13

Endofiniteness

Contents

In this chapter we study a particular finiteness condition for objects in a locally finitely presented category. An object X is called endofinite if the morphisms from any finitely presented object form a finite length module over $\operatorname{End}(X)$. For example, a ring is of finite representation type if and only if all its modules are endofinite. We present a remarkable classification of endofinite objects in purely numerical terms, using subadditive functions on finitely presented objects. A basic idea is to identify an object X with an exact functor $\bar{X}\colon \mathcal{C} \to \mathrm{Ab}$ for an appropriate abelian category $\mathcal{C}$. Then X is endofinite if and only if the quotient $\mathcal{C}/\operatorname{Ker}\bar{X}$ is a length category. Thus the study of endofinite objects is equivalent to the study of exact functors from length categories to abelian groups. Of particular interest are indecomposable endofinite objects that are not finitely presented; often they represent families of finitely presented objects.

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We illustrate this by looking at endofinite modules over Artin algebras. Other interesting examples arise from locally finitely presented categories such that the finitely presented objects form a uniserial category.

13.1 Endofinite Objects and Subadditive Functions

Let $\mathcal{A}$ be a locally finitely presented category and let $\operatorname{fp}\mathcal{A}$ be the full subcategory of finitely presented objects. In this section we study the notion of endofiniteness. An object $X \in \mathcal{A}$ is called *endofinite* if

$$\ell_{\operatorname{End}_{\mathcal{A}}(X)}(\operatorname{Hom}_{\mathcal{A}}(C,X)) < \infty \quad \text{for all} \quad C \in \operatorname{fp}\mathcal{A}.$$

We begin with a discussion of subadditive functions which are defined on additive categories with cokernels.

Subadditive Functions

Let $\mathcal{C}$ be an additive category with cokernels. A *subadditive function* $\chi\colon \mathcal{C} \to \mathbb{N}$ assigns to each object in $\mathcal{C}$ a non-negative integer such that

(SF1) $\chi(X \oplus Y) = \chi(X) + \chi(Y)$ for all $X, Y \in \mathcal{C}$, and
(SF2) $\chi(X) + \chi(Z) \geq \chi(Y)$ for each exact sequence $X \to Y \to Z \to 0$ in $\mathcal{C}$.

If the category $\mathcal{C}$ is abelian, then an *additive function* $\chi\colon \mathcal{C} \to \mathbb{N}$ assigns to each object in $\mathcal{C}$ a non-negative integer such that $\chi(X) + \chi(Z) = \chi(Y)$ for each exact sequence $0 \to X \to Y \to Z \to 0$.

The sum $\chi_1 + \chi_2$ of (sub)additive functions χ_1 and χ_2 is again (sub)additive. More generally, if $(\chi_i)_{i\in I}$ is a family of (sub)additive functions and if for any X in $\mathcal{C}$ the set $\{i \in I \mid \chi_i(X) \neq 0\}$ is finite, then we can define the *locally finite sum* $\sum_{i\in I} \chi_i$.

A (sub)additive function $\chi \neq 0$ is *irreducible* if χ cannot be written as a sum of two non-zero (sub)additive functions.

We give a quick proof of the following result using the localisation theory for abelian categories.

Lemma 13.1.1. *Let $\mathcal{C}$ be an abelian category. Every additive function $\mathcal{C} \to \mathbb{N}$ can be written uniquely as a locally finite sum of irreducible additive functions.*

Proof Fix an additive function $\chi\colon \mathcal{C} \to \mathbb{N}$. The objects X satisfying $\chi(X) = 0$ form a Serre subcategory of $\mathcal{C}$ which we denote by $\mathcal{S}_\chi$. The quotient category $\mathcal{C}/\mathcal{S}_\chi$ is an abelian length category since the length of each object X is bounded by $\chi(X)$. Let $\operatorname{Sp}\chi$ (the *spectrum* of χ) denote a representative set of simple

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objects in $\mathcal{C}/\mathcal{S}_\chi$. For each S in $\operatorname{Sp}\chi$ let $\chi_S\colon \mathcal{C} \to \mathbb{N}$ denote the map sending X to the multiplicity of S in a composition series of X in $\mathcal{C}/\mathcal{S}_\chi$. Clearly, χ_S is irreducible and we have the expression

$$\chi = \sum_{S \in \operatorname{Sp}\chi} \chi(S)\chi_S \tag{13.1.2}$$

which is unique by the Jordan–Hölder theorem. □

Lemma 13.1.3. *Let $\mathcal{C}$ be an additive category with cokernels and let $h\colon \mathcal{C} \to \operatorname{Fp}(\mathcal{C}, \mathrm{Ab})$ be the Yoneda embedding. Then the assignment $\chi \mapsto \chi \circ h$ induces an additive bijection between*

(1) *additive functions* $\operatorname{Fp}(\mathcal{C}, \mathrm{Ab}) \to \mathbb{N}$, *and*
(2) *subadditive functions* $\mathcal{C} \to \mathbb{N}$.

Proof The inverse map sends $\chi\colon \mathcal{C} \to \mathbb{N}$ to the map $\hat{\chi}$ that takes $F \in \operatorname{Fp}(\mathcal{C}, \mathrm{Ab})$ with presentation

$$0 \to \operatorname{Hom}_{\mathcal{C}}(Z, -) \to \operatorname{Hom}_{\mathcal{C}}(Y, -) \to \operatorname{Hom}_{\mathcal{C}}(X, -) \to F \to 0$$

to $\hat{\chi}(F) = \chi(X) - \chi(Y) + \chi(Z)$. □

Corollary 13.1.4. *Let $\mathcal{C}$ be an additive category with cokernels. Every subadditive function $\mathcal{C} \to \mathbb{N}$ can be written uniquely as a locally finite sum of irreducible subadditive functions.* □

Endofinite Functors

Let $\mathcal{C}$ be an essentially small abelian category. An exact functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is called *endofinite* if $F(X)$ has finite length as $\operatorname{End}(F)$-module for each object X. An endofinite exact functor F induces an additive function

$$\chi_F\colon \mathcal{C} \longrightarrow \mathbb{N}, \quad X \mapsto \ell_{\operatorname{End}(F)}(F(X)).$$

We need the following elementary lemma.

Lemma 13.1.5. *Let $\mathcal{A}$ be an abelian category and E an injective object. Then*

$$\ell_{\operatorname{End}_{\mathcal{A}}(E)}(\operatorname{Hom}_{\mathcal{A}}(X, E)) \le \ell_{\mathcal{A}}(X) \quad \text{for} \quad X \in \mathcal{A};$$

equality holds provided that $\operatorname{Hom}_{\mathcal{A}}(Y, E) = 0$ *implies* $Y = 0$ *for all* $Y \in \mathcal{A}$.

Proof Observe that $\ell_{\operatorname{End}_{\mathcal{A}}(E)}(\operatorname{Hom}_{\mathcal{A}}(X, E)) \le 1$ when X is simple. Now use induction on $\ell_{\mathcal{A}}(X)$. □

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Lemma 13.1.6. *Let* $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ *be an exact functor and let* $\mathcal{D} = \mathcal{C}/\mathcal{S}_F$, *where* $\mathcal{S}_F$ *denotes the Serre subcategory of objects* X *satisfying* $F(X) = 0$. *Then*

$$\ell_{\mathrm{End}(F)}(F(X)) \geq \ell_{\mathcal{D}}(X) \quad \textit{for} \quad X \in \mathcal{C};$$

equality holds when all objects in $\mathcal{D}$ *have finite length.*

Proof The first assertion is clear by induction on $\ell_{\mathcal{D}}(X)$. Now suppose that all objects in $\mathcal{D}$ have finite length. We consider the abelian category $\mathrm{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ of left exact functors $\mathcal{D}^{\mathrm{op}} \to \mathrm{Ab}$. The Yoneda functor

$$\mathcal{D} \longrightarrow \mathrm{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto h_X = \mathrm{Hom}_{\mathcal{D}}(-, X) \tag{13.1.7}$$

identifies $\mathcal{D}$ with the full subcategory of finite length objects. We write F as the composite $\mathcal{C}^{\mathrm{op}} \twoheadrightarrow \mathcal{D}^{\mathrm{op}} \xrightarrow{\bar{F}} \mathrm{Ab}$ and note that $\mathrm{End}(\bar{F}) \cong \mathrm{End}(F)$. Then $\bar{F}$ is an injective object in $\mathrm{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ by Corollary 11.2.15. Using Lemma 13.1.5 we compute

$$\ell_{\mathrm{End}(F)}(F(X)) = \ell_{\mathrm{End}(\bar{F})}(\mathrm{Hom}(h_X, \bar{F})) = \ell(h_X) = \ell_{\mathcal{D}}(X). \qquad \square$$

Proposition 13.1.8. *The assignment* $F \mapsto \chi_F$ *induces a bijection between the isomorphism classes of indecomposable endofinite exact functors* $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ *and the irreducible additive functions* $\mathcal{C} \to \mathbb{N}$.

