text{ for some } Y \in \mathcal{B}\}$$

and

$$\mathcal{T} = \{X \in \mathbf{P}(\mathcal{A}) \mid \operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(X, \bar{Y}) = 0 \text{ for all } Y \in \mathcal{B}\}.$$

We claim that this gives a torsion pair $(\mathcal{T}, \mathcal{F})$ for $\mathbf{P}(\mathcal{A})$. First observe that $\mathcal{F}$ is closed under filtered colimits, by Lemma 12.2.4. The inclusion $\mathcal{F} \hookrightarrow \mathbf{P}(\mathcal{A})$ has a left adjoint $f \colon \mathbf{P}(\mathcal{A}) \to \mathcal{F}$ which is constructed as follows. For $X \in \mathbf{P}(\mathcal{A})$ let $(Y_i)_{i \in I}$ be the set of quotient objects of X which are in $\mathcal{F}$. Define $f(X)$ to be the image and $t(X)$ the kernel of the canonical morphism $X \to \prod_{i \in I} Y_i$. Next observe that $\mathcal{S} = \mathcal{T} \cap \operatorname{fp} \mathbf{P}(\mathcal{A})$ is a Serre subcategory of $\operatorname{fp} \mathbf{P}(\mathcal{A})$. We write $\mathcal{T}' = \vec{\mathcal{S}}$ for the full subcategory consisting of the filtered colimits $\operatorname{colim} X_i$ with $X_i \in \mathcal{S}$ for all i. We claim that $\mathcal{T}' = \mathcal{T}$.

For each $X \in \operatorname{fp} \mathbf{P}(\mathcal{A})$ we show that $t(X) \in \mathcal{T}'$. To this end write $t(X) = \operatorname{colim} U_i$ as filtered colimit of its finitely generated subobjects. We need to show that $U_i \in \mathcal{S}$ for all i. Suppose that $U = U_i \notin \mathcal{S}$. Then there is a non-zero morphism $\phi \colon U \to \bar{Y}$ for some $Y \in \mathcal{B}$, and ϕ extends to a morphism $\psi \colon X \to \bar{Y}$ since $\bar{Y}$ is an exact functor; see Lemma 12.1.4. But the adjointness property of f implies that ψ factors through $X \to f(X)$. Therefore $\phi(U) = 0$, a contradiction to our assumption. Thus $t(X) \in \mathcal{T}'$. Now let $X = \operatorname{colim} X_i$ be an arbitrary object in $\mathbf{P}(\mathcal{A})$, written as a filtered colimit of objects in $\operatorname{fp} \mathbf{P}(\mathcal{A})$. We obtain an exact sequence

$$0 \longrightarrow \operatorname{colim} t(X_i) \longrightarrow \operatorname{colim} X_i \longrightarrow \operatorname{colim} f(X_i) \longrightarrow 0$$

with $\operatorname{colim} t(X_i) \in \mathcal{T}'$ and $\operatorname{colim} f(X_i) \in \mathcal{F}$, since both $\mathcal{T}'$ and $\mathcal{F}$ are closed under filtered colimits. We conclude that $\mathcal{T}' = \mathcal{T}$ and $(\mathcal{T}, \mathcal{F})$ is a torsion pair. Thus

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

for $X \in \mathcal{A}$ we have $X \in \mathcal{B}$ if and only if $\bar{X} \in \mathcal{F}$ if and only if $\operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(\mathcal{S}, \bar{X}) = 0$ if and only if $X \in \mathcal{S}^{\perp}$. It follows that $\mathcal{B}$ is definable. □

Example 12.2.6. Let $\mathcal{A}$ be a locally finitely presented category and $\mathcal{C} \subseteq \operatorname{fp}\mathcal{A}$ a full additive subcategory. Then $\vec{\mathcal{C}}$ is a definable subcategory of $\mathcal{A}$ if and only if $\mathcal{C}$ is covariantly finite in $\operatorname{fp}\mathcal{A}$.

Proof For any $\mathcal{C}$, it is easily checked that $\vec{\mathcal{C}}$ is closed under filtered colimits and pure subobjects. Thus it remains to check when $\vec{\mathcal{C}}$ is closed under products; see Example 11.1.24. □

Let us mention another important property of definable subcategories.

Proposition 12.2.7. *A definable subcategory of a locally finitely presented category is covariantly finite.*

Proof Let $\mathcal{B} \subseteq \mathcal{A}$ be a definable subcategory. We use the identification $\mathcal{B} \xrightarrow{\sim} \operatorname{Ex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}}, \operatorname{Ab})$ from Corollary 12.2.3 and view this as a subcategory of $\mathcal{A}$ via the canonical functor $p\colon \operatorname{Ab}(\mathcal{A}) \twoheadrightarrow \operatorname{Ab}(\mathcal{B})$. Now observe that

$$\operatorname{Lex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}}, \operatorname{Ab}) \subseteq \operatorname{Lex}(\operatorname{Ab}(\mathcal{A})^{\mathrm{op}}, \operatorname{Ab})$$

is covariantly finite, since the restriction p^* admits a left adjoint; see Proposition 11.1.31. On the other hand,

$$\operatorname{Ex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}}, \operatorname{Ab}) \subseteq \operatorname{Lex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}}, \operatorname{Ab})$$

is covariantly finite by Proposition 11.1.27.

Let X be an object in $\mathcal{A}$, viewed as an exact functor $\operatorname{Ab}(\mathcal{A})^{\mathrm{op}} \to \operatorname{Ab}$. Compose the approximations $X \to X^{\operatorname{Lex}\mathcal{B}}$ and $X^{\operatorname{Lex}\mathcal{B}} \to X^{\operatorname{Ex}\mathcal{B}}$, which are obtained from the above inclusions. This gives a left $\mathcal{B}$-approximation of X. □

We add one more closure property of definable subcategories.

Proposition 12.2.8. *A definable subcategory of a locally finitely presented category is closed under taking pure-injective envelopes.*

Proof Let $\mathcal{B} \subseteq \mathcal{A}$ be a definable subcategory. As before, we use the identification $\mathcal{B} \xrightarrow{\sim} \operatorname{Ex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}}, \operatorname{Ab})$ from Corollary 12.2.3 and view this as a subcategory of $\mathcal{A}$ via the canonical functor $\operatorname{Ab}(\mathcal{A}) \twoheadrightarrow \operatorname{Ab}(\mathcal{B})$. Also, we use that pure-injectives in $\mathcal{A}$ identify with injectives in $\operatorname{Lex}(\operatorname{Ab}(\mathcal{A})^{\mathrm{op}}, \operatorname{Ab})$, by Lemma 12.1.8. Then the assertion follows from the fact that

$$\operatorname{Lex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}}, \operatorname{Ab}) \subseteq \operatorname{Lex}(\operatorname{Ab}(\mathcal{A})^{\mathrm{op}}, \operatorname{Ab})$$

is closed under taking injective envelopes; see Corollary 2.2.15 and Proposition 11.1.31. □

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Change of Categories

We study functors between locally finitely presented categories. For any locally finitely presented category $\mathcal{A}$ we use the canonical identification $\mathcal{A} = \operatorname{Ex}(\operatorname{Ab}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$; see Lemma 12.1.4. Now let $\mathcal{A}$ and $\mathcal{B}$ be locally finitely presented categories. Then an exact functor $f\colon \operatorname{Ab}(\mathcal{B}) \to \operatorname{Ab}(\mathcal{A})$ induces a functor $f^*\colon \mathcal{A} \to \mathcal{B}$ by sending $X \in \mathcal{A}$ to $X \circ f$.

Theorem 12.2.9. *For a functor $F\colon \mathcal{A} \to \mathcal{B}$ between locally finitely presented categories the following are equivalent.*

(1) *F preserves filtered colimits and products.*
(2) *$F \cong f^*$ for some exact functor $f\colon \operatorname{Ab}(\mathcal{B}) \to \operatorname{Ab}(\mathcal{A})$.*

Moreover, in this case F preserves pure-injectivity, and F is fully faithful if and only if f induces an equivalence $\operatorname{Ab}(\mathcal{B})/(\operatorname{Ker} f) \xrightarrow{\sim} \operatorname{Ab}(\mathcal{A})$.

Proof One implication is clear since f^* preserves limits and colimits. Thus we assume that F preserves filtered colimits and products. The functor f is constructed as follows. The restriction $F|_{\operatorname{fp}\mathcal{A}}\colon \operatorname{fp}\mathcal{A} \to \mathcal{B} \hookrightarrow \mathbf{P}(\mathcal{B})$ extends to a left exact functor $\operatorname{fp}\mathbf{P}(\mathcal{A}) \to \mathbf{P}(\mathcal{B})$, and this extends to a filtered colimit preserving functor $\bar{F}\colon \mathbf{P}(\mathcal{A}) \to \mathbf{P}(\mathcal{B})$. Note that $\bar{F}$ extends $\mathcal{A} \xrightarrow{F} \mathcal{B} \hookrightarrow \mathbf{P}(\mathcal{B})$. Also $\bar{F}$ is left exact, since an exact sequence in $\mathbf{P}(\mathcal{A})$ can be written as a filtered colimit of exact sequences in $\operatorname{fp}\mathbf{P}(\mathcal{A})$; see Remark 11.1.20. Moreover, $\bar{F}$ preserves products since its restriction to the full subcategory of injective objects preserves products. Thus $\bar{F}$ preserves limits and therefore has a left adjoint $\bar{F}_\lambda\colon \mathbf{P}(\mathcal{B}) \to \mathbf{P}(\mathcal{A})$ by the special adjoint functor theorem [183, Theorem 10.6.5]. Note that $\bar{F}_\lambda$ restricts to a functor $f\colon \operatorname{Ab}(\mathcal{B}) = \operatorname{fp}\mathbf{P}(\mathcal{B}) \to \operatorname{fp}\mathbf{P}(\mathcal{A}) = \operatorname{Ab}(\mathcal{A})$, since $\bar{F}$ preserves filtered colimits. The functor f induces an adjoint pair $(f_!, f^*) = (\bar{F}_\lambda, \bar{F})$ of functors $\mathbf{P}(\mathcal{B}) \rightleftarrows \mathbf{P}(\mathcal{A})$. In particular $f^*|_{\mathcal{A}} \cong F$.

