8

Moduli of Stable Pairs

We bring together the moduli theory of Chapter 6 with K-flatness of Chapter 7 to obtain the moduli theory of stable pairs in full generality. The basic definitions originate in the papers Kollár and Shepherd-Barron (1988) and Alexeev (1996); the resulting moduli spaces are usually called KSBA moduli spaces.

In Section 8.1 we discuss a bookkeeping device called marking: we need to know not only what the boundary divisor Δ is, but also how it is written as a linear combination of effective $\mathbb{Z}$-divisors. In the cases considered in Chapter 6, there was always a unique, obvious marking; this is why the notion was not introduced before. Simple examples show that, without marking, we get infinite dimensional moduli spaces, already for pointed curves (8.2).

The notion of Kollár–Shepherd-Barron–Alexeev stability is introduced in Section 8.2. The proof that we get a good moduli theory, as defined in (6.10), follows the methods of Chapter 6 if the coefficients are rational (8.9), but a few more steps are need if they are irrational (8.15).

The end result is the following consequence of (8.9) and (8.15).

Theorem 8.1 *Fix a base scheme S of characteristic 0, a coefficient vector $\mathbf{a} = (a_1, \dots, a_r) \in [0,1]^r$, an integer n, and a real number v. Let $\mathcal{SP}(\mathbf{a}, n, v)$ denote the functor of marked, stable pairs of dimension n and volume v. Then $\mathcal{SP}(\mathbf{a}, n, v)$ is good moduli theory (6.10) and it has a coarse moduli space $\mathrm{SP}(\mathbf{a}, n, v)$, which is projective over S.*

A variant with floating coefficients is treated in Section 8.3 and the moduli theory of more general polarized pairs is discussed in Sections 8.4–8.5.

The construction of moduli spaces as quotients by group actions is treated in Section 8.6, and a short overview of descent is in Section 8.7.

https://doi.org/10.1017/9781009346115.010 Published online by Cambridge University Press

In Section 8.8, we discuss several unexpected problems that appear in positive characteristic. Quite likely, these necessitate substantial changes in the moduli theory of varieties of dimension ≥ 3 in positive and mixed characteristics.

Further Results

An early difficulty of KSBA theory was that good examples were not easy to write down. The first notable successes were Alexeev (2002); Hacking (2004). By now there is a rapidly growing body of fully understood cases.

Various moduli spaces are worked out in the papers Abramovich and Vistoli (2000); van Opstall (2005, 2006b,a); Hacking (2012); Alexeev (2015); Franciosi et al. (2015b, 2017, 2018); Alexeev (2016); Ascher and Gallardo (2018); Ascher and Bejleri (2019, 2021a,b); Ascher et al. (2020); Alexeev and Thompson (2021); Bejleri and Inchiostro (2021).

Examples of stable degenerations and their relations to other invariants are exhibited in Hassett (1999, 2000, 2001); Alexeev (2008); Tziolas (2009, 2010); Hacking and Prokhorov (2010); Hacking (2013, 2016); Urzúa (2016b,a); Rana (2017); Hacking et al. (2017); Rana and Urzúa (2019); Franciosi et al. (2022).

Computations of invariants of stable surfaces are given in Liu and Rollenske (2014); Franciosi et al. (2015a); Stern and Urzúa (2016); Tziolas (2017, 2022).

Special examples are computed in detail in Hacking et al. (2006, 2009); Thompson (2014); Ascher and Molcho (2016); Alexeev and Liu (2019a,b); Donaldson (2020).

Other approaches to the moduli spaces are discussed in Abramovich and Vistoli (2002); Alexeev and Knutson (2010); Abramovich and Hassett (2011); Abramovich et al. (2013, 2017); Abramovich and Chen (2014); Abramovich and Fantechi (2017).

Assumptions In this Chapter we work over a $\mathbb{Q}$-scheme. The definitions are set up in full generality, but some of the theorems fail in positive characteristic; see Section 8.8 for a discussion.

8.1 Marked Stable Pairs

So far, we have studied slc pairs (X, Δ), but usually did not worry too much about how Δ was written as a sum of divisors. As long as we look at a single variety, we can write Δ uniquely as $\sum a_i D_i$ where the D_i are prime divisors, and there is usually not much reason to do anything else. However, the situation changes when we look at families.