Proof We construct the inverse map. Let $\chi\colon \mathcal{C} \to \mathbb{N}$ be an irreducible additive function. Following the proof of Lemma 13.1.1, we consider the Serre subcategory $\mathcal{S}_\chi$ of $\mathcal{C}$ consisting of the objects X satisfying $\chi(X) = 0$. The quotient category $\mathcal{D} = \mathcal{C}/\mathcal{S}_\chi$ is a length category, and $\chi(X)$ equals the length of X in $\mathcal{D}$ for each object X, since χ is irreducible. Now consider the abelian category $\mathrm{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ of left exact functors $\mathcal{D}^{\mathrm{op}} \to \mathrm{Ab}$. The Yoneda functor (13.1.7) identifies $\mathcal{D}$ with the full subcategory of finite length objects. There is a unique simple object in $\mathrm{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ since χ is irreducible, and we denote by F an injective envelope. It follows that F is indecomposable, and the injectivity implies that F is exact. For each X in $\mathcal{D}$ we have

$$\ell_{\mathrm{End}(F)}(F(X)) = \ell_{\mathrm{End}(F)}(\mathrm{Hom}(h_X, F)) = \ell_{\mathcal{D}}(X) = \chi(X)$$

by Lemma 13.1.5.

Let $F'\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ be the composite of F with the quotient functor $\mathcal{C} \to \mathcal{D}$ and observe that $\mathrm{End}(F') \cong \mathrm{End}(F)$. Then F' has the desired properties: it is indecomposable endofinite exact and $\chi_{F'} = \chi$.

It remains to show for an indecomposable endofinite exact functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ that the function χ_F is irreducible. Set $\mathcal{D} = \mathcal{C}/\mathcal{S}_{\chi_F}$ and view F as an exact functor $\mathcal{D}^{\mathrm{op}} \to \mathrm{Ab}$. Note that $\mathrm{Hom}(h_S, F) = F(S) \neq 0$ for each simple object

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S in $\mathcal{D}$. The indecomposability of F implies that all simple objects in $\mathcal{D}$ are isomorphic, and the equation (13.1.2) then implies that χ is irreducible since for each simple object S

$$\chi_F(S) = \ell_{\mathrm{End}(F)}(F(S)) = \ell_{\mathcal{D}}(S) = 1$$

by Lemma 13.1.6. □

Endofinite Objects

Let $\mathcal{A}$ be a locally finitely presented category. We recall the embedding $\mathcal{A} \to \mathbf{P}(\mathcal{A})$ into the purity category that identifies an object $X \in \mathcal{A}$ with the exact functor $\bar{X}\colon \mathrm{Ab}(\mathcal{A})^{\mathrm{op}} \to \mathrm{Ab}$. This yields the abelian category $\mathrm{Ab}(X) = \mathrm{Ab}(\mathcal{A})/\mathcal{S}_X$, where $\mathcal{S}_X = \mathrm{Ker}\,\bar{X}$; it is a useful invariant, because endofiniteness of X is controlled by $\mathrm{Ab}(X)$.

Proposition 13.1.9. *An object X in $\mathcal{A}$ is endofinite if and only if every object in* $\mathrm{Ab}(X)$ *has finite length. In that case X is Σ-pure-injective and decomposes into a coproduct of indecomposable endofinite objects with local endomorphism rings.*

Proof We identify X with the exact functor $\bar{X}\colon \mathrm{Ab}(\mathcal{A})^{\mathrm{op}} \twoheadrightarrow \mathrm{Ab}(X)^{\mathrm{op}} \to \mathrm{Ab}$. Using the Yoneda embedding

$$\mathrm{fp}\,\mathcal{A} \longrightarrow \mathrm{Ab}(\mathcal{A}), \quad C \mapsto h_C = \mathrm{Hom}_{\mathrm{fp}\,\mathcal{A}}(C, -)$$

we have $\mathrm{Hom}_{\mathcal{A}}(C, X) = \bar{X}(h_C)$. Thus X is endofinite if and only if $\bar{X}$ is an endofinite functor. It follows from Lemma 13.1.6 that X is endofinite if and only if every object in $\mathrm{Ab}(X)$ has finite length. The second part of the assertion then follows from Theorem 12.3.4. □

Corollary 13.1.10. *An object X in $\mathcal{A}$ is endofinite if and only if the subgroups of finite definition of* $\mathrm{Hom}(C, X)$ *satisfy the ascending and descending chain conditions for every $C \in \mathrm{fp}\,\mathcal{A}$. In this case* $\mathrm{End}(X)$*-submodules and subgroups of finite definition of* $\mathrm{Hom}(C, X)$ *coincide.*

Proof The first part follows from the above proposition since the lattice of subgroups of finite definition of $\mathrm{Hom}(C, X)$ identifies with the lattice of subobjects of $\mathrm{Hom}(C, -)$ in $\mathrm{Ab}(X)$ by Lemma 12.3.1. For the second part, see Lemma 12.3.5. □

An endofinite object X gives rise to a subadditive function χ_X by setting

$$\chi_X(C) = \ell_{\mathrm{End}_{\mathcal{A}}(X)}(\mathrm{Hom}_{\mathcal{A}}(C, X)) \quad \text{for} \quad C \in \mathrm{fp}\,\mathcal{A}.$$

Note that $\chi_X(C) = \ell_{\mathrm{Ab}(X)}(h_C)$ by Lemma 13.1.6.

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Theorem 13.1.11. *Let $\mathcal{A}$ be a locally finitely presented category.*

(1) *Any subadditive function* $\operatorname{fp}\mathcal{A} \to \mathbb{N}$ *can be written uniquely as a locally finite sum of irreducible subadditive functions.*
(2) *The assignment* $X \mapsto \chi_X$ *induces a bijection between the isomorphism classes of indecomposable endofinite objects in* $\mathcal{A}$ *and the irreducible subadditive functions* $\operatorname{fp}\mathcal{A} \to \mathbb{N}$.
(3) *Let* $X \in \mathcal{A}$ *be endofinite and* $(X_i)_{i \in I}$ *a representative set of indecomposable direct summands of* X. *Then* $\chi_X = \sum_{i \in I} \chi_{X_i}$.

Proof (1) This is Corollary 13.1.4.

(2) Following the proof of Proposition 13.1.9, endofinite objects in $\mathcal{A}$ correspond to endofinite exact functors $\operatorname{Ab}(\mathcal{A})^{\mathrm{op}} \to \mathrm{Ab}$. Thus the bijective correspondence $X \mapsto \chi_X$ between endofinite objects and subadditive functions follows from Proposition 13.1.8.