It remains to show that f is exact. Observe that a sequence $\eta\colon 0 \to A \to B \to C \to 0$ in $\operatorname{fp}\mathbf{P}(\mathcal{A})$ is exact if and only if $\operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(\eta, \bar{X})$ is exact for every $X \in \mathcal{A}$, since every injective object in $\mathbf{P}(\mathcal{A})$ is of the form $\bar{X}$ for some $X \in \mathcal{A}$. Thus if a sequence η in $\operatorname{fp}\mathbf{P}(\mathcal{B})$ is exact, then $\operatorname{Hom}_{\mathbf{P}(\mathcal{B})}(\eta, \bar{F}(\bar{X}))$ is exact, and therefore the sequence $\operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(f(\eta), \bar{X})$ is exact. It follows that f is exact.

Having shown that f is exact, it follows that $f_!$ is exact, since every exact sequence in $\mathbf{P}(\mathcal{B})$ can be written as a filtered colimit of exact sequences in $\operatorname{fp}\mathbf{P}(\mathcal{B})$. Thus its right adjoint f^* preserves injectivity, and therefore F preserves pure-injectivity because of Lemma 12.1.8.

Next we apply Lemma 11.1.30. Thus $f^*\colon \mathbf{P}(\mathcal{A}) \to \mathbf{P}(\mathcal{B})$ is fully faithful if and only if f induces an equivalence $\operatorname{Ab}(\mathcal{B})/(\operatorname{Ker} f) \xrightarrow{\sim} \operatorname{Ab}(\mathcal{A})$. It remains to observe that f^* is fully faithful if and only if its restriction to the full

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

subcategories of injective objects is fully faithful; see Lemma 2.1.11. Here we use again that pure-injectives in $\mathcal{A}$ identify with injectives in $\mathbf{P}(\mathcal{A})$. □

Remark 12.2.10. Suppose $F\colon \mathcal{A} \to \mathcal{B}$ preserves filtered colimits and products. If a subcategory $\mathcal{B}' \subseteq \mathcal{B}$ is definable then $F^{-1}(\mathcal{B}') \subseteq \mathcal{A}$ is definable. On the other hand, if F is fully faithful then F maps definable subcategories to definable subcategories.

A consequence of the theorem is a characterisation of coherent functors, which is a special case of Theorem 2.5.26.

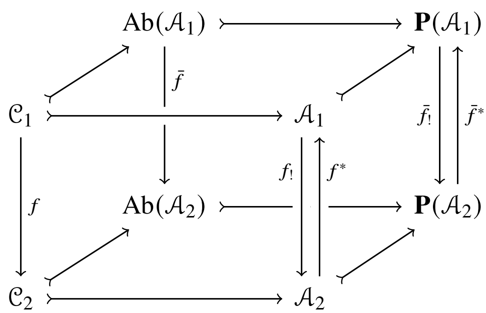
Corollary 12.2.11. *A functor* $F\colon \mathcal{A} \to \mathrm{Ab}$ *preserves filtered colimits and products if and only if it admits a presentation*

$$\mathrm{Hom}_{\mathcal{A}}(D,-) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(C,-) \longrightarrow F \longrightarrow 0$$

which is given by a morphism $C \to D$ *in* $\operatorname{fp}\mathcal{A}$.

Proof One direction is clear. Thus we assume that $F \cong f^*$ for some exact functor $f\colon \mathrm{Ab}(\mathrm{Ab}) \to \mathrm{Ab}(\mathcal{A})$. Then $f(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},-))$ is an object in $\mathrm{Fp}(\operatorname{fp}\mathcal{A}, \mathrm{Ab})$ which yields the presentation of F. □

Next we consider locally finitely presented categories $\mathcal{A}_i$ for $i = 1,2$ such that each $\mathcal{C}_i := \operatorname{fp}\mathcal{A}_i$ is abelian. Fix an exact functor $f\colon \mathcal{C}_1 \to \mathcal{C}_2$. This induces an adjoint pair $(f_!, f^*)$ of functors $\mathcal{A}_1 \rightleftarrows \mathcal{A}_2$ and also an exact functor $\bar{f}\colon \mathrm{Ab}(\mathcal{A}_1) \to \mathrm{Ab}(\mathcal{A}_2)$. We collect these functors in the following commutative diagram, where all vertical downward functors are exact.



Lemma 12.2.12. *Suppose* $f\colon \mathcal{C}_1 \to \mathcal{C}_2$ *induces an equivalence* $\mathcal{C}_1/(\operatorname{Ker} f) \xrightarrow{\sim} \mathcal{C}_2$. *Then* $\bar{f}$ *also induces an equivalence* $\mathrm{Ab}(\mathcal{A}_1)/(\operatorname{Ker}\bar{f}) \xrightarrow{\sim} \mathrm{Ab}(\mathcal{A}_2)$, *and both* f^* *and* $\bar{f}^*$ *are fully faithful.*

Proof We apply Lemma 11.1.30. If $f\colon \mathcal{C}_1 \to \mathcal{C}_2$ induces an equivalence $\mathcal{C}_1/(\operatorname{Ker} f) \xrightarrow{\sim} \mathcal{C}_2$ then f^* is fully faithful, and therefore also $\bar{f}^*$ is fully faithful.

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Restricting the left adjoint $\bar{f}_!$ to subcategories of finitely presented objects, it follows that $\mathrm{Ab}(\mathcal{A}_1)/(\mathrm{Ker}\,\bar{f}) \xrightarrow{\sim} \mathrm{Ab}(\mathcal{A}_2)$. □

12.3 Indecomposable Pure-Injective Objects

In this section we focus on properties of indecomposable pure-injective objects in locally finitely presented categories. In particular, we investigate when objects decompose into indecomposable pure-injectives.

We keep our set-up and fix a locally finitely presented category $\mathcal{A}$. We set $\mathcal{C} = \mathrm{fp}\,\mathcal{A}$ and $\mathrm{Ab}(\mathcal{A}) = \mathrm{Fp}(\mathcal{C}, \mathrm{Ab})^{\mathrm{op}}$, so that objects $X \in \mathcal{A}$ identify with exact functors $\bar{X}\colon\ \mathrm{Ab}(\mathcal{A})^{\mathrm{op}} \to \mathrm{Ab}$. We set

$$\mathrm{Ab}(X) := \mathrm{Ab}(\mathcal{A})/\mathrm{Ker}\,\bar{X}.$$

Subgroups of Finite Definition

Fix an object $X \in \mathcal{A}$. We consider the exact functor

$$\mathrm{Ab}(\mathcal{A}) \twoheadrightarrow \mathrm{Ab}(X) \longrightarrow \mathrm{Mod}\,\mathrm{End}(X), \qquad F \mapsto \bar{F}(X) = \bar{X}(F)$$

and study its image. This leads to the notion of a subgroup of finite definition. In fact, for each object in $\mathrm{fp}\,\mathcal{A}$ the collection of these subgroups forms a lattice which provides a useful invariant of X.

Given a morphism $\phi\colon C \to C'$ in $\mathrm{fp}\,\mathcal{A}$, we denote by X_ϕ the image of the induced map $\mathrm{Hom}(C', X) \to \mathrm{Hom}(C, X)$ and call it a *subgroup of finite definition* of $\mathrm{Hom}(C, X)$. Thus

$$\bar{X}(\mathrm{Im}\,\mathrm{Hom}(\phi, -)) = X_\phi = \mathrm{Im}\,\mathrm{Hom}(\phi, X).$$

Note that any subgroup X_ϕ of finite definition is an $\mathrm{End}(X)$-submodule.

Lemma 12.3.1. *The subgroups of finite definition of* $\mathrm{Hom}(C, X)$ *are closed under finite sums and intersections. Thus they form a lattice, which is anti-isomorphic to the lattice of subobjects of* $\mathrm{Hom}(C, -)$ *in* $\mathrm{Ab}(X)$.

Proof Given morphisms $\phi_i\colon C \to C_i$ $(i = 1, 2)$ in $\mathcal{C}$, the pushout is given by an exact sequence $C \xrightarrow{\phi_1+\phi_2} C_1 \oplus C_2 \to C' \to 0$. Then

$$X_{\phi_1} + X_{\phi_2} = X_{\phi_1+\phi_2} \qquad \text{and} \qquad X_{\phi_1} \cap X_{\phi_2} = X_\phi$$

for $\phi\colon C \to C_i \to C'$. Also observe that $X_{\phi_2} \subseteq X_{\phi_1}$ if $X_{\phi_2} = X_{\psi\phi_1}$ for some $\psi\colon C_1 \to C'$.

Any subobject of $\mathrm{Hom}(C, -)$ in $\mathrm{Fp}(\mathcal{C}, \mathrm{Ab})$ is of the form $F = \mathrm{Im}\,\mathrm{Hom}(\phi, -)$

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

for some morphism $\phi\colon C \to C'$. The assignment $F \mapsto \bar{X}(F)$ induces an inclusion preserving map from the lattice of subobjects of $\operatorname{Hom}(C,-)$ in $\operatorname{Fp}(\mathcal{C},\mathrm{Ab})$ to the lattice of subgroups of finite definition of $\operatorname{Hom}(C,X)$. Clearly, this is surjective, and for $F' \subseteq F$ we have $F' = F$ in $\operatorname{Ab}(X)$ if and only if $\bar{X}(F') = \bar{X}(F)$. Finally note that $X_{\phi'} \subseteq X_{\phi}$ implies $\operatorname{Im}\operatorname{Hom}(\phi',-) \subseteq \operatorname{Im}\operatorname{Hom}(\phi,-)$, since we may assume $\phi' = \psi\phi$ for some ψ. □

Lemma 12.3.2. *Given a pure-injective object X in $\mathcal{A}$, every cyclic* $\operatorname{End}(X)$*-submodule of* $\operatorname{Hom}(C,X)$ *is the intersection of subgroups of finite definition.*

Proof We use the embedding $\mathcal{A} \to \mathbf{P}(\mathcal{A})$ that takes X to $\bar{X}$. Let $\phi\colon C \to X$ be a morphism and write $\operatorname{Ker}\bar{\phi} = \sum_i K_i$ as a sum of finiteley generated subobjects in $\mathbf{P}(\mathcal{A})$. For each i choose a morphism $\phi_i\colon C \to C_i$ with $\operatorname{Ker}\bar{\phi}_i = K_i$. Then $\phi \in \bigcap_i X_{\phi_i}$. On the other hand, every morphism $C \to X$ in $\bigcap_i X_{\phi_i}$ necessarily factors through ϕ since X is pure-injective. □

Σ-Pure-Injectivity

For a definable subcategory $\mathcal{B} \subseteq \mathcal{A}$ the abelian category $\operatorname{Ab}(\mathcal{B})$ is an important invariant. We illustrate this by the following result.