8.2 (Is $D = \frac{1}{n}(nD)$?) Assume that we have an slc family over an irreducible base $f\colon (X,\Delta) \to S$ with generic point $g \in S$. Then the natural approach is to write $\Delta_g = \sum a_i D^i_g$, where the D^i_g are prime divisors on the generic fiber X_g. For any other point $s \in S$ this gives a decomposition $\Delta_s = \sum a_i D^i_s$, where D^i_s is the specialization of D^i_g. Note that the D^i_s need not be prime divisors. They can have several irreducible components with different multiplicities and two different D^i_s, D^j_s can have common irreducible components. Thus $\Delta_s = \sum a_i D^i_s$ is not the "standard" way to write Δ_s.

Let us now turn this around. We fix a proper slc pair (X_0, Δ_0) and aim to understand all deformations of it. A first suggestion could be the following:

8.2.1 (*Naive definition*) An slc deformation of (X_0, Δ_0) over a local scheme $(0 \in S)$ is a proper slc morphism $f\colon (X,\Delta) \to S$ whose central fiber $(X,\Delta)_0$ is isomorphic to (X_0, Δ_0).

As an example, start with $(\mathbb{P}^1_{xy}, (x = 0))$. Pick $n \geq 1$ and variables t_i. Then

$$\left(\mathbb{P}^1_{xy} \times \mathbb{A}^n_{\mathbf{t}}, \tfrac{1}{n}(x^n + t_{n-1}x^{n-1}y + \cdots + t_0 y^n = 0)\right) \tag{8.2.2}$$

is a deformation of $(\mathbb{P}^1_{xy}, (x = 0))$ over $\mathbb{A}^n$ by the naive definition (8.2.1). We get a deformation space of dimension $n-2$ using $\mathbf{Aut}(\mathbb{P}^1, (0{:}1))$. Letting n vary results in an infinite dimensional deformation space.

The polynomial in (8.2.2) is irreducible over $k(t_0, \dots, t_{n-1})$, thus our recipe says that we should write $\Delta = \frac{1}{n} D_g$ (where D_g is irreducible). Then the special fiber is written as $(x = 0) = \frac{1}{n}(x^n = 0)$.

The situation becomes even less clear if we take two deformations as in (8.2.2) for two different values n, m and glue them together over the origin. The family is locally stable. One side says that the fiber over the origin should be $\frac{1}{n}(x^n = 0)$, the other side that it should be $\frac{1}{m}(x^m = 0)$.

As (8.2) suggests, some bookkeeping is necessary to control the multiplicities of the divisorial part of a pair (X,Δ) in families. This is the role of the marking we introduce next.

Once we control how a given $\mathbb{R}$-divisor Δ is written as a linear combination of $\mathbb{Z}$-divisors, we obtain finite dimensional moduli spaces.

Definition 8.3 (Marked pairs) A *marking* of an effective Weil $\mathbb{R}$-divisor Δ is a way of writing $\Delta = \sum a_i D_i$, where the D_i are effective $\mathbb{Z}$-divisors and $0 < a_i \in \mathbb{R}$. We call $\mathbf{a} = (a_1, \dots, a_r)$ the *coefficient vector.*

A *marked pair* is a pair (X,Δ), plus a marking $\Delta = \sum a_i D_i$.

We allow the D_i to be empty; this has the advantage that the restriction of a marking to an open subset is again a marking. However, in other contexts, this is not natural and we will probably sometimes disregard empty divisors.

Observe that $\Delta = \sum a_i D_i$ and $\Delta = \sum(\frac{1}{2}a_i)(2D_i)$ are different as markings. This seems rather pointless for one pair but, as we observed in (8.2), it is a meaningful distinction when we consider deformations of a pair.

Note that, for a given (X, Δ), markings are combinatorial objects that are not constrained by the geometry of X. If $\Delta = \sum_i b_i B_i$ and the B_i are distinct prime divisors, then the markings correspond to ways of writing the vector $(b_1, \dots, b_r)$ as a positive linear combination of nonnegative integral vectors.

Comments Working with such markings is a rather natural thing to do. For example, plane, curves C of degree d can be studied using the log CY pair $(\mathbb{P}^2, \Delta_C := \frac{3}{d}C)$ as in Hacking (2004). Thus, even if C is reducible, we want to think of the $\mathbb{Q}$-divisor Δ_C as $\frac{3}{d}C$; hence as a marked divisor with $\mathbf{a} = (\frac{3}{d})$. Similarly, in most cases when we choose the boundary divisor Δ, it has a natural marking.