(3) We identify X with the induced exact functor $\operatorname{Ab}(X)^{\mathrm{op}} \to \mathrm{Ab}$. Then $\chi_X(C) = \ell_{\operatorname{Ab}(X)}(h_C)$ for all $C \in \operatorname{fp}\mathcal{A}$, by Lemma 13.1.6. Let $\operatorname{Sp}\chi_X$ denote a representative set of simple objects in $\operatorname{Ab}(X)$. For $S \in \operatorname{Sp}\chi_X$, consider the irreducible subadditive function $\chi_S \colon \operatorname{fp}\mathcal{A} \to \mathbb{N}$ that maps C to the multiplicity of S in a composition series of h_C in $\operatorname{Ab}(X)$. Then we have $\chi_S = \chi_{X_i}$ for a unique $i \in I$ by the first part of this proof, and therefore $\chi_X = \sum_S \chi_S = \sum_i \chi_{X_i}$ by the identity (13.1.2). □

Remark 13.1.12. Let $X \in \mathcal{A}$ be endofinite. Then the isomorphism classes of indecomposable direct summands of X are in canonical bijection to the isomorphism classes of simple objects in $\operatorname{Ab}(X)$. If X_i corresponds to $S_i \in \operatorname{Ab}(X)$, then $\operatorname{End}(X_i)/J(\operatorname{End}(X_i)) \cong \operatorname{End}(S_i)$.

Proof The first assertion follows from the proof of part (3) of Theorem 13.1.11. Thus an indecomposable summand X_i of X identifies with an injective envelope of a simple object in $\operatorname{Lex}(\operatorname{Ab}(X)^{\mathrm{op}}, \mathrm{Ab})$; see also Proposition 13.1.8 and its proof. For an abelian category and a simple object T with injective envelope $E = E(T)$, we have $\operatorname{soc} E = T$ and the assignment $\phi \mapsto \phi|_{\operatorname{soc} E}$ yields a surjective homomorphism $\operatorname{End}(E) \to \operatorname{End}(T)$ with kernel $J(\operatorname{End}(E))$. □

For an object X in $\mathcal{A}$ let $\operatorname{Add} X$ denote the full subcategory formed by all coproducts of copies of X and their direct summands.

For subadditive functions χ', χ we write $\chi' \leq \chi$ if $\chi - \chi'$ is a subadditive function.

Corollary 13.1.13. *Let* $X \in \mathcal{A}$ *be an endofinite object. Then* $\operatorname{Add} X$ *is a definable subcategory of* $\mathcal{A}$*, consisting of all endofinite objects* $Y \in \mathcal{A}$ *such that* $\chi_Y \leq \chi_X$.

Proof Recall that $\mathrm{Ab}(X) = \mathrm{Ab}(\mathcal{A})/\mathcal{S}_X$, where $\mathcal{S}_X = \operatorname{Ker}\bar{X}$. Let $\mathcal{B}$ denote the smallest definable subcategory of $\mathcal{A}$ containing X; it identifies with the category $\mathrm{Ex}(\mathrm{Ab}(X)^{\mathrm{op}}, \mathrm{Ab})$ by Corollary 12.2.3. All objects in $\mathcal{B}$ are endofinite by Proposition 13.1.9. Let $(X_i)_{i\in I}$ be a representative set of indecomposable direct summands of X. Then it follows from Theorem 13.1.11 and its proof that the objects in $\mathcal{B}$ are precisely those of the form $Y = \coprod_{i\in I} Y_i$ with Y_i a coproduct of copies of X_i. In particular $\chi_Y \le \chi_X$. Conversely, if $\chi_Y \le \chi_X$ for some object $Y \in \mathcal{A}$, then $\mathcal{S}_X \subseteq \mathcal{S}_Y$. Thus $\bar{Y}\colon \mathrm{Ab}(\mathcal{A})^{\mathrm{op}} \to \mathrm{Ab}$ factors through $\mathrm{Ab}(\mathcal{A})^{\mathrm{op}} \twoheadrightarrow \mathrm{Ab}(X)^{\mathrm{op}}$ and therefore $Y \in \mathcal{B}$. □

Corollary 13.1.14. *For an indecomposable object $X \in \mathcal{A}$ the following are equivalent.*

(1) *The object X is endofinite.*
(2) *The coproducts of copies of X form a definable subcategory.*
(3) *Every product of copies of X is a coproduct of copies of X.*

Proof (1) ⇒ (2): Apply Corollary 13.1.13.

(2) ⇒ (3): This is clear, since a definable subcategory is closed under products.

(3) ⇒ (1): The object X is Σ-pure-injective and every object in $\mathrm{Ab}(X)$ is noetherian, by Theorem 12.3.4. In fact, this result also implies that $\{X\}$ is a Ziegler closed subset of $\operatorname{Ind}\mathcal{A}$, by Theorem 12.2.2. Let $\mathrm{Ab}(X)_0$ denote the full subcategory of finite length objects. If this is a proper subcategory, then $\mathrm{Ab}(X)_0$ corresponds to a proper Ziegler closed subset of $\{X\}$, by Theorem 12.2.2. This is impossible, and therefore all objects in $\mathrm{Ab}(X)$ have finite length. Thus X is endofinite by Proposition 13.1.9. □

Corollary 13.1.15. *Let $X_1, \dots, X_n$ be endofinite objects in $\mathcal{A}$. Then $X_1 \oplus \cdots \oplus X_n$ is endofinite.*

Proof Set $X = X_1 \oplus \cdots \oplus X_n$. Then we have ${}^{\perp}X = \bigcap_i {}^{\perp}X_i$ in $\mathrm{Fp}(\mathrm{fp}\,\mathcal{A}, \mathrm{Ab})$. Thus if $\mathrm{Ab}(X_i)$ is a length category for each i, then $\mathrm{Ab}(X)$ is a length category, by the lemma below. Now the assertion is clear from the characterisation of endofiniteness of X via $\mathrm{Ab}(X)$ in Proposition 13.1.9. □

Lemma 13.1.16. *Let $\mathcal{C}$ be an abelian category and $\mathcal{C}_1, \dots, \mathcal{C}_n$ Serre subcategories of $\mathcal{C}$. If each localisation $\mathcal{C}/\mathcal{C}_i$ is a length category, then $\mathcal{C}/\bigcap_i \mathcal{C}_i$ is a length category.*

Proof The product $\prod_i \mathcal{C}/\mathcal{C}_i$ is a length category since $\ell(X) = \sum_i \ell(X_i)$ for each object $X = (X_i)$. The kernel of the canonical functor $\mathcal{C} \to \prod_i \mathcal{C}/\mathcal{C}_i$ equals $\bigcap_i \mathcal{C}_i$. This yields a faithful and exact functor $\mathcal{C}/\bigcap_i \mathcal{C}_i \to \prod_i \mathcal{C}/\mathcal{C}_i$. Clearly,

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the length of each object in $\mathcal{C}/\bigcap_i \mathcal{C}_i$ is bounded by the length of its image in $\prod_i \mathcal{C}/\mathcal{C}_i$. □

Finite Type

Let $\mathcal{A}$ be a locally finitely presented category. We wish to characterise the fact that all objects in $\mathcal{A}$ are endofinite. This requires a study of representable functors of finite length and we begin with some preparations.

Let $\mathcal{C}$ be an additive category. Let us call $\mathcal{C}$ *left Hom-finite* if for all objects X, Y in $\mathcal{C}$ the $\operatorname{End}(Y)$-module $\operatorname{Hom}(X, Y)$ has finite length. Clearly, this property implies that $\mathcal{C}$ is a Krull–Schmidt category, assuming that $\mathcal{C}$ is idempotent complete.

Lemma 13.1.17. *Let $\mathcal{C}$ be a Krull–Schmidt category. Then the following are equivalent.*

(1) *The category $\mathcal{C}$ is left Hom-finite.*
(2) *For all indecomposable objects X, Y the $\operatorname{End}(Y)$-module $\operatorname{Hom}(X, Y)$ has finite length.*
(3) *Every object in $\mathcal{C}$ has a left artinian endomorphism ring.*

Proof (1) ⇔ (2): Fix a pair of objects X, Y in $\mathcal{C}$. First observe that for any decomposition $X = \bigoplus_i X_i$ we have

$$\ell_{\operatorname{End}(Y)}(\operatorname{Hom}(X, Y)) = \sum_i \ell_{\operatorname{End}(Y)}(\operatorname{Hom}(X_i, Y)).$$

Now fix a decomposition $Y = \bigoplus_j Y_j^{n_j}$, $n_j > 0$ such that the Y_j are indecomposable and pairwise non-isomorphic. Set $Y' = \bigoplus_j Y_j$. Then

$$\begin{aligned}\ell_{\operatorname{End}(Y)}(\operatorname{Hom}(X, Y)) &= \ell_{\operatorname{End}(Y')}(\operatorname{Hom}(X, Y')) \\ &= \sum_j \ell_{\operatorname{End}(Y_j)}(\operatorname{Hom}(X, Y_j))\end{aligned}$$

since

$$\operatorname{End}(Y')/J(\operatorname{End}(Y')) \cong \prod_j \operatorname{End}(Y_j)/J(\operatorname{End}(Y_j)).$$

Now the assertion follows.