Theorem 12.3.3. *Let $\mathcal{A}$ be a locally finitely presented category. For a definable subcategory $\mathcal{B}$ of $\mathcal{A}$ the following are equivalent.*

(1) *Every object in $\mathcal{B}$ is pure-injective.*
(2) *Every object in $\mathcal{B}$ decomposes into a coproduct of indecomposable objects with local endomorphism rings.*
(3) *Every object in* $\operatorname{Ab}(\mathcal{B})$ *is noetherian.*

Proof We begin with some preparations. Identify $\mathcal{B}$ with $\operatorname{Ex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}},\mathrm{Ab})$; see Corollary 12.2.3. Thus we identify an object $X \in \mathcal{B}$ with the exact functor $\bar{X}\colon \operatorname{Ab}(\mathcal{A})^{\mathrm{op}} \twoheadrightarrow \operatorname{Ab}(\mathcal{B})^{\mathrm{op}} \xrightarrow{\bar{X}_{\mathcal{B}}} \mathrm{Ab}$. Set $\mathbf{P}(\mathcal{B}) = \operatorname{Lex}(\operatorname{Ab}(\mathcal{B})^{\mathrm{op}},\mathrm{Ab})$ and note that $\operatorname{fp}\mathbf{P}(\mathcal{B})$ identifies with $\operatorname{Ab}(\mathcal{B})$ by Theorem 11.1.15. Now $X \in \mathcal{B}$ is pure-injective if and only if $\bar{X}$ is injective in $\mathbf{P}(\mathcal{A})$ if and only if $\bar{X}_{\mathcal{B}}$ is injective in $\mathbf{P}(\mathcal{B})$; see Lemma 12.1.8 and Proposition 11.1.31 plus the subsequent remark.

(1) ⇔ (3): Apply Corollary 11.2.15, which says that all functors of the form $\bar{X}_{\mathcal{B}}$ are injective if and only if all objects in $\operatorname{Ab}(\mathcal{B})$ are noetherian.

(2) ⇔ (3): Observe that all objects in $\operatorname{Ab}(\mathcal{B})$ are noetherian if and only if $\mathbf{P}(\mathcal{B})$ is locally noetherian; see Proposition 11.2.5. Now apply Theorem 11.2.12. □

An object X in $\mathcal{A}$ is called Σ-*pure-injective* if every coproduct of copies of X

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

is pure-injective. A Σ-pure-injective object admits a decomposition into indecomposable objects. In fact, there is a host of useful properties that characterise Σ-pure-injectivity.

Theorem 12.3.4. *For an object X in $\mathcal{A}$ the following are equivalent.*

(1) *The object X is Σ-pure-injective.*
(2) *Every object in* $\mathrm{Ab}(X)$ *is noetherian.*
(3) *The object X is pure-injective and the direct summands of products of copies of X form a definable subcategory.*
(4) *The canonical monomorphism $X^{(\mathbb{N})} \to X^{\mathbb{N}}$ splits.*
(5) *Every product of copies of X decomposes into a coproduct of indecomposable objects with local endomorphism rings.*
(6) *There exists an object Y such that every product of copies of X is a pure subobject of a coproduct of copies of Y.*
(7) *The subgroups of finite definition of* $\mathrm{Hom}(C, X)$ *satisfy the descending chain condition for every $C \in \operatorname{fp} \mathcal{A}$.*

Proof We apply Theorem 12.3.3 by taking for $\mathcal{B}$ the smallest definable subcategory of $\mathcal{A}$ containing X. In particular, we have $\mathrm{Ab}(X) = \mathrm{Ab}(\mathcal{B})$. As in the proof of Theorem 12.3.3, we consider $\mathbf{P}(\mathcal{B})$ and characterise the fact that it is locally noetherian.

(1) $\Rightarrow$ (2): We adapt the proof of Theorem 11.2.12. The pure-injectivity of all coproducts of copies of X implies that all coproducts of copies of $\bar{X}$ are injective in $\mathbf{P}(\mathcal{B})$. It follows that all objects in $\mathrm{Ab}(X)$ are noetherian.

(2) $\Rightarrow$ (3): If follows from Theorem 12.3.3 that X is pure-injective. In fact, the object $\bar{X}$ in $\mathbf{P}(\mathcal{B})$ is an injective cogenerator. Thus each $Y \in \mathcal{B}$ is a pure subobject of some product of copies of X. The pure monomorphism splits since Y is pure-injective, again by Theorem 12.3.3.

(3) $\Rightarrow$ (1): All objects in $\mathcal{B}$ are pure-injective. Thus all coproducts of copies of X are pure-injective.

(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6): This follows from Proposition 11.2.16 applied to $\bar{X}$ in $\mathbf{P}(\mathcal{B})$. The assumption on $\bar{X}$ in this proposition is satisfied by Lemma 11.1.26.

(2) $\Leftrightarrow$ (7): Every object in $\mathrm{Ab}(X)$ is noetherian if and only if $\mathrm{Hom}(C, -)$ is noetherian in $\mathrm{Ab}(X)$ for all $C \in \operatorname{fp} \mathcal{A}$. Now apply Lemma 12.3.1. □

Given an object X in $\mathcal{A}$, every subgroup of finite definition of $\mathrm{Hom}(C, X)$ is an $\mathrm{End}(X)$-submodule. Therefore X is Σ-pure-injective, provided $\mathrm{Hom}(C, X)$ is an artinian module over $\mathrm{End}(X)$ for all $C \in \operatorname{fp} \mathcal{A}$. We note the following partial converse.

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Lemma 12.3.5. *Let $X \in \mathcal{A}$ be Σ-pure-injective. Then every finitely generated* $\operatorname{End}(X)$*-submodule of* $\operatorname{Hom}(C, X)$ *is a subgroup of finite definition.*

Proof It suffices to show this for cyclic submodules since subgroups of finite definition are closed under finite sums by Lemma 12.3.1. But for cyclic submodules this follows from Lemma 12.3.2 since subgroups of finite definition of $\operatorname{Hom}(C, X)$ satisfy the descending chain condition by Theorem 12.3.4. □

Example 12.3.6. Let Q be a quiver and k a commutative ring. If X is a k-linear representation such that X_i is an artinian k-module for each vertex $i \in Q_0$, then X is Σ-pure-injective.

Proof For a finitely presented representation C, we have an epimorphism $\bigoplus_{j=1}^{n} P(i_j) \to C$ where $P(i_1), \dots, P(i_n)$ is a finite number of standard projectives corresponding to vertices $i_j \in Q_0$. We have $\operatorname{Hom}(P(i), X) \cong X_i$ for each $i \in Q_0$. Thus $\operatorname{Hom}(C, X)$ identifies with an $\operatorname{End}(X)$-submodule of $\bigoplus_{j=1}^{n} X_{i_j}$ and is therefore artinian, so satisfies the descending chain condition for subgroups of finite definition. □

Product-Complete Objects

We consider a particular class of Σ-pure-injective objects. For an object X let $\operatorname{Add} X$ denote the full subcategory consisting of all direct summands of coproducts of copies of X. Analogously, let $\operatorname{Prod} X$ denote the full subcategory consisting of all direct summands of products of copies of X.

An object satisfying the equivalent conditions of the following proposition is called *product-complete*.

Proposition 12.3.7. *Let $\mathcal{A}$ be a locally finitely presented category. For an object X the following are equivalent.*

(1) $\operatorname{Prod} X = \operatorname{Add} X$.
(2) $\operatorname{Add} X$ *is a definable subcategory of* $\mathcal{A}$.
(3) *X is Σ-pure-injective and the indecomposable direct summands of X form a Ziegler closed set.*

Proof (1) ⇒ (2): It follows from Theorem 12.3.4 that X is Σ-pure-injective. The same result implies that $\operatorname{Add} X$ is a definable subcategory.

(2) ⇒ (3): As before, Theorem 12.3.4 implies that X is Σ-pure-injective. The indecomposable objects in $\operatorname{Add} X$ form a Ziegler closed set by Theorem 12.2.2.

(3) ⇒ (1): The definable subcategory generated by X equals $\operatorname{Prod} X$ by Theorem 12.3.4, since X is Σ-pure-injective. Since all objects in $\operatorname{Prod} X$ decompose into indecomposable objects, it follows that $\operatorname{Prod} X = \operatorname{Add} X$. □

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Prüfer Objects

We consider a class of Σ-pure-injective objects which generalises the notion of a Prüfer module over a Dedekind domain.

Let k be a commutative noetherian ring and $\mathcal{A}$ a k-linear locally finitely presented category. An object $X \in \mathcal{A}$ is called a *Prüfer object* if there is an endomorphism $\phi\colon X \to X$ such that for each $C \in \operatorname{fp}\mathcal{A}$

(Pr1) each morphism $C \to X$ is annihilated by some power of ϕ, and

(Pr2) the kernel of $\operatorname{Hom}(C, X) \xrightarrow{\phi\circ -} \operatorname{Hom}(C, X)$ is a finite length k-module.

Example 12.3.8. Let $A = k$ be a Dedekind domain and $\mathfrak{p}$ a maximal ideal. Then the Prüfer module

$$A_{\mathfrak{p}^\infty} = \bigcup_{n\geq 0} A/\mathfrak{p}^n = E(A/\mathfrak{p})$$

is a Prüfer object in $\operatorname{Mod} A$, because the canonical morphism $A/\mathfrak{p}^2 \twoheadrightarrow A/\mathfrak{p}$ extends to an epimorphism $\phi\colon A_{\mathfrak{p}^\infty} \twoheadrightarrow A_{\mathfrak{p}^\infty}$ with $\operatorname{Ker}\phi^n = A/\mathfrak{p}^n$.

Proposition 12.3.9. *A Prüfer object is Σ-pure-injective.*

Proof Let X be a Prüfer object with endomorphism $\phi\colon X \to X$. We show that $\operatorname{Hom}(C, X)$ is an artinian $\operatorname{End}(X)$-module for each $C \in \operatorname{fp}\mathcal{A}$. Then the subgroups of finite definition of $\operatorname{Hom}(C, X)$ satisfy the descending chain condition, and therefore the assertion follows from Theorem 12.3.4.

Consider the polynomial ring $k[t]$ in one variable and the homomorphism $k[t] \to \operatorname{End}(X)$ given by $t \mapsto \phi$. Fix $C \in \operatorname{fp}\mathcal{A}$ and set $C_n = \operatorname{Ker}\operatorname{Hom}(C, \phi^n)$ for $n \geq 0$. An induction shows C_n has finite length as a k-module, since it fits into an exact sequence $0 \to C_{n-1} \to C_n \to C_1$. Also, the socle of the $k[t]$-module $\operatorname{Hom}(C, X)$ has finite length because it is annihilated by t and therefore contained in C_1. It remains to apply the lemma below. □

Lemma 12.3.10. *Let A be a commutative noetherian ring. Then an A-module M is artinian if and only if M is a directed union of finite length submodules and* $\operatorname{soc} M$ *has finite length.*

Proof Suppose that M is artinian. If $M = \bigcup_i M_i$ is written as a directed union of finitely generated submodules, then each M_i has finite length. The module $\operatorname{soc} M$ is semisimple and artinian, and therefore has finite length.