However, when a part of Δ is forced upon us, for instance coming from the exceptional divisor of a resolution, there is frequently no “natural” marking, though usually it is possible to choose a marking that works well enough.

If (X, Δ) is slc and $a_i > \frac{1}{2}$ for every i, then the marking is almost determined by Δ. For example, if the a_i are distinct then the obvious marking of $\Delta = \sum a_i D_i$ is the unique one. If all the $a_i = 1$, then the markings of $\sum_{i \in I} D_i$ correspond to partitions of I.

If we allow $a_i = \frac{1}{2}$, then an irreducible divisor D can have three different markings: $[D]$, $\frac{1}{2}[2D]$, or $\frac{1}{2}[D] + \frac{1}{2}[D]$. The smaller the a_i, the more markings are possible.

Definition 8.4 (Families of marked pairs) Fix a real vector $\mathbf{a} = (a_1, \dots, a_r)$. A *family of marked pairs* with coefficient vector $\mathbf{a} = (a_1, \dots, a_r)$ consists of

(8.4.1) a flat morphism $f: X \to S$ with demi-normal fibers (11.36),

(8.4.2) an effective, relative, Mumford $\mathbb{R}$-divisor Δ, plus

(8.4.3) a marking $\Delta = \sum a_i D_i$, where the D_i are effective, relative, Mumford $\mathbb{Z}$-divisors (4.68).

As we discussed in Section 4.1, the relative Mumford assumption on the D_i assures that markings can be pulled back by base-change morphisms $W \to S$.

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However, being relative Mumford is not automatic. This means that not all markings of Δ give a family of marked pairs.

Examples 8.5 (Marking and stability) Given a family of pairs $f : (X, \Delta) \to S$, it can happen that it is KSBA-stable for one choice of the marking, but not for other markings. Although we define KSBA-stability only in the next section, these examples influenced the precise definitions of KSBA-stability, especially (8.13), so this is their right place.

(8.5.1) If S is normal then, by (4.4), every marking of Δ yields a family of marked pairs $f : (X, \Delta) \to S$.

(8.5.2) Assume that Δ is a $\mathbb{Q}$-divisor and S is reduced. There is a smallest $N \in \mathbb{N}^{>0}$ such that $N\Delta$ is a $\mathbb{Z}$-divisor. In characteristic 0, $D := N\Delta$ is a relative Mumford divisor by (4.39), thus $\Delta = \frac{1}{N}D$ gives a marking of (X, Δ).

(8.5.3) For $\mathbb{R}$-divisors, there are markings $\Delta = \sum_i \lambda_i \Delta_i$ where the λ_i are $\mathbb{Q}$-linearly independent; see (11.47). In characteristic 0 we get the same stable families, independent of the choice of the λ_i by (11.43.4) and (4.39).

(8.5.4) The simplest case is when $\Delta = cD$ with a single irreducible D. The only possible markings are $\Delta = \sum a_i(m_i D)$ for some $m_i \in \mathbb{N}$ and $c = \sum a_i m_i$. In characteristic 0 we get the same stable families, independent of the choices by (4.39), but not in positive characteristic, see (8.76).

(8.5.5) Let B be a curve with a single node b and $B^\circ := B \setminus \{b\}$. Let $b_1, b_2 \in \bar{B}$ be the preimages of the node in the normalization $\bar{B}$. Set $\bar{S} = \mathbb{P}^1 \times \bar{B}$.

Let $\bar{E}_i := \{p_i\} \times \bar{B}$ sections for $i = 1, 2, 3$ and $\bar{\Delta}$ their sum. If we use the marking with only 1 divisor $D_1^\circ := \sum_i E_i^\circ$, then we can use any of the six automorphisms of $\mathbb{P}^1$ that preserve $\{p_1, p_2, p_3\}$ to descend $(\bar{S}, \bar{\Delta})$ to a family of marked pairs over B. If we use the marking with three divisors $D_i^\circ := E_i^\circ$, then the identity gives the only descent.

8.2 Kollár–Shepherd-Barron–Alexeev Stability

Now we come to the main theorem of the book, the existence of a good moduli theory for all marked stable pairs (X, Δ) in characteristic 0.