(1) ⇔ (3): One direction is clear. So fix objects $X, Y \in \mathcal{C}$ and suppose that $\Lambda = \operatorname{End}(X \oplus Y)$ is left artinian. Thus $\ell({}_\Lambda\Lambda)$ is finite. Let $e \in \Lambda$ be the idempotent given by projecting onto Y. Observe that $\ell_{e\Lambda e}(eM) \leq \ell_\Lambda(M)$ for every left Λ-module M. Now $\operatorname{Hom}(X, Y)$ is a direct summand of $e\Lambda = \operatorname{Hom}(X \oplus Y, Y)$ and has therefore finite length over $e\Lambda e = \operatorname{End}(Y)$. □

Now suppose that $\mathcal{C}$ is a Krull–Schmidt category. Let $\operatorname{ind}\mathcal{C}$ denote a representative set of the isoclasses of indecomposable objects. For an additive functor $F\colon \mathcal{C} \to \mathrm{Ab}$ we define its *support*

$$\operatorname{Supp}(F) = \{X \in \operatorname{ind}\mathcal{C} \mid FX \neq 0\}$$

and let $\ell(F)$ denote the composition length of F in $\operatorname{Add}(\mathcal{C}, \mathrm{Ab})$.

Lemma 13.1.18. *For an additive functor* $F\colon \mathcal{C} \to \mathrm{Ab}$ *we have*

$$\ell(F) = \sum_{X \in \operatorname{ind}\mathcal{C}} \ell_{\operatorname{End}(X)}(FX).$$

Proof The assignment

$$F \mapsto \tilde{\ell}(F) := \sum_{X \in \operatorname{ind}\mathcal{C}} \ell_{\operatorname{End}(X)}(FX)$$

satisfies $\tilde{\ell}(F) = \tilde{\ell}(F') + \tilde{\ell}(F'')$ for every exact sequence $0 \to F' \to F \to F'' \to 0$, and $\tilde{\ell}(F) \neq 0$ for every $F \neq 0$. Thus $\ell(F) = \infty$ implies $\tilde{\ell}(F) = \infty$.

Now suppose that $\ell(F) < \infty$. If F is a simple functor and $FX \neq 0$ for some $X \in \operatorname{ind}\mathcal{C}$, then we have $\operatorname{Supp}(F) = \{X\}$ and $\ell(F) = 1 = \ell_{\operatorname{End}(X)}(FX)$. From this the assertion follows by induction on $\ell(F)$. □

Remark 13.1.19. Let $\mathcal{F} = \operatorname{Fp}(\mathcal{C}, \mathrm{Ab})$ be abelian and $F \in \mathcal{F}$. Consider the embedding $\mathcal{F} \to \bar{\mathcal{F}} = \operatorname{Add}(\mathcal{C}, \mathrm{Ab})$. Then $\ell_{\mathcal{F}}(F) = \ell_{\bar{\mathcal{F}}}(F)$ since the embedding is right exact and every simple object in $\mathcal{F}$ is simple in $\bar{\mathcal{F}}$.

Theorem 13.1.20. *For a locally finitely presented category $\mathcal{A}$ the following are equivalent.*

1. (1) *Every object in $\mathcal{A}$ is endofinite.*
2. (2) *The abelian category* $\operatorname{Ab}(\mathcal{A})$ *is a length category.*
3. (3) *For all $C \in \operatorname{fp}\mathcal{A}$ the functor* $\operatorname{Hom}(C, -)\colon \operatorname{fp}\mathcal{A} \to \mathrm{Ab}$ *has finite length.*
4. (4) *For all $C \in \operatorname{fp}\mathcal{A}$ the endomorphism ring* $\operatorname{End}(C)$ *is left artinian and there are, up to isomorphism, only finitely many indecomposable objects $D \in \operatorname{fp}\mathcal{A}$ such that* $\operatorname{Hom}(C, D) \neq 0$.

In this case each object in $\mathcal{A}$ decomposes into a coproduct of indecomposable finitely presented objects.

Proof (1) ⇔ (2): For every object $X \in \mathcal{A}$ we have the quotient $\operatorname{Ab}(\mathcal{A}) \twoheadrightarrow \operatorname{Ab}(X)$. Note that $\operatorname{Ab}(\mathcal{A}) \xrightarrow{\sim} \operatorname{Ab}(U)$ for $U = \prod_{X \in \operatorname{Ind}\mathcal{A}} X$; see Theorem 12.2.2. Now apply Proposition 13.1.9. Thus when U is endofinite, then $\operatorname{Ab}(\mathcal{A})$ is a length category. On the other hand, when $\operatorname{Ab}(\mathcal{A})$ is a length category, then $\operatorname{Ab}(X)$ is a length category for every $X \in \mathcal{A}$.

(2) ⇔ (3): Clearly, $\mathrm{Ab}(\mathcal{A})$ is a length category if and only if for each $C \in \mathrm{fp}\,\mathcal{A}$ the representable functor $\mathrm{Hom}(C, -)$ has finite length, keeping in mind Remark 13.1.19.

(3) ⇔ (4): We apply Lemma 13.1.17 and Lemma 13.1.18. Then each representable functor $\mathrm{fp}\,\mathcal{A} \to \mathrm{Ab}$ has finite length if and only if $\mathrm{fp}\,\mathcal{A}$ is left Hom-finite and the support of each representable functor is finite.

To prove the final assertion, observe that each endofinite object decomposes into a coproduct of indecomposables by Proposition 13.1.9. Thus it remains to show that each indecomposable $X \in \mathcal{A}$ is finitely presented. Let $X = \operatorname{colim} X_i$ be written as a filtered colimit of objects in $\mathrm{fp}\,\mathcal{A}$. There is a unique simple object $S \in \mathrm{fp}\,\mathbf{P}(\mathcal{A})$ such that $E(S) = \bar{X}$. The inclusion $S \to \bar{X}$ factors though $\bar{X}_i$, and we may assume that X_i is indecomposable. Thus $\bar{X}_i \cong E(S)$, and therefore $X \cong X_i$. □

Example 13.1.21. Let Λ denote the Kronecker algebra and consider the subcategory $\mathcal{I} \subseteq \mathrm{mod}\,\Lambda$ of postinjective Kronecker representations (finite direct sums of indecomposable representations with dimension vector $(n+1, n)$). Then $\vec{\mathcal{I}}$ satisfies the conditions of the above theorem.

Properties of Endofinite Objects

Let $\mathcal{A}$ be a locally finitely presented category. The indecomposable endofinite objects may be viewed as points of $\mathrm{Ind}\,\mathcal{A}$. These give rise to discrete subsets.

Proposition 13.1.22. *Let $X \in \mathcal{A}$ be an endofinite object. Then each subset $\mathcal{U} \subseteq \mathrm{Add}\,X \cap \mathrm{Ind}\,\mathcal{A}$ is a closed subset of $\mathrm{Ind}\,\mathcal{A}$.*

Proof It follows from Corollary 13.1.13 that the coproducts of copies of objects in $\mathcal{U}$ form a definable subcategory of $\mathcal{A}$. Thus $\mathcal{U}$ is a closed subset of $\mathrm{Ind}\,\mathcal{A}$ by Theorem 12.2.2. □

We have the following immediate consequence.