For the other implication consider an injective envelope $\operatorname{soc} M \to E(\operatorname{soc} M)$ which extends to a morphism $\alpha\colon M \to E(\operatorname{soc} M)$. We claim that $\operatorname{Ker}\alpha = 0$. Otherwise $\operatorname{Ker}\alpha$ has a simple submodule, because it is a directed union of finite length submodules. This is impossible, and therefore α is a monomorphism.

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

It remains to observe that $E(\operatorname{soc} M)$ is artinian, since it is a finite direct sum of modules of the form $E(A/\mathfrak{p})$ for some maximal ideal $\mathfrak{p} \in \operatorname{Spec} A$, cf. Lemma 2.4.19. □

Suppose that $\mathcal{A}$ is abelian. Then a Prüfer object with endomorphism $\phi\colon X \to X$ is given by a sequence of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X_n & \xrightarrow{\phi_n} & X_{n-1} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & X_{n+1} & \xrightarrow{\phi_{n+1}} & X_n & \longrightarrow & 0 \end{array}$$

where $X_n = \operatorname{Ker} \phi^n$ and the vertical morphisms are the inclusions. In particular, $X = \bigcup_{n \geq 0} X_n$ and $\phi = \bigcup_{n \geq 0} \phi_n$.

Compactness

Recall that a topological space T is *quasi-compact* if for any family $(V_i)_{i \in I}$ of closed subsets $\bigcap_{i \in I} V_i = \varnothing$ implies $\bigcap_{i \in J} V_i = \varnothing$ for some finite subset $J \subseteq I$.

The correspondence in Theorem 12.2.2 provides an inclusion reversing isomorphism between the lattice of closed subsets of $\operatorname{Ind} \mathcal{A}$ and the lattice of Serre subcategories of $\operatorname{Ab}(\mathcal{A})$. This yields a criterion for when the space $\operatorname{Ind} \mathcal{A}$ is quasi-compact.

An abelian category $\mathcal{C}$ is *finitely generated* if there exists an object $X \in \mathcal{C}$ such that $\mathcal{C}$ equals the smallest Serre subcategory containing X.

Proposition 12.3.11. *Let $\mathcal{A}$ be a locally finitely presented category. The space* $\operatorname{Ind} \mathcal{A}$ *is quasi-compact if and only if the abelian category* $\operatorname{Ab}(\mathcal{A})$ *is finitely generated.*

Proof Combine the correspondence in Theorem 12.2.2 with Lemma 12.1.16, using the bijection $\operatorname{Ind} \mathcal{A} \xrightarrow{\sim} \operatorname{Sp} \mathbf{P}(\mathcal{A})$. □

Corollary 12.3.12. *Suppose there exists an object $G \in \operatorname{fp} \mathcal{A}$ such that every object in $\operatorname{fp} \mathcal{A}$ is a quotient of G^n for some integer $n \geq 1$. Then $\operatorname{Ind} \mathcal{A}$ is quasi-compact.*

Proof The abelian category $\operatorname{Ab}(\mathcal{A})$ is generated by $\operatorname{Hom}_{\mathcal{A}}(G, -)$, since each object in $\operatorname{Fp}(\operatorname{fp} \mathcal{A}, \operatorname{Ab})$ is a quotient of some representable functor $\operatorname{Hom}_{\mathcal{A}}(C, -)$ which embeds into $\operatorname{Hom}_{\mathcal{A}}(G^n, -)$ when $G^n \twoheadrightarrow C$. □

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Left Almost Split Morphisms

A morphism $\phi\colon X \to Y$ is *left almost split* if it is not a split monomorphism, and if every morphism $X \to X'$ which is not a split monomorphism factors through ϕ.

We fix a locally finitely presented abelian category $\mathcal{A}$. In the following we use freely the fact that the functor $\mathrm{ev}\colon \mathcal{A} \to \mathbf{P}(\mathcal{A})$ identifies the pure-injective objects in $\mathcal{A}$ with the injective objects in the purity category $\mathbf{P}(\mathcal{A})$; see Lemma 12.1.8.

Theorem 12.3.13. *For an indecomposable pure-injective object $X \in \mathcal{A}$ the following are equivalent.*

(1) *X is the source of a left almost split morphism in $\mathcal{A}$.*
(2) *$\bar{X}$ is an injective envelope of a simple object in $\mathbf{P}(\mathcal{A})$.*
(3) *If X is isomorphic to a direct summand of a product $\prod_{i\in I} Y_i$ of indecomposable objects in $\mathcal{A}$, then $X \cong Y_i$ for some $i \in I$.*

Proof (1) $\Rightarrow$ (2): Let $\phi\colon X \to Y$ be left almost split. Choose a finitely generated subobject $0 \neq C \subseteq \operatorname{Ker} \bar{\phi}$, a maximal subobject $U \subseteq C$, and an injective envelope $C/U \to \bar{X}'$. Then the induced morphism $C \to \bar{X}'$ factors through $C \hookrightarrow \bar{X}$ via a morphism $\alpha\colon X \to X'$. We claim that α is a split monomorphism. Otherwise it factors through ϕ, which is impossible since $\bar{\phi}(C) = 0$. Thus α is an isomorphism, and $\bar{X}$ is an injective envelope of a simple object in $\mathbf{P}(\mathcal{A})$.

(2) $\Rightarrow$ (1): Let $S \hookrightarrow \bar{X}$ be an injective envelope of a simple object S in $\mathbf{P}(\mathcal{A})$. We choose a left $\mathcal{A}$-approximation $\bar{X}/S \to \bar{Y}$ which yields a morphism $\phi\colon X \to Y$. This is possible by Proposition 11.1.27, because we identify $\mathcal{A}$ with the full subcategory of exact functors in $\mathbf{P}(\mathcal{A})$. We claim that ϕ is left almost split. Clearly, ϕ is not a split monomorphism since $\bar{\phi}$ is not a monomorphism. Let $\alpha\colon X \to X'$ be a morphism which is not a split monomorphism. Thus $\bar{\alpha}$ is not a monomorphism, and therefore $\bar{\alpha}(S) = 0$. Thus $\bar{\alpha}$ factors through $\bar{X} \to \bar{X}/S$, and therefore through the approximation $\bar{X}/S \to \bar{Y}$ via a morphism $\beta\colon Y \to X'$. Thus $\alpha = \beta\phi$.

(2) $\Rightarrow$ (3): Let $S \hookrightarrow \bar{X}$ be an injective envelope of a simple object S in $\mathbf{P}(\mathcal{A})$. If X is isomorphic to a direct summand of a product $\prod_{i\in I} Y_i$ of indecomposable objects in $\mathcal{A}$, then $\operatorname{Hom}(S, \bar{Y}_i) \neq 0$ for some $i \in I$. This yields a monomorphism $S \to \bar{X} \to \prod_{i\in I} \bar{Y}_i \to \bar{Y}_i$, and therefore $X \to Y_i$ is a pure monomorphism, which splits since X is pure-injective. Thus $X \cong Y_i$ since Y_i is indecomposable.

(3) $\Rightarrow$ (2): Let $\mathcal{U} = \operatorname{Ind} \mathcal{A} \setminus \{X\}$ and consider the canonical morphisms

$$\phi\colon X \longrightarrow \prod_{Y\in\mathcal{U}} Y^{\operatorname{Hom}(X,Y)}.$$

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

The condition (3) implies $\operatorname{Ker}\bar{\phi} \neq 0$. Choose a finitely generated non-zero subobject $U \subseteq \operatorname{Ker}\bar{\phi}$ and a maximal subobject $V \subseteq U$ in $\mathbf{P}(\mathcal{A})$. Set $S = U/V$. The composite $U \twoheadrightarrow S \rightarrowtail E(S)$ extends to a morphism $\bar{X} \to E(S)$ which does not factor through $\bar{\phi}$. Thus $\bar{X} \cong E(S)$ by the construction of ϕ. □

Corollary 12.3.14. *The subset $\mathcal{U} \subseteq \operatorname{Ind}\mathcal{A}$ of indecomposables which are the source of a left almost split morphism is dense.*

Proof Set $U = \prod_{X \in \mathcal{U}} X$. This is an injective cogenerator of $\mathbf{P}(\mathcal{A})$ by the above theorem, and therefore ${}^{\perp}\mathcal{U} = \{0\}$. Thus $\mathcal{U}$ is dense by Theorem 12.2.2. □

Call a point $X \in \operatorname{Ind}\mathcal{A}$ *isolated* if the set $\{X\}$ is open.

Corollary 12.3.15. *If $X \in \operatorname{Ind}\mathcal{A}$ is isolated, then X is the source of a left almost split morphism. The converse holds when X is finitely presented.*

Proof Set $\mathcal{U} = \operatorname{Ind}\mathcal{A} \setminus \{X\}$. Then ${}^{\perp}\mathcal{U} \subseteq \operatorname{Ab}(\mathcal{A})$ is a proper Serre subcategory by Theorem 12.2.2 when X is isolated. Choose $C \in \operatorname{Ab}(\mathcal{A}) \setminus {}^{\perp}\mathcal{U}$ and a maximal subobject $U \subseteq C$. Then $\bar{X}$ is an injective envelope of C/U, since $\operatorname{Hom}_{\mathbf{P}(\mathcal{A})}(C, \bar{Y}) = 0$ for all $Y \in \mathcal{U}$.

Now suppose that $S \hookrightarrow \bar{X}$ is an injective envelope of a simple object S in $\mathbf{P}(\mathcal{A})$. If X is finitely presented, then S is finitely presented in $\mathbf{P}(\mathcal{A})$. Thus $\{S\}^{\perp} = \mathcal{U}$ is closed. □

Fp-Injective Objects

Let $\mathcal{A}$ be a locally finitely presented abelian category and set $\mathcal{C} = \operatorname{fp}\mathcal{A}$. An object $X \in \mathcal{A}$ satisfying $\operatorname{Ext}^1_{\mathcal{A}}(-, X)|_{\operatorname{fp}\mathcal{A}} = 0$ is called *fp-injective*.