The principle is that, once we have K-flatness to replace flatness in Section 6.2, the rest of the arguments should go through with small changes. This is indeed true for rational coefficients, so we start with that case.

For irrational coefficients, it is less clear how to cook up ample line bundles, so the existence of embedded moduli spaces needs more work.

KSBA Stability with Rational Coefficients

Fix a rational coefficient vector $\mathbf{a} = (a_1, \ldots, a_r)$ and let $\mathrm{lcd}(\mathbf{a})$ denote the least common denominator of the a_i.

8.6 (Stable objects) These are marked pairs $(X, \Delta = \sum_i a_i D_i)$ with coefficient vector **a** such that

(8.6.1) (X, Δ) is slc,

(8.6.2) X is projective and $K_X + \Delta$ is ample.

8.7 (Stable families) A family $f\colon (X, \Delta = \sum_i a_i D_i) \to S$ is *KSBA-stable* if the following hold:

(8.7.1) $f\colon X \to S$ is flat, finite type, pure dimensional.

(8.7.2) The D_i are K-flat families of relative, Mumford, $\mathbb{Z}$-divisors (7.1).

(8.7.3) The fibers (X_s, Δ_s) are slc.

(8.7.4) $\omega_{X/S}^{[m]}(m\Delta - B)$ is a flat family of divisorial sheaves, provided $\mathrm{lcd}(\mathbf{a}) \mid m$ and $B = \sum_{j \in J} D_j$ with $a_j = 1$ for $j \in J$.

(8.7.5) f is proper and $K_{X/S} + \Delta$ is f-ample.

The first four of these conditions define *locally KSBA-stable* families.

8.8 (Explanation) These conditions are mostly straightforward generalizations of (6.16.1–3). We discussed K-flatness in Chapter 7.

The main question is (8.7.4). We should think of it as the minimal assumption, which should be made more stronger whenever possible, without changing the reduced structure of the moduli space.

The main case is $B = 0$. For $\omega_{X/S}^{[m]}(m\Delta)$ to make sense, $m\Delta$ must be a $\mathbb{Z}$-divisor. If the D_i have no multiple or common irreducible components, this holds only if m is a multiple of $\mathrm{lcd}(\mathbf{a})$. The nonzero choices of B in (8.7.4) are discussed in (6.23).

We could also ask about the sheaves $\omega_{X/S}^{[m]}(\sum \lfloor m a_i \rfloor D_i)$, as in (6.22.3). As we saw in (2.41), they are not flat families of divisorial sheaves in general, but (2.79) discusses various examples where they are. Thus, on a case-by-case basis, a strengthening of (8.6.4) is possible and useful. This was one of the themes of Chapter 6.

Theorem 8.9 *KSBA-stability with rational coefficients, as defined in (8.6–8.7), is a good moduli theory (6.10).*

Proof We need to check the conditions (6.10.1–5).

Separatedness (6.10.1) follows from (2.50); valuative-properness (6.10.2) is proved in (2.51) and (7.4.2). Assumption (8.7.4) follows from (2.79.1) if $B = 0$

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and from (2.79.8) when $B \neq 0$. Representability is proved in (7.65) and (3.31). Boundedness holds by (6.8.1) and (6.14).

Once we know that $m(K_X + \Delta)$ is very ample for every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)$ for some fixed m, embedded moduli spaces (6.10.3) are constructed in (8.52). However, the universal family over $\mathrm{C^mESP}(\mathbf{a}, n, \mathbb{P}^N_{\mathbb{Q}})$ satisfies (8.7.3) only for multiples of m. We can then handle the other values as in the proof of (6.24).

The coarse moduli space exists by (6.6). □

As in (6.25), we get the following from (8.7.3).

Proposition 8.10 *For KSBA-stable families as in (8.6–8.7), let m be a multiple of* $\mathrm{lcd}(\mathbf{a})$. *Then* $\chi(X, \omega_X^{[m]}(m\Delta))$ *and* $h^0(X, \omega_X^{[m]}(m\Delta))$ *are both deformation invariant.* □

KSBA Stability with Arbitrary Coefficients

Fix a coefficient vector $\mathbf{a} = (a_1, \dots, a_r)$ where $a_i \in [0, 1]$ are arbitrary real numbers. By (11.43.1), if $K_X + \sum_i a_i D_i$ is $\mathbb{R}$-Cartier, then we can get many $\mathbb{Q}$-Cartier divisors. We start by listing them.