Corollary 13.1.23. *Suppose that $\mathrm{Ind}\,\mathcal{A}$ is quasi-compact. If X is an endofinite object in $\mathcal{A}$, then the number of isomorphism classes of indecomposable direct summands of X is finite.*

Proof The indecomposable direct summands of X form a discrete space by the above proposition. This space is necessarily finite if it is quasi-compact. □

Next we study the endomorphism ring of an endofinite object. Recall that for an object C in an abelian category the height $\mathrm{ht}(C)$ is bounded by its composition length $\ell(C)$.

Proposition 13.1.24. *Let X be an endofinite object and J the Jacobson radical of its endomorphism ring. Then $\bigcap_{n\geq 0} J^n = 0$. Moreover, for $n \geq 0$ we have*

$$J^n = 0 \iff \mathrm{ht}(F) \leq n \text{ for all } F \in \mathrm{Ab}(X)$$
$$\iff \mathrm{ht}(\mathrm{Hom}(C,X)) \leq n \text{ for all } C \in \mathrm{fp}\,\mathcal{A}.$$

Proof As before we identify X with an exact functor $\bar{X}\colon \mathrm{Ab}(X)^{\mathrm{op}} \to \mathrm{Ab}$ in $\mathrm{Lex}(\mathrm{Ab}(X)^{\mathrm{op}}, \mathrm{Ab})$. This is a locally finite Grothendieck category and then the assertion follows from Proposition 11.2.8 and Remark 11.2.9. □

Corollary 13.1.25. *Suppose there exists an object $G \in \mathrm{fp}\,\mathcal{A}$ such that every object in $\mathrm{fp}\,\mathcal{A}$ is a quotient of G^n for some integer $n \geq 1$. Then for every endofinite object in $\mathcal{A}$ the Jacobson radical of its endomorphism ring is nilpotent.*

Proof Let $X \in \mathcal{A}$ be endofinite. For $C \in \mathrm{fp}\,\mathcal{A}$ an epimorphism $G^n \to C$ induces a monomorphism $\mathrm{Hom}(C,X) \to \mathrm{Hom}(G^n,X)$. Thus $\mathrm{ht}(\mathrm{Hom}(C,X)) \leq \mathrm{ht}(\mathrm{Hom}(G,X))$. □

Uniserial Categories

Let $\mathcal{A}$ be a locally finitely presented abelian category and set $\mathcal{C} = \mathrm{fp}\,\mathcal{A}$. Our aim is to describe all indecomposable objects in $\mathcal{A}$ when $\mathcal{C}$ is an abelian category which is uniserial. In particular, we see that all indecomposables are endofinite.

Fix an object $X \in \mathcal{A}$. The composition length of X is denoted by $\ell(X)$, and the height $\mathrm{ht}(X)$ is the smallest $n \geq 0$ such that $\mathrm{soc}^n(X) = X$.

Now suppose that every object in $\mathrm{fp}\,\mathcal{A}$ has finite length. Let $X = \mathrm{colim}\, X_i$ be written as a filtered colimit of finitely presented objects. Then $\mathrm{soc}^n(X) = \mathrm{colim}\,\mathrm{soc}^n(X_i)$ for all $n \geq 0$, and therefore $X = \bigcup_{n\geq 0} \mathrm{soc}^n(X)$.

Recall that $\mathcal{C}$ is *uniserial* if $\mathcal{C}$ is a length category and each indecomposable object has a unique composition series.

Lemma 13.1.26. *Let $\mathcal{C}$ be a length category. Then $\mathcal{C}$ uniserial if and only if $\mathrm{ht}(X) = \ell(X)$ for every indecomposable $X \in \mathcal{C}$.*

Proof Let $X \in \mathcal{C}$ be indecomposable. If $\mathrm{ht}(X) = \ell(X)$, then the socle series of X is the unique composition series of X.

Now assume $\mathrm{ht}(X) \neq \ell(X)$. Then there exists some $n \geq 0$ such that

$$\mathrm{soc}^{n+1}(X)/\mathrm{soc}^n(X) = S_1 \oplus \cdots \oplus S_r$$

with all S_i simple and $r > 1$. Choose n minimal and let $\mathrm{soc}^n(X) \subseteq U_i \subseteq X$ be given by $U_i/\mathrm{soc}^n(X) = S_i$. Then we have at least r different composition series

$$0 = \mathrm{soc}^0(X) \subseteq \mathrm{soc}^1(X) \subseteq \cdots \subseteq \mathrm{soc}^n(X) \subseteq U_i \subseteq \cdots \subseteq \mathrm{soc}^{n+1}(X) \subseteq \cdots$$

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of X. □

Lemma 13.1.27. *Let $\mathcal{C}$ be a uniserial category. Then $\mathcal{C}$ is left Hom-finite.*

Proof We apply Lemma 13.1.17 and need to show for all X, Y in $\mathcal{C}$ that the $\operatorname{End}(Y)$-module $\operatorname{Hom}(X, Y)$ has finite length, and it suffices to assume that Y is indecomposable. We claim that

$$\ell_{\operatorname{End}(Y)}(\operatorname{Hom}(X, Y)) \leq \ell(X).$$

Using induction on $\ell(X)$ the claim reduces to the case that X is simple. So let $S = \operatorname{soc}(Y)$ and write $E = E(Y)$ for its injective envelope. Note that $\operatorname{soc}^n(E) = Y$ for $n = \ell(Y)$, by Lemma 13.1.26. Thus any endomorphism $E \to E$ restricts to a morphism $Y \to Y$. Write $i \colon S \to Y$ for the inclusion. Then any morphism $j \colon S \to Y$ induces an endomorphism $f \colon E \to E$ such that $f|_Y \circ i = j$. Thus the $\operatorname{End}(Y)$-module $\operatorname{Hom}(S, Y)$ is cyclic, and it is annihilated by the radical of $\operatorname{End}(Y)$. Therefore $\operatorname{Hom}(S, Y)$ is simple. □

Theorem 13.1.28. *Let $\mathcal{A}$ be a locally finitely presented category and suppose that $\operatorname{fp}\mathcal{A}$ is a uniserial category. Then every non-zero object in $\mathcal{A}$ has an indecomposable direct summand that is finitely presented or injective.*

Proof From Lemma 13.1.26 it follows that for every indecomposable injective object E in $\mathcal{A}$ the subobjects form a chain

$$0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$$

with $E_n = \operatorname{soc}^n(E)$ in $\mathcal{C}$ for all $n \geq 0$ and $E = \bigcup_{n \geq 0} E_n$. Note that $E = E_{\ell(E)}$ when $\ell(E) < \infty$.

Fix $X \neq 0$ in $\mathcal{A}$ and choose a simple subobject $S \subseteq X$. Let $U \subseteq X$ be a maximal subobject containing S such that $S \subseteq U$ is essential; this exists by Zorn's lemma. Then U is injective or belongs to $\mathcal{C}$. In the first case we are done. So assume $U \in \mathcal{C}$. We claim that the inclusion $U \to X$ is a pure monomorphism. To see this, choose a morphism $C \to X/U$ with $C \in \mathcal{C}$. This yields the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & X/U & \longrightarrow & 0 \end{array}$$

Write $V = \bigoplus V_i$ as a direct sum of indecomposable objects. Then there exists an index i such that the composite $S \hookrightarrow U \to V_i \to X$ is non-zero. Thus $S \to V_i$ is essential and $V_i \to X$ is a monomorphism. It follows from the maximality of U that $U \to V_i$ is an isomorphism. Therefore the top row splits, and this yields

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the claim. It remains to observe that every object in $\mathcal{C}$ is pure-injective since $\mathcal{C}$ is left Hom-finite; see Proposition 13.1.9 and Lemma 13.1.27. □

Corollary 13.1.29. *Every indecomposable object in $\mathcal{A}$ is endofinite.* □

Corollary 13.1.30. *Every indecomposable object in $\mathcal{A}$ is the source of a left almost split morphism.*

Proof Let X be indecomposable. If X is injective, then $X \twoheadrightarrow X/\operatorname{soc}(X)$ is left almost split. If X is not injective, then X is finitely presented and there exists a monomorphism $X \to X'$ such that X' is indecomposable and $\ell(X') = \ell(X)+1$. It is easily checked that $X \to X' \oplus X/\operatorname{soc}(X)$ is left almost split. □

13.2 Endofinite Modules

Let Λ be a ring. We consider the category of Λ-modules and set $\mathcal{A} = \operatorname{Mod}\Lambda$. Note that $\mathcal{A}$ is locally finitely presented with $\operatorname{fp}\mathcal{A} = \operatorname{mod}\Lambda$. In this section we apply the general theory of endofinite objects and study the Λ-modules which are endofinite.