Lemma 12.3.16. *An fp-injective object is injective if and only if it is pure-injective.*

Proof If X is fp-injective then any exact sequence $0 \to X \to Y \to Z \to 0$ is pure-exact. Thus X is injective if and only if X is pure-injective. □

Now suppose that $\mathcal{C} = \operatorname{fp}\mathcal{A}$ is an abelian category. We write $\operatorname{Eff}(\mathcal{C}, \operatorname{Ab})$ for the Serre subcategory of $\operatorname{Fp}(\mathcal{C}, \operatorname{Ab})$ given by all functors F with presentation

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{C}}(C, -) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(B, -) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(A, -) \longrightarrow F \longrightarrow 0$$

coming from an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$.

Proposition 12.3.17. *We have*

$$\operatorname{Eff}(\mathcal{C}, \operatorname{Ab})^{\perp} = \{X \in \mathcal{A} \mid X \textit{ is fp-injective}\}.$$

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Therefore the fp-injective objects form a definable subcategory and

$$\mathrm{Eff}(\mathcal{C},\mathrm{Ab})^{\perp}\cap\mathrm{Ind}\,\mathcal{A}=\mathrm{Inj}\,\mathcal{A}\cap\mathrm{Ind}\,\mathcal{A}.$$

Proof Identifying $\mathcal{A}=\mathrm{Lex}(\mathcal{C}^{\mathrm{op}},\mathrm{Ab})$, the first assertion follows as a reformulation of Lemma 11.1.26. The subcategory $\mathrm{Eff}(\mathcal{C},\mathrm{Ab})^{\perp}$ is definable by definition, and the last equality then follows from the fact that pure-injectivity and injectivity coincide for fp-injective objects, by Lemma 12.3.16. □

Corollary 12.3.18. *A locally finitely presented abelian category is locally noetherian if and only if the injective objects form a definable subcategory.*

Proof When $\mathcal{A}$ is locally noetherian then every fp-injective object is injective; this follows from Baer's criterion. Thus the injective objects form a definable subcategory. Conversely, if the injectives form a definable subcategory, then they are closed under coproducts and therefore $\mathcal{A}$ is locally noetherian, by Theorem 11.2.12. □

12.4 Pure-Injective Modules

Let Λ be a ring. We consider the category of Λ-modules and set $\mathcal{A}=\mathrm{Mod}\,\Lambda$. Note that $\mathcal{A}$ is locally finitely presented with $\mathrm{fp}\,\mathcal{A}=\mathrm{mod}\,\Lambda$. In this section we give an explicit description of the embedding $\mathcal{A}\to\mathbf{P}(\mathcal{A})$ into the purity category, and we identify $\mathrm{Ab}(\mathcal{A})$ with the free abelian category over Λ. Also, we consider the set of indecomposable pure-injectives $\mathrm{Ind}\,\Lambda:=\mathrm{Ind}(\mathrm{Mod}\,\Lambda)$, which is called the *Ziegler spectrum* of Λ.

The Free Abelian Category

The *free abelian category* over Λ is by definition

$$\mathrm{Ab}(\Lambda):=\mathrm{Fp}(\mathrm{mod}\,\Lambda,\mathrm{Ab})^{\mathrm{op}}=\big(\mathrm{mod}\,\big((\mathrm{mod}\,\Lambda)^{\mathrm{op}}\big)\big)^{\mathrm{op}}.$$

We identify Λ with the representable functor $\mathrm{Hom}_{\Lambda}(\Lambda,-)$ in $\mathrm{Ab}(\Lambda)$. The following universal property of $\mathrm{Ab}(\Lambda)$ justifies its name.

Proposition 12.4.1. *For a ring Λ the category $\mathrm{Ab}(\Lambda)$ is abelian. Given an object X in an abelian category $\mathcal{A}$ and a ring homomorphism $\phi\colon\Lambda\to\mathrm{End}_{\mathcal{A}}(X)$, there exists a unique (up to isomorphism) exact functor $\mathrm{Ab}(\Lambda)\to\mathcal{A}$ sending Λ to X and inducing the homomorphism ϕ.*

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Proof The category $\mathrm{Ab}(\Lambda)$ is abelian by Lemma 2.1.6 since $\operatorname{mod}\Lambda$ has cokernels. A homomorphism $\phi\colon \Lambda \to \operatorname{End}_{\mathcal{A}}(X)$ extends uniquely to an additive functor $\phi_0\colon \operatorname{proj}\Lambda \to \mathcal{A}$, and therefore uniquely to a right exact functor $\phi_1\colon \operatorname{mod}\Lambda \to \mathcal{A}$, by Lemma 2.1.7. Then ϕ_1 extends uniquely to a left exact functor $\phi_2\colon \mathrm{Ab}(\Lambda) \to \mathcal{A}$, again by Lemma 2.1.7. The functor ϕ_2 is exact by Lemma 2.1.8. Clearly, ϕ_2 agrees on Λ with ϕ and is uniquely determined, up to isomorphism. □

The universal property of $\mathrm{Ab}(\Lambda)$ yields an equivalence

$$\mathrm{Ab}(\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Ab}(\Lambda^{\mathrm{op}})$$

extending the identity $\Lambda^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$. We give an explicit description. To this end define for $F \in \mathrm{Ab}(\Lambda^{\mathrm{op}})$

$$F^\vee\colon \operatorname{mod}\Lambda \longrightarrow \mathrm{Ab}, \qquad X \mapsto \operatorname{Hom}(F, X \otimes_\Lambda -).$$

Then we have for $X \in \operatorname{mod}\Lambda$ and $Y \in \operatorname{mod}(\Lambda^{\mathrm{op}})$

$$(X \otimes_\Lambda -)^\vee = \operatorname{Hom}_\Lambda(X,-) \qquad \text{and} \qquad (\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(Y,-))^\vee = - \otimes_\Lambda Y.$$

Lemma 12.4.2. *The assignment $F \mapsto F^\vee$ yields mutually inverse equivalences between $\mathrm{Ab}(\Lambda)^{\mathrm{op}}$ and $\mathrm{Ab}(\Lambda^{\mathrm{op}})$.*

Proof Given $F \in \mathrm{Ab}(\Lambda)$ and $G \in \mathrm{Ab}(\Lambda^{\mathrm{op}})$, we have

$$\operatorname{Hom}(F, G^\vee) \cong \operatorname{Hom}(G, F^\vee).$$

This is clear for $F = \operatorname{Hom}_\Lambda(X,-)$ and follows for arbitrary F by exactness, since a presentation

$$\operatorname{Hom}_\Lambda(X_1,-) \longrightarrow \operatorname{Hom}_\Lambda(X_0,-) \longrightarrow F \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow F^\vee \longrightarrow X_0 \otimes_\Lambda - \longrightarrow X_1 \otimes_\Lambda -$$

in $\mathrm{Ab}(\Lambda^{\mathrm{op}})$. Thus $F^{\vee\vee} \cong F$ since this holds for all representable functors. □

For $\mathcal{A} = \operatorname{Mod}\Lambda$ we have by definition $\mathrm{Ab}(\mathcal{A}) = \mathrm{Ab}(\Lambda)$. The following gives an explicit description of the purity category $\mathbf{P}(\mathcal{A})$.

Proposition 12.4.3. *The assignment $F \mapsto \operatorname{Hom}((-)^\vee, F)$ induces an equivalence*

$$\operatorname{Add}(\operatorname{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab}) \xrightarrow{\sim} \operatorname{Lex}(\operatorname{Fp}(\operatorname{mod}\Lambda, \mathrm{Ab}), \mathrm{Ab}) = \mathbf{P}(\mathcal{A}). \qquad (12.4.4)$$

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Proof Both categories are locally finitely presented. We have

$$\mathrm{fp}\,\mathrm{Add}(\mathrm{mod}(\Lambda^{\mathrm{op}}),\mathrm{Ab}) = \mathrm{Fp}(\mathrm{mod}(\Lambda^{\mathrm{op}}),\mathrm{Ab}) = \mathrm{Ab}(\Lambda^{\mathrm{op}})^{\mathrm{op}}$$

and

$$\mathrm{fp}\,\mathbf{P}(\mathcal{A}) = \mathrm{Fp}(\mathrm{mod}\,\Lambda,\mathrm{Ab})^{\mathrm{op}} = \mathrm{Ab}(\Lambda);$$

see Proposition 11.1.9 and Theorem 11.1.15. Thus the assertion follows from the second part of Theorem 11.1.15 and the equivalence $\mathrm{Ab}(\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Ab}(\Lambda^{\mathrm{op}})$. □

Corollary 12.4.5. *The embedding* $\mathrm{ev}\colon \mathcal{A} \to \mathbf{P}(\mathcal{A})$ *identifies with the functor*

$$\mathcal{A} \longrightarrow \mathrm{Add}(\mathrm{mod}(\Lambda^{\mathrm{op}}),\mathrm{Ab}), \qquad X \mapsto X \otimes_\Lambda -$$

via the equivalence (12.4.4). *In particular,* Λ*-modules identify with exact functors* $\mathrm{Ab}(\Lambda)^{\mathrm{op}} \to \mathrm{Ab}$*, by sending a* Λ*-module* X *to the functor*

$$\mathrm{Ab}(\Lambda) \ni F \longmapsto \mathrm{Hom}(F^\vee, X \otimes_\Lambda -).$$

Proof We need to check that there is a natural isomorphism

$$\mathrm{ev}(X) = \bar{X} \cong \mathrm{Hom}((-)^\vee, X \otimes_\Lambda -)$$

for every Λ-module X. For $F = \mathrm{Hom}_\Lambda(C,-)$ we have

$$\bar{X}(F) = \mathrm{Hom}_\Lambda(C,X) \cong \mathrm{Hom}(C \otimes_\Lambda -, X \otimes_\Lambda -) \cong \mathrm{Hom}(F^\vee, X \otimes_\Lambda -).$$

The functors $\bar{X}$ and $\mathrm{Hom}((-)^\vee, X \otimes_\Lambda -)$ are both exact; so we have the isomorphism for all $F \in \mathrm{Ab}(\mathcal{A})$.

The second assertion is an immediate consequence. □

Corollary 12.4.6. *A sequence* $0 \to X \to Y \to Z \to 0$ *of* Λ*-modules is pure-exact if and only if the induced sequence*

$$0 \longrightarrow X \otimes_\Lambda C \longrightarrow Y \otimes_\Lambda C \longrightarrow Z \otimes_\Lambda C \longrightarrow 0$$

is exact for every Λ^{op}*-module* C.

Proof Combine Lemma 12.1.6 and Corollary 12.4.5. □

A Criterion for Pure-Injectivity

Let k be a commutative ring and Λ a k-algebra. We fix a minimal injective cogenerator E over k and set $D(X) := \mathrm{Hom}_k(X,E)$ for every k-module X. This induces Matlis duality between right and left Λ-modules.