Definition 8.11 Fix $\mathbf{a} = (a_1, \dots, a_r)$ with linear $\mathbb{Q}$-envelope $\mathrm{LEnv}_{\mathbb{Q}}(1, \mathbf{a}) \subset \mathbb{Q}^{r+1}$ as in (11.44). For $\Delta = \sum_{i=1}^r a_i D_i$, set

$$\mathrm{LEnv}_{\mathbb{Z}}(K_X + \Delta) := \{m_0 K_X + \textstyle\sum m_i D_i \colon (m_0, \dots, m_r) \in \mathrm{LEnv}_{\mathbb{Q}}(1, \mathbf{a}) \cap \mathbb{Z}^{r+1}\}.$$

Let us mention two extreme cases.
(8.11.1) If all $a_i \in \mathbb{Q}$, then $\mathrm{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ consists of all $\mathbb{Z}$-multiples of $\mathrm{lcd}(\mathbf{a})(K_X + \Delta)$.
(8.11.2) If $\{1, a_1, \dots, a_r\}$ is a $\mathbb{Q}$-linearly independent set, then $\mathrm{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ consist of all $\mathbb{Z}$-linear combinations $m_0 K_X + \sum m_i D_i$.

It is very important that, by (11.44) and (11.43.1), if $K_X + \Delta$ is $\mathbb{R}$-Cartier, then all elements of $\mathrm{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ are $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisors. (There may be other linear combinations that are $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisors.)

The stable objects are the same as before, but the definition of stable families again looks different.

8.12 (Stable objects) We parametrize marked pairs $(X, \Delta = \sum_i a_i D_i)$ with coefficient vector $\mathbf{a}$ such that
(8.12.1) (X, Δ) is slc,
(8.12.2) X is projective and $K_X + \Delta$ is ample.

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8.13 (Stable families) A family $f\colon (X, \Delta = \sum_i a_i D_i) \to S$ is *KSBA-stable* if the following hold

(8.13.1) $f\colon X \to S$ is flat, finite type, pure dimensional.

(8.13.2) The D_i are K-flat families of relative, Mumford, $\mathbb{Z}$-divisors (7.1).

(8.13.3) The fibers (X_s, Δ_s) are slc.

(8.13.4) $\omega_{X/S}^{[m_0]}(\sum m_i D_i - B)$ is a flat family of divisorial sheaves, whenever $(m_0, \ldots, m_r) \in \mathrm{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ and $B = \sum_{j \in J} D_j$, where $a_j = 1$ for $j \in J$.

(8.13.5) f is proper and $K_{X/S} + \Delta$ is f-ample.

The first four of these conditions define *locally KSBA-stable* families.

8.14 (Explanation) These conditions are mostly straightforward generalizations of (8.7), again the main question is assumption (8.13.4).

If the a_i are rational, then, by (8.11.1), $\mathrm{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ consists of the integer multiples of $\mathrm{lcd}(\mathbf{a})(K_X + \Delta)$, so (8.13.4) specializes to (8.7.4). If $1, a_1, \ldots, a_r$ are $\mathbb{Q}$-linearly independent, then, by (8.11.2), we specialize to (6.38).

For the intermediate cases we follow the philosophy behind KSB stability as in Section 6.2. Whenever we can prove to have a flat family of divisorial sheaves over DVR's, we require this property over all schemes.

Working with all of $\mathrm{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ is (almost) necessary for our proof. We are using several rational perturbations of $K_X + \Delta$ to get enough ample $\mathbb{Q}$-divisors. These span $\mathrm{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ (at least with $\mathbb{Q}$-coefficients).

The choice of B in (8.13.4) is discussed in (6.23).

The sheaves $\omega_{X/S}^{[m]}(\sum \lfloor m a_i \rfloor D_i)$ are not easy to understand. As we already noted in (8.8), they are not always flat families of divisorial sheaves, though the latter holds for infinitely many m, depending on the coefficient vector $\mathbf{a}$. Unfortunately, the method of (11.50) is ineffective, it is not at all clear how to produce such values m.

Theorem 8.15 *KSBA-stability, as defined in (8.12–8.13), is a good moduli theory (6.10).*

Proof We need to check the conditions (6.10.1–5).