Properties of Endofinite Modules

For a Λ-module X with $\Gamma = \operatorname{End}_\Lambda(X)$ we denote by $\ell_\Lambda(X)$ its composition length and call $\operatorname{endol}_\Lambda(X) := \ell_\Gamma(X)$ the *endolength* of X. Clearly, $\operatorname{endol}_\Lambda(X) < \infty$ if and only if X is an endofinite object in $\operatorname{Mod}\Lambda$, since for any epimorphism $\Lambda^n \twoheadrightarrow C$ we have

$$\ell_\Gamma(\operatorname{Hom}_\Lambda(C, X)) \leq n \cdot \operatorname{endol}_\Lambda(X).$$

Given a bimodule ${}_\Sigma X_\Lambda$ we have $\operatorname{endol}_\Lambda(X) \leq \ell_\Sigma(X)$. Thus when Λ is a k-algebra over some commutative ring k, then a Λ-module of finite length over k is endofinite. In particular, when Λ is an Artin k-algebra we have

$$\operatorname{endol}_\Lambda(X) \leq \ell_k(X) \leq \ell_\Lambda(X) \cdot \ell_k(\Lambda/J(\Lambda)).$$

The following summarises the basic properties of endofinite modules.

Proposition 13.2.1. *Endofinite modules are Σ-pure-injective. The class of endofinite modules is closed under finite direct sums, and arbitrary products or coproducts of copies of one module. If $X' \subseteq X$ is a pure submodule of an endofinite module X, then X' is a direct summand and* $\operatorname{endol}(X') \leq \operatorname{endol}(X)$.

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Proof Endofinite modules are Σ-pure-injective by Proposition 13.1.9. Being closed under finite direct sums follows from Corollary 13.1.15. Being closed under arbitrary (co)products of copies of one module follows from Corollary 13.1.13. The same result shows that endofinite modules are closed under pure submodules, because a definable subcategory is closed under pure subobjects, by Theorem 12.2.5. Also, we have for a pure submodule $X' \subseteq X$

$$\operatorname{endol}(X') = \chi_{X'}(\Lambda) \leq \chi_X(\Lambda) = \operatorname{endol}(X)$$

by Corollary 13.1.13. □

The next result establishes the decomposition of endofinite modules into indecomposables. For a cardinal α let $X^{(\alpha)}$ denote a coproduct of α copies of X.

Proposition 13.2.2. *A Λ-module X is endofinite if and only if there are pairwise non-isomorphic indecomposable endofinite Λ-modules $X_1, \dots, X_n$ and cardinals $\alpha_i > 0$ such that $X = \bigoplus_{i=1}^n X_i^{(\alpha_i)}$. In that case*

$$\operatorname{endol}(X) = \sum_{i=1}^n \operatorname{endol}(X_i).$$

Proof Let X be endofinite. Then X decomposes into a coproduct of indecomposable modules because X is Σ-pure-injective; see Theorem 12.3.4. The number of isomorphism classes of indecomposable summands in such a decomposition is finite by Corollary 13.1.23, since $\operatorname{Ind}\Lambda$ is quasi-compact; see Corollary 12.3.12. The summation formula for $\operatorname{endol}_\Lambda(X)$ then follows from Theorem 13.1.11.

Now let $X_1, \dots, X_n$ be indecomposable modules which are endofinite. Then $\bigoplus_{i=1}^n X_i^{(\alpha_i)}$ is endofinite for any choice of cardinals α_i, by Proposition 13.2.1. □

Examples of Endofinite Modules

We collect various examples of endofinite modules.

Example 13.2.3. A ring Λ is right artinian if and only if every injective Λ-module is endofinite.

Proof Let I be an injective Λ-module and set $\Gamma = \operatorname{End}_\Lambda(I)$. Given a Λ-module X, we have $\ell_\Gamma(\operatorname{Hom}_\Lambda(X, I)) \leq 1$ when X is simple, and therefore by induction

$$\ell_\Gamma(\operatorname{Hom}_\Lambda(X, I)) \leq \ell_\Lambda(X),$$

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with equality when I is an injective cogenerator. Now the assertion follows by setting $X = \Lambda$. □

Example 13.2.4. Let Λ be a commutative noetherian ring and $\mathfrak{p} \in \operatorname{Spec}\Lambda$. Then the injective envelope $E(\Lambda/\mathfrak{p})$ is endofinite if and only if $\mathfrak{p}$ is minimal.

Proof All injective Λ-modules decompose into indecomposables because Λ is noetherian. Also, we use the bijection $\mathfrak{q} \mapsto E(\Lambda/\mathfrak{q})$ between $\operatorname{Spec}\Lambda$ and the set of isoclasses of indecomposable injective Λ-modules (Corollary 2.4.15).

Now fix $\mathfrak{p} \in \operatorname{Spec}\Lambda$ and consider the specialisation closed set $\mathcal{V} = \{\mathfrak{q} \in \operatorname{Spec}\Lambda \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$. This yields a split torsion pair

$$(\{X \in \operatorname{Inj} A \mid \operatorname{Ass} X \subseteq \mathcal{V}\}, \{X \in \operatorname{Inj} A \mid \operatorname{Ass} X \cap \mathcal{V} = \varnothing\})$$

(Corollary 2.4.16). Then $\{X \in \operatorname{Inj} A \mid \operatorname{Ass} X \cap \mathcal{V} = \varnothing\}$ is closed under products and consists of coproducts of copies of $E(\Lambda/\mathfrak{p})$ if and only if $\mathfrak{p}$ is minimal. From this the assertion follows because of the characterisation of indecomposable endofinite objects in Corollary 13.1.14. □

The next example takes up the construction of pure-injective modules over group algebras from Proposition 12.4.20.

Example 13.2.5. Let G be a finite group and k a field. If $\mathfrak{p}$ is a prime ideal in $R = H^*(G, k)$ and $I_\mathfrak{p} = E(R/\mathfrak{p})$ the corresponding indecomposable injective R-module, then we may assume the corresponding kG-module $T(I_\mathfrak{p})$ to be indecomposable, by removing all non-zero injective summands.

If $\mathfrak{p}$ is a minimal prime, then $T(I_\mathfrak{p})$ is endofinite. This follows from the above Example 13.2.4 and the characterisation of indecomposable endofinite objects in Corollary 13.1.14, because the functor T preserves products and coproducts.

Example 13.2.6. The indecomposable endofinite Λ-modules X with $\operatorname{End}_\Lambda(X)$ a division ring correspond bijectively to a ring epimorphism $\phi\colon \Lambda \to \Gamma$ such that Γ is simple artinian. The correspondence sends X to $\Lambda \to \operatorname{End}_{\operatorname{End}_\Lambda(X)}(X)$ and ϕ to the restriction of the simple Γ-module.

Example 13.2.7. For a commutative ring Λ the modules of endolength one correspond bijectively to prime ideals, by taking $\mathfrak{p} \in \operatorname{Spec}\Lambda$ to the quotient field $Q(\Lambda/\mathfrak{p})$.

The following is an analogue of Example 12.3.6.

Example 13.2.8. Let Q be a quiver and k a commutative ring. If X is a k-linear representation such that X_i is a finite length k-module for each vertex $i \in Q_0$, then X is endofinite.