A Λ-module Q is *pure-injective* if Q is a pure-injective object in $\mathrm{Mod}\,\Lambda$, so

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

every pure monomorphism $X \to Y$ induces a surjective map $\operatorname{Hom}_\Lambda(Y, Q) \to \operatorname{Hom}_\Lambda(X, Q)$.

Proposition 12.4.7. *For a Λ-module X the following are equivalent.*

(1) *The module X is pure-injective.*
(2) *The natural morphism $\phi_X \colon X \to D^2(X)$ given by $\phi_X(x)(\alpha) = \alpha(x)$ for $x \in X$ and $\alpha \in D(X)$ is a split monomorphism.*
(3) *There is a bimodule ${}_\Lambda Y_\Gamma$ for some ring Γ and an injective Γ-module I such that X is isomorphic to a direct summand of* $\operatorname{Hom}_\Gamma(Y, I)$.

Proof (1) $\Rightarrow$ (2): The composite $D(X) \xrightarrow{\phi_{D(X)}} D^3(X) \xrightarrow{D(\phi_X)} D(X)$ is the identity. Thus $D(\phi_X)$ is a split epimorphism, and therefore

$$D(\phi_X \otimes_\Lambda C) \cong \operatorname{Hom}_\Lambda(C, D(\phi_X))$$

is an epimorphism for every left Λ-module C. It follows that $\phi_X \otimes_\Lambda C$ is a monomorphism, and therefore ϕ_X is a pure monomorphism by Corollary 12.4.6. We conclude that ϕ_X splits when X is pure-injective.

(2) $\Rightarrow$ (3): Take ${}_\Lambda Y_\Gamma = {}_\Lambda D(X)_k$. Then X is isomorphic to a direct summand of $\operatorname{Hom}_k(Y, E) = D^2(X)$.

(3) $\Rightarrow$ (1): The functor

$$\operatorname{Hom}_\Lambda(-, \operatorname{Hom}_\Gamma(Y, I)) \cong \operatorname{Hom}_\Gamma(- \otimes_\Lambda Y, I)$$

sends pure-exact sequences to exact sequences, by the description of pure-exact sequences in Corollary 12.4.6. Thus $\operatorname{Hom}_\Gamma(Y, I)$ is pure-injective. □

Corollary 12.4.8. *Every Λ-module X admits a pure monomorphism into a pure-injective module of the form $X \to \prod_{i \in I} D(Y_i)$ that is given by a family of finitely presented Λ^{op}-modules $(X_i)_{i \in I}$.*

Proof Choose an epimorphism $\coprod_{i \in I} Y_i \to D(X)$ and take the composite $X \to D^2(X) \to \prod_{i \in I} D(Y_i)$. □

Example 12.4.9. Let ${}_\Gamma X_\Lambda$ be a bimodule and suppose that X is artinian over Γ. Then X is a Σ-pure-injective Λ-module, because the descending chain condition for subgroups of finite definition is satisfied; see Theorem 12.3.4.

Duality

There is no global duality between right and left Λ-modules, but there is a bijective correspondence between specific classes of modules. This is based on the equivalence $\operatorname{Ab}(\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{Ab}(\Lambda^{\mathrm{op}})$ given by $F \mapsto F^\vee$, which induces a

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

bijection between Serre subcategories. Applying the bijective correspondence between Serre subcategories of $\mathrm{Ab}(\Lambda)$ and definable subcategories of $\operatorname{Mod}\Lambda$ from Theorem 12.2.2, we obtain for definable subcategories a bijection

$$\operatorname{Mod}\Lambda \supseteq \mathcal{X} \longmapsto (({}^{\perp}\mathcal{X})^{\vee})^{\perp} \subseteq \operatorname{Mod}\Lambda^{\mathrm{op}}.$$

For a Λ-module X we consider the Serre subcategory

$${}^{\perp}X := \{F \in \mathrm{Ab}(\Lambda) \mid \mathrm{Hom}(F^{\vee}, X \otimes_{\Lambda} -) = 0\}$$

and observe that $({}^{\perp}X)^{\perp} \subseteq \operatorname{Mod}\Lambda$ is the smallest definable subcategory containing X. We call a pair of Λ-modules $(X_{\Lambda}, {}_{\Lambda}Y)$ a *dual pair* if the following equivalent conditions are satisfied:

$$({}^{\perp}X)^{\vee} = {}^{\perp}Y \quad\iff\quad {}^{\perp}X = ({}^{\perp}Y)^{\vee}.$$

Proposition 12.4.10. *Let ${}_{\Gamma}X_{\Lambda}$ be a Γ-Λ-bimodule and fix an injective cogenerator $I \in \operatorname{Mod}\Gamma$. Then $(X, \mathrm{Hom}_{\Gamma}(X, I))$ is a dual pair of Λ-modules.*

Proof Choose $F \in \mathrm{Ab}(\Lambda)$ with presentation

$$\mathrm{Hom}_{\Lambda}(C', -) \longrightarrow \mathrm{Hom}_{\Lambda}(C, -) \longrightarrow F \longrightarrow 0$$

given by a morphism in $\phi\colon C \to C'$ in $\operatorname{mod}\Lambda$. Then $F^{\vee} \in \mathrm{Ab}(\Lambda^{\mathrm{op}})$ has the presentation

$$0 \longrightarrow F^{\vee} \longrightarrow C \otimes_{\Lambda} - \longrightarrow C' \otimes_{\Lambda} -$$

and we have

$$\begin{aligned}
F \in {}^{\perp}X &\iff \mathrm{Hom}_{\Lambda}(\phi, X) \text{ is an epimorphism}\\
&\iff \mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\Lambda}(\phi, X), I) \text{ is a monomorphism}\\
&\iff \phi \otimes_{\Lambda} \mathrm{Hom}_{\Gamma}(X, I) \text{ is a monomorphism}\\
&\iff F^{\vee} \in {}^{\perp}\mathrm{Hom}_{\Gamma}(X, I)\\
&\iff F \in ({}^{\perp}\mathrm{Hom}_{\Gamma}(X, I))^{\vee}.
\end{aligned}$$

□

Example 12.4.11. Let Λ be a k-algebra over a commutative ring k. Then a Λ-module X together with its Matlis dual $D(X)$ form a dual pair $(X, D(X))$.

Pure-Semisimplicity

A ring Λ is called *right pure-semisimple* when every pure-exact sequence of Λ-modules is split exact.

Proposition 12.4.12. *For a ring Λ the following are equivalent.*

(1) *The ring Λ is right pure-semisimple.*

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

(2) *Every Λ-module is pure-injective.*

(3) *Every Λ-module decomposes into a coproduct of indecomposable modules with local endomorphism rings.*

(4) *Every object in* $\mathrm{Ab}(\Lambda)$ *is noetherian.*

Proof Apply Theorem 12.3.3. □

Modules of Finite Projective Dimension

Let Λ be a ring. We consider for fixed $n \in \mathbb{N}$ the full subcategory of Λ-modules X such that the projective dimension $\operatorname{proj.dim} X$ is bounded by n.

Proposition 12.4.13. *Let Λ be a ring that is right perfect and left coherent. Then for each $n \in \mathbb{N}$ the Λ-modules of projective dimension at most n form a definable subcategory of* $\operatorname{Mod}\Lambda$.

Recall that Λ is *right perfect* if every flat module is projective. The ring Λ is *right coherent* if the category of finitely presented Λ-modules is abelian. For example, every right artinian ring is right perfect and right coherent.

Proof Because Λ is right perfect, we have $\operatorname{proj.dim} X \le n$ if and only if $\operatorname{Tor}^{\Lambda}_{n+1}(X,-) = 0$. We can test the vanishing of $\operatorname{Tor}^{\Lambda}_{n+1}(X,-)$ on finitely presented left modules, since $\operatorname{Tor}^{\Lambda}_{n+1}(X,-)$ preserves filtered colimits and every module is a filtered colimit of finitely presented modules. Because Λ is left coherent, a finitely presented left Λ-module Y admits a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Y \longrightarrow 0$$

such that each P_i is finitely generated. It follows that $\operatorname{Tor}^{\Lambda}_{n+1}(-,Y)$ preserves products, since each functor $-\otimes_{\Lambda} P_i$ preserves products and taking products of abelian groups is exact. In particular

$$\bigcap_{Y \in \operatorname{mod}(\Lambda^{\mathrm{op}})} \operatorname{Ker} \operatorname{Tor}^{\Lambda}_{n+1}(-,Y)$$

is a definable subcategory of $\operatorname{Mod}\Lambda$, which equals the subcategory of modules of projective dimension at most n. □

A consequence of Theorem 12.2.2 is then the fact that we may test on $\operatorname{Ind}\Lambda$ the *finitistic dimension* of Λ, that is, the supremum of all finite projective dimensions $\operatorname{proj.dim} X$.

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

The Ziegler Spectrum of an Artin Algebra

Let Λ be an Artin algebra over a commutative artinian ring k. We write $D = \operatorname{Hom}_k(-, E)$ for the Matlis duality over k given by a minimal injective cogenerator E.

We consider $\mathcal{A} = \operatorname{Mod}\Lambda$ and identify $\mathbf{P}(\mathcal{A}) = \operatorname{Add}(\operatorname{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$. Finitely presented and finite length modules over Λ coincide because Λ is artinian. We find some further finiteness conditions that are equivalent.

Proposition 12.4.14. *The assignment* $X \mapsto \operatorname{soc}(X \otimes_\Lambda -)$ *induces a bijection between*

- *the isomorphism classes of indecomposable finite length* Λ*-modules, and*
- *the isomorphism classes of simple objects in* $\operatorname{Add}(\operatorname{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$.

Proof Write $(\operatorname{mod}\Lambda, \operatorname{mod}k)$ for the category of k-linear functors $\operatorname{mod}\Lambda \to \operatorname{mod}k$ and observe that $F \mapsto D(F) := D \circ F$ induces an equivalence

$$(\operatorname{mod}\Lambda, \operatorname{mod}k)^{\mathrm{op}} \xrightarrow{\sim} ((\operatorname{mod}\Lambda)^{\mathrm{op}}, \operatorname{mod}k).$$

Next observe that every simple functor in $((\operatorname{mod}\Lambda)^{\mathrm{op}}, \operatorname{mod}k)$ is of the form

$$S_Y = \operatorname{Hom}_\Lambda(-, Y)/\operatorname{Rad}_\Lambda(-, Y)$$

for some indecomposable finitely presented Λ-module Y since $\operatorname{End}_\Lambda(Y)$ is local.