Separatedness and valuative-properness (6.10.1–2) are as for (8.9). Embedded moduli spaces (6.10.3) are worked out in (8.21). Representability holds by (7.65) and (3.31). Boundedness is discussed in (6.8.2). □

Let us note the following strengthening of (2.65) and (2.69).

Theorem 8.16 *Let $f\colon (X, \Delta = \sum_{i \in I} a_i D_i) \to S$ be a KSBA stable family. Let $B = \sum_{j \in J} D_j$ be a divisor, where $a_j = 1$ for $j \in J$ and L an f-semi-ample*

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divisorial sheaf (3.25) on X. Then $R^i f_*(L^{[-1]}(-B))$ *and* $R^i f_*(\omega_{X/S} \hat{\otimes} L(B))$ *are locally free and compatible with base change for every i.*

Proof As in the proof of (2.65), a suitable cyclic cover reduces the first part to the case when $L = \mathscr{O}_X$, which follows from Kovács and Schwede (2016, 5.1). (Note that the latter is stated over a smooth base, but that is not used in the proof. Also, $\mathscr{O}_X(-B)$ is a flat family of divisorial sheaves by (8.13.4), thus (X_s, B_s) is a Du Bois pair for every $s \in S$.) This implies the second part as in (2.69). □

KSBA Stability, Stronger Version

K-flatness is designed to work for all boundary divisors $\Delta = \sum_i a_i D_i$, thus it cannot capture the stronger properties of those D_i that have coefficient $> \frac{1}{2}$. The notion of strong KSBA-stability takes care of this. The resulting moduli space has the same underlying reduced subscheme, but a smaller nilpotent structure.

8.17 (Stable objects) $(X, \Delta = \sum_{i\in I} a_i D_i)$, same as in (8.12).

8.18 (Stable families) Families $f\colon (X, \Delta = \sum_{i\in I} a_i D_i) \to S$ as in (8.13), with the following additional assumption taken from (2.82).

(8.18.1) Let $J \subset I$ be any subset such that $a_j > \frac{1}{2}$ for every $j \in J$ and set $D_J := \cup_{j\in J} D_j$. Then $f|_{D_J}\colon D_J \to S$ is flat with reduced fibers.

Note It is possible that some variant of (6.27.3) could be added, but (2.83) does not seem strong enough for this.

The proof of (8.15) carries over without changes to give the following.

Theorem 8.19 *Strong KSBA-stability, as defined in (8.17–8.18), is a good moduli theory (6.10).* □

Example 8.20 To see that we do get a smaller scheme structure, even for surfaces, start with the A_1 singularity $S_0 := (y^2 - x^2 + z^2 = 0)$ and the nodal curve $C_0 := (z = y^2 - x^2 = 0)$. Then (S_0, C_0) is lc. Over $k[\varepsilon]$, consider the trivial deformation $S := (y^2 - x^2 + z^2 = 0)$. For C_0 we take the simplest K-flat, but nonflat deformation. Using the notation of (7.70.5), it is given by $y^2 - x^2 = 0$ and $z = \frac{y}{x}\varepsilon$ in the chart $(x \neq 0)$. The closure is given by

$$C = (y^2 - x^2 = zx - y\varepsilon = zy - x\varepsilon = z^2 = z\varepsilon = 0).$$

Then $(S, C) \to \operatorname{Spec} k[\varepsilon]$ is locally stable as in (8.13), but C is not flat over $\operatorname{Spec} k[\varepsilon]$. Hence (8.18.1) is not satisfied.

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8.21 (Construction of embedded moduli spaces) A way of approximating an $\mathbb{R}$-Cartier pair with $\mathbb{Q}$-Cartier pairs is given in (11.47).

Depending on the vector **a**, we have $\mathbb{Q}$-linear maps $\sigma_j^m : \mathbb{R} \to \mathbb{Q}$, extended to divisors by $\sigma_j^m(\sum a_i D_i) := \sum \sigma_j^m(a_i) D_i$, with the following properties.

(8.21.1) If $K_{X/S} + \Delta$ is $\mathbb{R}$-Cartier then the $K_{X/S} + \sigma_j^m(\Delta)$ are $\mathbb{Q}$-Cartier.

(8.21.2) $\lim_{m\to\infty} \sigma_j^m(\Delta) = \Delta$.