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Example 13.2.9. Let Λ be a ring and X an indecomposable Σ-pure injective Λ-module. Then the Ziegler closure of $\{X\}$ contains an endofinite module; see Proposition 14.1.19.

Finite Representation Type

A right artinian ring Λ is said to be of *finite representation type* if the number of isomorphism classes of finitely presented indecomposable Λ-modules is finite.

Theorem 13.2.10. *For a ring Λ the following conditions are equivalent.*

(1) *The ring Λ is right artinian and of finite representation type.*
(2) *Every Λ-module is endofinite.*
(3) *Every object in* $\mathrm{Ab}(\Lambda)$ *is of finite length.*

In this case each Λ-module decomposes into a coproduct of finitely presented indecomposables.

Proof Most implications as well as the final assertion are direct consequences of Theorem 13.1.20.

(1) $\Rightarrow$ (3): See [9, Theorem 3.1].

(2) $\Leftrightarrow$ (3): See Theorem 13.1.20.

(2) $\Rightarrow$ (1): The ring Λ is right artinian, thanks to Example 13.2.3. From Theorem 13.1.20 it follows that for all $C \in \operatorname{mod}\Lambda$ there are, up to isomorphism, only finitely many indecomposable objects $D \in \operatorname{mod}\Lambda$ such that $\operatorname{Hom}(C, D) \neq 0$. Taking $C = \Lambda_\Lambda$, it follows that Λ is of finite representation type. □

Note that the finite representation type is left-right symmetric because of the duality $\mathrm{Ab}(\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Ab}(\Lambda^{\mathrm{op}})$.

Noetherian Algebras

Let Λ be a noetherian k-algebra over a commutative ring k. In this case there is a natural finiteness condition for an endofinite Λ-module.

Lemma 13.2.11. *An endofinite Λ-module is noetherian if and only if it is artinian.*

Proof Let X be an endofinite Λ-module and set $\Gamma = \operatorname{End}_\Lambda(X)$. Suppose first that X_Λ is noetherian. Then Γ is a noetherian k-algebra and therefore any finite length Γ-module is of finite length over k. It follows that X is of finite length over Λ.

Suppose that X_Λ is artinian. Then the socle series of X_Λ is finite since each

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term is a Γ-submodule of X. It remains to note that a semisimple artinian module has finite length. □

The following is an analogue of Theorem 12.4.15 for Artin algebras.

Theorem 13.2.12. *For an indecomposable endofinite Λ-module X the following are equivalent.*

(1) *X is finitely presented.*
(2) *X is of finite length.*
(3) *X is the source of a left almost split morphism.*
(4) *X is isolated.*

Proof (1) $\Rightarrow$ (2): Follows from Lemma 13.2.11.

(2) $\Rightarrow$ (3): Adapt the proof of Theorem 12.4.15.

(3) $\Rightarrow$ (2): It follows from Theorem 12.3.13 that $\bar{X}$ is the injective envelope of a simple object $C \in \mathbf{P}(\operatorname{Mod}\Lambda)$. The Λ-module X is a direct summand of $\prod_{i \in I} DY_i$ for some collection of finitely presented left Λ-modules Y_i, where $D = \operatorname{Hom}_k(-, E)$ denotes the Matlis duality for k-modules; see Corollary 12.4.8. We have $\operatorname{Hom}(C, \bar{X}_{i_0}) \neq 0$ for some $i_0 \in I$ where $X_i = DY_i$. Then it follows that X is a direct summand of DY_{i_0}. We have $E = \coprod_S E(S)$, where S runs though the simple k-modules, and therefore $DY = \coprod_s \operatorname{Hom}_k(Y, E(S))$ when Y is finitely generated. It follows as before that X is a direct summand of $\operatorname{Hom}_k(Y_{i_0}, E(S_0))$ for some simple S_0 and therefore artinian over k, since the k-module $E(S_0)$ is artinian; cf. Lemma 2.4.19. Then the above Lemma 13.2.11 implies that X_Λ is of finite length.

(2) $\Rightarrow$ (1): Clear.

(3) $\Leftrightarrow$ (4): Apply Corollary 12.3.15. □

The above theorem suggests that we can single out the indecomposable endofinite modules which have infinite length; these are called *generic*.

Quasi-Frobenius Rings

Right artinian rings are characterised by the fact that all projective and all injective modules are endofinite. The following theorem describes the right artinian rings such that both classes of modules coincide.

Theorem 13.2.13. *For a ring Λ the following conditions are equivalent.*

(1) *Projective and injective Λ-modules coincide.*
(2) *The category* $\operatorname{Mod}\Lambda$ *of Λ-modules is a Frobenius category.*
(3) *The ring Λ is right noetherian and the module Λ_Λ is injective.*

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(4) *The ring* Λ *is right artinian and* $\operatorname{mod}\Lambda$ *is a Frobenius category.*

Proof (1) $\Rightarrow$ (2): This is clear, since $\operatorname{Mod}\Lambda$ is a category with enough projective and enough injective objects

(2) $\Rightarrow$ (3): The category of injective modules is closed under coproducts, and therefore Λ is a right noetherian ring (Theorem 11.2.12).

(3) $\Rightarrow$ (4): The module Λ_Λ decomposes into finitely many indecomposables with local endomorphism rings. Let $P_1, \dots, P_n$ represent the isomorphism classes of indecomposable summands, and set $S_i = P_i/\operatorname{rad} P_i$. Then $S_1, \dots, S_n$ represent the isomorphism classes of simple Λ-modules, and $P_1, \dots, P_n$ represent the isomorphism classes of their injective envelopes. Here one uses that the P_i are injective. For any i, any product of copies of P_i decomposes into indecomposables, and the socle of each summand is isomorphic to $\operatorname{soc} P_i$. Thus the characterisation of indecomposable endofinite objects in Corollary 13.1.14 implies that P_i is endofinite. But then Λ is endofinite, and therefore all injective Λ-modules are endofinite. This implies that Λ is right artinian, by Example 13.2.3. It is clear that $\operatorname{mod}\Lambda$ has enough projective and enough injective objects. Moreover, the indecomposable projectives and injectives coincide, and therefore all projectives and injectives in $\operatorname{mod}\Lambda$ coincide, by the Krull–Remak–Schmidt theorem. Thus $\operatorname{mod}\Lambda$ is a Frobenius category.

(4) $\Rightarrow$ (1): Since Λ is right artinian, all projective and all injective Λ-modules are direct sums of indecomposable modules. These indecomposables belong to $\operatorname{mod}\Lambda$, and therefore projective and injective Λ-modules coincide. □

The Space of Indecomposables

We study indecomposable endofinite Λ-modules as points of the spectrum $\operatorname{Ind}\Lambda$. We use the embedding $\operatorname{Mod}\Lambda \to \mathbf{P}(\operatorname{Mod}\Lambda)$ and identify each Λ-module X with the corresponding exact functor $\bar{X}\colon \operatorname{Ab}(\Lambda)^{\mathrm{op}} \to \operatorname{Ab}$; see Corollary 12.4.5.