Let $X \in \operatorname{mod}\Lambda$ be indecomposable. The functor

$$D(X \otimes_\Lambda -) \cong \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(-, D(X))$$

has a unique simple quotient $S_{D(X)}$. Thus $X \otimes_\Lambda -$ has $D(S_{D(X)})$ as a unique simple subobject in $\mathbf{P}(\mathcal{A})$, and this implies

$$\operatorname{soc}(X \otimes_\Lambda -) \cong D(S_{D(X)}).$$

Let $S \in \mathbf{P}(\mathcal{A})$ be simple. Then $S = D(S_Y)$ for some indecomposable $Y \in \operatorname{mod}\Lambda^{\mathrm{op}}$. We have $D\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(-, Y) \cong D(Y) \otimes_\Lambda -$, and this implies

$$\operatorname{soc}(D(Y) \otimes_\Lambda -) \cong S.$$ □

Theorem 12.4.15. *For an indecomposable pure-injective* Λ*-module* X *the following are equivalent.*

(1) *X is finitely presented.*
(2) *X is the source of a left almost split morphism.*
(3) *X is isolated.*

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Proof We use the embedding $\mathcal{A} \to \mathbf{P}(\mathcal{A})$ which sends Y to $\bar{Y} = Y \otimes_\Lambda -$; see Corollary 12.4.5.

(1) ⇔ (2): The module X is finitely presented if and only if $\bar{X}$ is the injective envelope of a simple object in $\mathbf{P}(\mathcal{A})$, by Proposition 12.4.14, and this happens if and only if X is the source of a left almost split morphism, by Theorem 12.3.13.

(2) ⇔ (3): Apply Corollary 12.3.15, using also the first part of the proof. □

Corollary 12.4.16. *An Artin algebra of infinite representation type has an indecomposable pure-injective module of infinite length.*

Proof The space $\operatorname{Ind}\Lambda$ is quasi-compact by Corollary 12.3.12; so it cannot consist of infinitely many isolated points. □

The Zariski Spectrum

Let Λ be a commutative noetherian ring. We consider the *Zariski spectrum* $\operatorname{Spec}\Lambda$ consisting of all prime ideals, where a subset is *Zariski closed* if it is of the form

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}\Lambda \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

for some ideal $\mathfrak{a}$ of Λ. Recall that the assignment $\mathfrak{p} \mapsto E(\Lambda/\mathfrak{p})$ yields a bijection

$$\Phi\colon \operatorname{Spec}\Lambda \xrightarrow{\sim} \operatorname{Sp}(\operatorname{Mod}\Lambda)$$

onto the spectrum consisting of a representative set of the isomorphism classes of indecomposable injective Λ-modules (Corollary 2.4.15).

Let us compare via Φ the Zariski topology on $\operatorname{Spec}\Lambda$ with the topology on $\operatorname{Sp}(\operatorname{Mod}\Lambda)$ which is defined in Lemma 12.1.12.

Proposition 12.4.17. *For a subset $\mathcal{V} \subseteq \operatorname{Spec}\Lambda$ the following conditions are equivalent.*

(1) $\Phi(\mathcal{V})$ *is closed.*
(2) $\Phi(\mathcal{V})$ *is closed under products. If $X \subseteq \prod_{i\in I} Y_i$ for some indecomposable injective module X and a family of modules $Y_i \in \Phi(\mathcal{V})$, then $X \in \Phi(\mathcal{V})$.*
(3) $(\operatorname{Spec}\Lambda) \setminus \mathcal{V}$ *is specialisation closed.*
(4) $\mathcal{V} = \bigcap_{i\in I} \mathcal{U}_i$ *for a family of Zariski open subsets $\mathcal{U}_i \subseteq \operatorname{Spec}\Lambda$.*

Proof (1) ⇔ (2): This follows from Lemma 12.1.14.

(1) ⇔ (3): By definition, $\Phi(\mathcal{V})$ is closed if it is of the form $\mathcal{C}^\perp$ for some Serre subcategory $\mathcal{C} \subseteq \operatorname{mod}\Lambda$. Such a Serre subcategory corresponds to a specialisation closed subset of $\operatorname{Spec}\Lambda$ via $\mathcal{C} \mapsto \operatorname{Supp}\mathcal{C}$; see Proposition 2.4.8. Using the theory of associated primes, the map Φ identifies $(\operatorname{Spec}\Lambda) \setminus (\operatorname{Supp}\mathcal{C})$ with $\mathcal{C}^\perp$ by Corollary 2.4.16.

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

(3) $\Leftrightarrow$ (4): A subset of $\operatorname{Spec}\Lambda$ is specialisation closed if and only if it is the union of Zariski closed subsets. □

Corollary 12.4.18. *The assignment* $\mathfrak{p} \mapsto E(\Lambda/\mathfrak{p})$ *gives a map* $\Phi\colon \operatorname{Spec}\Lambda \to \operatorname{Ind}\Lambda$ *that identifies* $\operatorname{Spec}\Lambda$ *with a Ziegler closed subset of* $\operatorname{Ind}\Lambda$*. For a subset* $\mathcal{V} \subseteq \operatorname{Spec}\Lambda$ *the following conditions are equivalent.*

(1) $\Phi(\mathcal{V})$ *is Ziegler closed.*
(2) $\Phi(\mathcal{V})$ *is closed under products. If* $X \subseteq \prod_{i\in I} Y_i$ *for some indecomposable injective module* X *and a family of modules* $Y_i \in \Phi(\mathcal{V})$*, then* $X \in \Phi(\mathcal{V})$*.*
(3) $(\operatorname{Spec}\Lambda) \setminus \mathcal{V}$ *is specialisation closed.*
(4) $\mathcal{V} = \bigcap_{i\in I} \mathcal{U}_i$ *for a family of Zariski open subsets* $\mathcal{U}_i \subseteq \operatorname{Spec}\Lambda$*.*

Proof The injective Λ-modules form a definable subcategory of $\operatorname{Mod}\Lambda$ since Λ is noetherian, by Corollary 12.3.18. Clearly, every injective module is pure-injective, and it follows that Φ identifies $\operatorname{Spec}\Lambda$ with a Ziegler closed subset of $\operatorname{Ind}\Lambda$.

The second part of the assertion follows from Proposition 12.4.17 since the topology on $\operatorname{Sp}(\operatorname{Mod}\Lambda)$ is the restriction of the Ziegler topology on $\operatorname{Ind}\Lambda$. □

Remark 12.4.19. For a commutative noetherian ring Λ, Zariski and Ziegler topology on $\operatorname{Spec}\Lambda$ are related via *Hochster duality* as follows. The prime ideal spectrum of any commutative ring with its Zariski topology is a *spectral space*. For a spectral space there is a dual topology with closed subsets of the form $\bigcap_{i\in I} \mathcal{U}_i$ for any family of quasi-compact and open subsets $\mathcal{U}_i$ (the Ziegler closed subsets). The dual space is again spectral, and its Hochster dual topology coincides with the original Zariski topology of $\operatorname{Spec}\Lambda$.

Injective Cohomology Representations

Let G be a finite group and k a field. We consider modules over the group algebra kG and note that kG is a self-injective algebra. The *group cohomology*

$$R := H^*(G,k) := \operatorname{Ext}^*_{kG}(k,k)$$

is by definition the Ext-algebra of the trivial representation k; it is a graded commutative and noetherian k-algebra by a theorem of Golod, Venkov and Evens [29, Corollary 3.10.2]. We consider only graded modules over R. Let R^+ denote the unique maximal ideal consisting of positive degree elements and call an R-module *torsion* if each element is annihilated by some power of R^+. The torsion modules form a localising subcategory which is denoted by $\operatorname{Mod}_0 R$.

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

We extend the functor $H^*(G,-) = \mathrm{Ext}^*_{kG}(k,-)$ from kG-modules to the category $\mathbf{K}(\mathrm{Inj}\, kG)$ of complexes of injective kG-modules. Set

$$\mathrm{Hom}^*_{\mathbf{K}(kG)}(X,Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{K}(kG)}(X,\Sigma^n Y)$$

for each pair of complexes X, Y and

$$H^*(G,X) = \mathrm{Hom}^*_{\mathbf{K}(kG)}(ik,X)$$

where ik denotes an injective resolution of the trival representation. Note that $\mathrm{End}^*_{\mathbf{K}(kG)}(ik) \cong R$. The functor $H^*(G,-)$ is cohomological and preserves coproducts; it induces the following commutative diagram.

$$\begin{array}{ccccc} \mathrm{Loc}(kG) & \rightarrowtail & \mathbf{K}(\mathrm{Inj}\, kG) & \twoheadrightarrow & \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\, kG) \\ \downarrow & & \downarrow{\scriptstyle H^*(G,-)} & & \downarrow \\ \mathrm{Mod}_0\, R & \rightarrowtail & \mathrm{Mod}\, R & \twoheadrightarrow & \mathrm{Mod}\, R/\mathrm{Mod}_0\, R \end{array}$$

The upper row of the diagram is taken from Proposition 9.1.10, where $\mathrm{Loc}(kG)$ denotes the localising subcategory generated by kG, viewed as a complex concentrated in degree zero, and keeping in mind that kG is self-injective. Note that $H^*(G,X)$ is torsion for each $X \in \mathrm{Loc}(kG)$, since $H^*(G,kG)$ is torsion (in fact just k in degree zero). Thus the diagram does commute.

We wish to explain that the functor $H^*(G,-)$ admits a partial adjoint when we restrict to injective R-modules.

Given a pair of R-modules M, N we write

$$\mathrm{Hom}^*_R(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_R(M,N)$$

for the graded abelian group of R-linear morphisms $\phi\colon M \to N$ satisfying $\phi(M^i) \subseteq M^{i+n}$ for $\phi \in \mathrm{Hom}^n_R(M,N)$ and $i,n \in \mathbb{Z}$. The module M is called *torsion free* if $\mathrm{Hom}^*_R(-,M)$ vanishes on $\mathrm{Mod}_0\, R$. Recall that the category of injective R-modules is closed under coproducts since R is noetherian.

Also, we use the triangle equivalence $Z^0\colon \mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\, kG) \xrightarrow{\sim} \mathrm{StMod}\, kG$ and view it as an identification (Proposition 4.4.18). A quasi-inverse maps a kG-module X to a complete resolution tX.