Lemma 13.2.14. *The endolength of a* Λ*-module* X *equals the length of the object* $F = \operatorname{Hom}_\Lambda(\Lambda, -)$ *in* $\operatorname{Ab}(X)$*. For an integer* $n \geq 0$*, we have* $\operatorname{endol}(X) \leq n$ *if and only if for every chain of subobjects*

$$0 = F_{n+1} \subseteq \cdots \subseteq F_1 \subseteq F_0 = \operatorname{Hom}_\Lambda(\Lambda, -)$$

in $\operatorname{Ab}(\Lambda)$ *we have* $\bar{X}(F_i/F_{i+1}) = 0$ *for some* $0 \leq i \leq n$*.*

Proof Observe that $\bar{X}(F) = X$. Thus the proof of Proposition 13.1.8 shows

$$\operatorname{endol}(X) = \ell_{\operatorname{End}(\bar{X})}(\bar{X}(F)) = \ell_{\operatorname{Ab}(X)}(F).$$

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Every chain of subobjects

$$0 = F_{n+1} \subseteq \cdots \subseteq F_1 \subseteq F_0 = F$$

in $\mathrm{Ab}(X)$ is isomorphic to the image of a chain of subobjects

$$0 = F_{n+1} \subseteq \cdots \subseteq F_1 \subseteq F_0 = F$$

in $\mathrm{Ab}(\Lambda)$ under the canonical functor $\mathrm{Ab}(\Lambda) \twoheadrightarrow \mathrm{Ab}(X)$. Clearly, $\ell_{\mathrm{Ab}(X)}(F) \leq n$ if and only if for every such chain $F_i/F_{i+1} = 0$ in $\mathrm{Ab}(X)$ for some $0 \leq i \leq n$. It remains to observe that $F_i/F_{i+1} = 0$ in $\mathrm{Ab}(X)$ if and only if $\bar{X}(F_i/F_{i+1}) = 0$. □

Proposition 13.2.15. *Let Λ be a ring. Then $\mathcal{U}_n = \{X \in \mathrm{Ind}\,\Lambda \mid \mathrm{endol}(X) \leq n\}$ is a closed subset of $\mathrm{Ind}\,\Lambda$ for every integer $n \geq 0$.*

Proof For a chain $\phi = (F_i)_{0\leq i\leq n+1}$ of subobjects

$$0 = F_{n+1} \subseteq \cdots \subseteq F_1 \subseteq F_0 = \mathrm{Hom}_\Lambda(\Lambda, -)$$

in $\mathrm{Ab}(\Lambda)$ we set $\mathcal{U}_{\phi,i} = \{X \in \mathrm{Ind}\,\Lambda \mid \bar{X}(F_i/F_{i+1}) = 0\}$ and $\mathcal{U}_\phi = \bigcup_{i=0}^n \mathcal{U}_{\phi,i}$. These are closed subsets of $\mathrm{Ind}\,\Lambda$, and therefore the intersection $\mathcal{U} = \bigcap_\phi \mathcal{U}_\phi$ is also closed, where ϕ runs through all chains $\phi = (F_i)_{0\leq i\leq n+1}$. It remains to observe that $\mathcal{U} = \mathcal{U}_n$ by the preceding lemma. □

A compactness argument provides for Artin algebras the existence of indecomposable endofinite modules which are of infinite length. Such modules are called *generic*.

Corollary 13.2.16. *Let Λ be an Artin algebra and set $\mathrm{ind}\,\Lambda = \mathrm{Ind}\,\Lambda \cap \mathrm{mod}\,\Lambda$. If a subset of $\{X \in \mathrm{ind}\,\Lambda \mid \mathrm{endol}(X) \leq n\}$ is infinite for some fixed $n \in \mathbb{N}$, then its closure contains a point $Y \in \mathrm{Ind}\,\Lambda \setminus \mathrm{ind}\,\Lambda$ with $\mathrm{endol}(Y) \leq n$.*

Proof The space $\mathcal{U}_n$ is quasi-compact since $\mathrm{Ind}\,\Lambda$ is quasi-compact; see Corollary 12.3.12. On the other hand, $\mathcal{U}_n \cap \mathrm{ind}\,\Lambda$ is discrete by Theorem 12.4.15. Thus $\mathcal{U}_n \subseteq \mathrm{ind}\,\Lambda$ implies that $\mathcal{U}_n$ is finite. □

Generic modules also arise from Prüfer modules; see Example 14.1.21.

Duality

Let Λ be a ring. There is a bijective correspondence between indecomposable endofinite right and left Λ-modules. Recall that a Λ-module X determines the following Serre subcategory of $\mathrm{Ab}(\Lambda)$

$${}^\perp X = \{F \in \mathrm{Ab}(\Lambda) \mid \mathrm{Hom}(F^\vee, X \otimes_\Lambda -) = 0\}.$$

For a module X we set $\Delta(X) = \mathrm{End}(X)/J(\mathrm{End}(X))$.

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Lemma 13.2.17. *Let* $\mathcal{S} \subseteq \mathrm{Ab}(\Lambda)$ *be a Serre subcategory such that* $\mathrm{Ab}(\Lambda)/\mathcal{S}$ *is a length category with a unique simple object* S*. Then there is up to isomorphism a unique indecomposable* Λ*-module* X *with* ${}^{\perp}X = \mathcal{S}$*. Moreover,* $\mathrm{endol}(X)$ *equals the length of* $\mathrm{Hom}_\Lambda(\Lambda, -)$ *in* $\mathrm{Ab}(\Lambda)/\mathcal{S}$*, and* $\Delta(X) \cong \mathrm{End}(S)$.

Proof Let $\mathcal{S}^{\perp}$ denote the definable subcategory corresponding to $\mathcal{S}$. Then $X \in \mathcal{S}^{\perp}$ is endofinite if $\mathrm{Ab}(\Lambda)/\mathcal{S}$ is a length category, by Proposition 13.1.9. The assertion about $\mathrm{endol}(X)$ then follows from Lemma 13.2.14, and for the rest see Remark 13.1.12. □

Theorem 13.2.18. *Let* Λ *be a ring. There is a bijection* $X \mapsto DX$ *between the isomorphism classes of indecomposable endofinite right and left* Λ*-modules. It is determined by any of the following conditions.*

(1) ${}^{\perp}DX = ({}^{\perp}X)^{\vee}$.
(2) *Let* Γ *be a ring such that* ${}_\Gamma X_\Lambda$ *is a bimodule. If* I *is an injective* Γ*-module, then* $\mathrm{Hom}_\Gamma(X, I)$ *is a coproduct of copies of* DX.

Moreover, we have $D^2X \cong X$*,* $\mathrm{endol}(DX) = \mathrm{endol}(X)$*, and* $\Delta(DX) \cong \Delta(X)^{\mathrm{op}}$.

Proof We apply the equivalence $\mathrm{Ab}(\Lambda)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Ab}(\Lambda^{\mathrm{op}})$ and combine this with Lemma 13.2.17. Observe for a Serre subcategory $\mathcal{S} \subseteq \mathrm{Ab}(\Lambda)$ that we have an induced equivalence

$$(\mathrm{Ab}(\Lambda)/\mathcal{S})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Ab}(\Lambda^{\mathrm{op}})/\mathcal{S}^{\vee}.$$

If $\mathcal{S} = {}^{\perp}X$ for a Λ-module X, then X is endofinite if and only if $\mathrm{Ab}(\Lambda)/\mathcal{S}$ is a length category, by Proposition 13.1.9. If X is indecomposable and endofinite, then DX is given by $\mathcal{S}^{\perp}$, or as a direct summand of $\mathrm{Hom}_\Gamma(X, I)$, see Proposition 12.4.10. Clearly, $D^2X \cong X$ since $\mathcal{S}^{\vee\vee} = \mathcal{S}$, and the rest follows from Lemma 13.2.17. □

Example 13.2.19. Let Λ be a noetherian k-algebra and $X \mapsto \mathrm{Hom}_k(X, E)$ the Matlis duality over k. For an indecomposable endofinite Λ-module X of finite length we have $DX = \mathrm{Hom}_k(X, E)$.

Notes

Modules of finite endolength were introduced by Crawley-Boevey [56, 57]. Of particular interest are generic modules, that is, the indecomposable endofinite modules that are not of finite length; they can be used to describe the representation type of an algebra, because generic modules parametrise families of

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finite dimensional representations [56]. In the more general context of locally finitely presented categories, endofinite objects were introduced and studied in [58]. This work contains the classification in terms of subadditive functions. The existence of generic modules over Artin algebras is closely related to the 'strongly unbounded representation type'. This yields a link to the second Brauer–Thrall conjecture [117] which is explained in [57]. The existence proof given here uses the compactness of the Ziegler spectrum and follows Herzog [111]. The characterisation of indecomposable endofinite modules of finite length over noetherian algebras is taken from [57]. The duality between indecomposable endofinite right and left modules is due to Herzog [110]. The study of quasi-Frobenius and self-injective rings goes back to Eilenberg and Nakayama [73], generalising work of Brauer and Nesbitt for finite dimensional algebras [40].

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