Proposition 12.4.20. *There is a fully faithful functor $T\colon \mathrm{Inj}\, R \to \mathbf{K}(\mathrm{Inj}\, kG)$ with a natural isomorphism*

$$\mathrm{Hom}^*_R(H^*(G,-),I) \cong \mathrm{Hom}^*_{\mathbf{K}(kG)}(-,T(I))$$

for each $I \in \mathrm{Inj}\, R$. The functor preserves products and coproducts. Moreover,

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

T restricts to a fully faithful functor

$$\{I \in \operatorname{Inj} R \mid I \text{ torsion free}\} \longrightarrow \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} kG) \xrightarrow{\sim} \operatorname{StMod} kG$$

and $T(I)$ is a Σ-pure-injective kG-module for each torsion free $I \in \operatorname{Inj} R$.

Proof The triangulated category $\mathbf{K}(\operatorname{Inj} kG)$ is compactly generated (Proposition 9.3.12) and therefore every cohomological functor $\mathbf{K}(\operatorname{Inj} kG)^{\mathrm{op}} \to \mathrm{Ab}$ that preserves coproducts is representable, by Brown's representability theorem (Theorem 3.4.16). This yields for each $I \in \operatorname{Inj} R$ an object $T(I)$ that represents $\operatorname{Hom}^*_R(H^*(G,-), I)$. Given $J \in \operatorname{Inj} R$ we compute

$$\begin{aligned} \operatorname{Hom}^*_{\mathbf{K}(kG)}(T(I), T(J)) &\cong \operatorname{Hom}^*_R(H^*(G, T(I)), J) \\ &\cong \operatorname{Hom}^*_R(\operatorname{Hom}^*_{\mathbf{K}(kG)}(ik, T(I)), J) \\ &\cong \operatorname{Hom}^*_R(\operatorname{Hom}^*_R(H^*(G, ik), I), J) \\ &\cong \operatorname{Hom}^*_R(\operatorname{Hom}^*_R(R, I), J) \\ &\cong \operatorname{Hom}^*_R(I, J). \end{aligned}$$

Thus T is fully faithful. Clearly, T preserves products. To show that it preserves coproducts consider for any family (I_α) of injective R-modules the canonical morphism $\phi\colon \coprod T(I_\alpha) \to T(\coprod I_\alpha)$. Apply $\operatorname{Hom}^*_{\mathbf{K}(kG)}(iX, -)$ where iX is the injective resolution of a finitely generated kG-module X. Then $\operatorname{Hom}^*_{\mathbf{K}(kG)}(iX, \phi)$ is an isomorphism since iX is compact, and it follows that ϕ is an isomorphism since the objects of the form iX generate $\mathbf{K}(\operatorname{Inj} kG)$ (Proposition 9.3.12). If $I \in \operatorname{Inj} R$ is torsion free, then

$$H^*T(I) \cong \operatorname{Hom}^*_{\mathbf{K}(kG)}(kG, T(I)) \cong \operatorname{Hom}^*_R(H^*(G, kG), I) = 0,$$

and therefore $T(I)$ is acyclic.

Next we use the identification $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} kG) \xrightarrow{\sim} \operatorname{StMod} kG$ and show that $T(I)$ is a Σ-pure-injective kG-module for each torsion free $I \in \operatorname{Inj} R$. We apply the criterion of Theorem 12.3.4 and show that the canonical monomorphism $T(I)^{(\mathbb{N})} \to T(I)^{\mathbb{N}}$ splits in $\operatorname{Mod} kG$. Clearly, the canonical monomorphism $I^{(\mathbb{N})} \to I^{\mathbb{N}}$ splits since R is noetherian. The functor T preserves products and coproducts. Thus $T(I)^{(\mathbb{N})} \to T(I)^{\mathbb{N}}$ splits in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} kG)$ and therefore also in $\operatorname{StMod} kG$. It remains to apply the lemma below. □

Lemma 12.4.21. *Let $\mathcal{A}$ be a Frobenius category. Then a monomorphism in $\mathcal{A}$ splits if and only if it splits in* $\operatorname{St}\mathcal{A}$. □

Let $\mathfrak{p}$ be a non-maximal prime ideal in R and $n \in \mathbb{Z}$. Then $I_\mathfrak{p} = E(R/\mathfrak{p})$ and its twist $I_\mathfrak{p}(n)$ are indecomposable injective R-modules. We may assume

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

that the corresponding kG-module $T(I_{\mathfrak{p}}(n))$ is indecomposable, by removing all non-zero injective summands.

For a kG-module X one defines its *Tate cohomology*

$$\hat{H}^n(G,X) := \widehat{\operatorname{Ext}}^n_{kG}(k,X) := H^n \operatorname{Hom}_{kG}(k,tX) \qquad (n \in \mathbb{Z})$$

and more generally

$$\widehat{\operatorname{Ext}}^n_{kG}(-,X) := H^n \operatorname{Hom}_{kG}(-,tX) \qquad (n \in \mathbb{Z})$$

where tX denotes a complete resolution of X (cf. Lemma 4.4.19). Note that $\hat{H}^*(G,X)$ is naturally an R-module via restriction along

$$\operatorname{Ext}^*_{kG}(k,k) \longrightarrow \widehat{\operatorname{Ext}}^*_{kG}(k,k).$$

Corollary 12.4.22. *Let $I \in \operatorname{Inj} R$ be torsion free. The kG-module $T(I)$ satisfies $\hat{H}^*(G,T(I)) \cong I$ and is uniquely determined (up to isomorphism in* StMod kG*) by the isomorphism*

$$\operatorname{Hom}^*_R(\hat{H}^*(G,-),I) \cong \widehat{\operatorname{Ext}}^*_{kG}(-,T(I)).$$

Moreover, after removing all non-zero injective summands, $T(I)$ admits a unique decomposition into indecomposable modules of the form $T(I_{\mathfrak{p}}(n))$, with $\mathfrak{p}$ a prime ideal in R and $n \in \mathbb{Z}$.

Proof The first part follows from the defining isomorphism for $T(I)$. More precisely, taking a complete resolution tk of k we have

$$I \cong \operatorname{Hom}^*_R(H^*(G,tk),I) \cong \operatorname{Hom}^*_{\mathbf{K}(kG)}(tk,T(I)) \cong \hat{H}^*(G,T(I)),$$

where the first isomorphism is induced by $R = H^*(G,ik) \to H^*(G,tk)$ since $H^*(G,pk)$ is torsion, the second isomorphism defines $T(I)$, and the third isomorphism is from Lemma 4.4.19. Similarly, we have for $X \in \operatorname{StMod} kG$

$$\begin{aligned}\operatorname{Hom}^*_R(\hat{H}^*(G,X),I) &\cong \operatorname{Hom}^*_R(H^*(G,tX),I)\\ &\cong \operatorname{Hom}^*_{\mathbf{K}(kG)}(tX,T(I))\\ &\cong \widehat{\operatorname{Ext}}^*_{kG}(X,T(I)),\end{aligned}$$

and the uniqueness of $T(I)$ then follows from Yoneda's lemma.

The module $T(I)$ is Σ-pure-injective and therefore decomposes uniquely into indecomposables, by Theorem 12.3.4. Then one uses the description of the indecomposable injective R-modules via Spec R (Corollary 2.4.15). □

Let us denote by Proj R the set of all homogeneous prime ideals of R except the maximal ideal consisting of positive degree elements.

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

Corollary 12.4.23. *Taking a prime ideal* $\mathfrak{p}$ *to* $T(I_{\mathfrak{p}})$ *yields an injective map*

$$\operatorname{Proj} H^*(G,k) \longrightarrow \operatorname{Ind} kG. \qquad \square$$

Corollary 12.4.24. *The modules of the form* $T(I)$ *(*$I \in \operatorname{Inj} H^*(G,k)$ *torsion free) form a definable subcategory of* $\operatorname{Mod} kG$.

Proof The torsion free injective $H^*(G,k)$-modules form a definable subcategory, because they are closed under products, coproducts, and direct summands, keeping in mind that $H^*(G,k)$ is noetherian (Corollary 12.3.18). Then this category equals $\operatorname{Add} I_0 = \operatorname{Prod} I_0$ for some product-complete module I_0. It follows that $T(I_0)$ is a product-complete kG-module, since T preserves products and coproducts. Thus the image of the functor T is definable, by Proposition 12.3.7. □

Notes

The notion of a pure subgroup (Servanzuntergruppe) of an abelian group was introduced by Prüfer [163]. For modules over arbitrary rings the concept of purity is due to Cohn [53]. Pure-injective modules are also known as algebraically compact modules [119, 200]. It was shown by Kiełpiński [124] and independently by Stenström [196] that every module admits a pure-injective envelope. For the characterisation of Σ-pure-injective modules, see Gruson and Jensen [99], Zimmermann [204], and Zimmermann-Huisgen [205], building on work of Chase [49, 50], but also Garavaglia [86] in a model theoretic setting. The space of indecomposable pure-injective modules is known as the Ziegler spectrum because it was introduced by Ziegler in his work on the model theory of modules [203]. For an Artin algebra of infinite representation type the existence of a large indecomposable module was established by Auslander [10]. Product-complete modules were introduced in joint work with Saorín [134].

The study of pure-exactness via the embedding of a module category into a bigger Grothendieck category (the purity category) goes back to work of Gruson and Jensen [100]; see also Simson [192]. The systematic treatment of purity for locally finitely presented categories is due to Crawley-Boevey [58]. Definable subcategories were introduced by Crawley-Boevey [59] for module categories, and more generally for locally finitely presented categories in [126]. The related notion of an elementary subcategory and its connection with Ziegler closed subsets appear already in [203]. For the correspondence between definable subcategories and Ziegler closed subsets, see Herzog [111] and [125].

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press

The free abelian category on a category was introduced by Freyd [75]. The characterisation of pure-injective modules via duality is taken from Auslander's work [11], where it arises from the description of morphisms determined by objects. Almost split morphisms were introduced by Auslander and Reiten [15]; for the connection with simple functors, see [11]. The characterisation of indecomposable pure-injectives which are the source of a left almost split morphism combines results from [10] and [57].

Any ring of finite representation type is known to be right and left pure-semisimple, by a result of Ringel and Tachikawa [175]. In fact, pure-semisimple rings were introduced by Simson, and no ring is known which is right pure-semisimple but not of finite representation type [191, 193].

For commutative rings the connection between the Zariski spectrum and the Ziegler spectrum was clarified by Prest [160] in terms of Hochster's duality for spectral spaces [113].

The construction of pure-injective modules for group algebras via group cohomology is taken from work with Benson [32]. For example these modules play a role in the study of local duality for representations of finite groups, and more generally of finite group schemes [31].

For more material and further references about infinite length modules and purity, see [133, 162].

https://doi.org/10.1017/9781108979108.020 Published online by Cambridge University Press