(8.21.3) Δ is a convex $\mathbb{R}$-linear combination of the $\sigma_j^m(\Delta)$ for every fixed m.

(8.21.4) If $(X, \Delta) \to S$ is stable then so are the $\bigl(X, \sigma_j^m(\Delta)\bigr) \to S$ for $m \gg 1$.

Since $\mathcal{SP}(\mathbf{a}, n, v)$ is bounded by (6.8.2), there is a fixed M such that

(8.21.5) the $\bigl(X, \sigma_j^M(\Delta)\bigr)$ are stable for every j and every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)$.

The volume of $\bigl(X, \sigma_j^M(\Delta)\bigr)$ may depend on (X, Δ), but it is locally constant in families, so only finitely many values can occur for $\mathcal{SP}(\mathbf{a}, n, v)$. Denote this set by $V \subset \mathbb{R}$.

Let $\mathcal{SP}(\sigma_*^M(\mathbf{a}), n, V)$ be the moduli functor of all pairs (X, Δ) of dimension n, for which all the $\bigl(X, \sigma_j^M(\Delta)\bigr)$ are stable and $\operatorname{vol}\bigl(X, \sigma_1^M(\Delta)\bigr) \in V$. We claim that this is a good moduli theory.

Indeed, first $\mathcal{SP}(\sigma_1^M(\mathbf{a}), n, V)$ is a good moduli theory by (8.9). Then we have to add the conditions that the $K_{X/S} + \sigma_j^M(\Delta)$ are $\mathbb{Q}$-Cartier for $j \neq 1$; these are representable by (4.29). Finally, once the $K_{X/S} + \sigma_j^M(\Delta)$ are $\mathbb{Q}$-Cartier, ampleness of these is an open condition. Thus we have the moduli space $\mathrm{SP}(\sigma_*^M(\mathbf{a}), n, V)$.

Since Δ is a convex $\mathbb{R}$-linear combination of the $\sigma_j^m(\Delta)$, $\mathrm{SP}(\mathbf{a}, n, v)$ is a closed subspace of $\mathrm{SP}(\sigma_*^M(\mathbf{a}), n, V)$ by (11.4.4). □

8.3 Stability with Floating Coefficients

Much of the technical subtlety of the KSBA approach is caused by the presence of boundary divisors that are not $\mathbb{Q}$-Cartier. As we discussed in Section 6.4, one way to avoid these is to work with marked pairs $(X, \sum_{i\in I} a_i D_i)$ as in (8.3), where the $a_i \in \mathbb{R}$ are $\mathbb{Q}$-linearly independent. However, in many important cases, the a_i are dictated by geometric considerations and they are rational.

By working on a $\mathbb{Q}$-factorialization $\pi\colon X' \to X$, we can achieve that the D_i' are $\mathbb{Q}$-Cartier. The price we pay is that $K_X + \sum a_i D_i'$ is only nef, giving a non-separated moduli space. We can restore separatedness if we know which linear combinations $-\sum_i c_i D_i'$ are π-ample. (The negative sign works better later.)

If we fix c_i, then $-\sum_i (c_i + \eta_i) D_i'$ is also π-ample for all $|\eta_i| \ll |c_i|$. Thus we can choose the η_i such that the $(a_i - c_i - \eta_i)$ are $\mathbb{Q}$-linearly independent; we are then back to the situation of Alexeev stability, as in Section 6.4. However, we do not wish to fix the c_i.

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Using floating coefficients was considered early on by Alexeev, Hassett, and Kovács. There is a short discussion in Alexeev (2006), but the first significant example of it is treated in Alexeev (2015). The general type case with a floating coefficient is worked out in Filipazzi and Inchiostro (2021).

Keeping in mind the chambers discussed in (6.39), it is clear that one cannot float several coefficients independently. A solution is to fix an ordering of the index set I; this is natural in many cases, but not always.

The key observation is that, for a normal pair $(X, \sum_{i=1}^r a_i D_i)$, there is at most one small modification $\pi\colon X' \to X$ such that

- $-\sum_{i=1}^r \varepsilon_i D_i'$ is π-ample for all $0 < \varepsilon_1 \ll \cdots \ll \varepsilon_r$.

(The notation means that there is a $\delta > 0$ such that ampleness holds whenever $\varepsilon_i \leq \delta\varepsilon_{i+1}$ for every i. By (11.43), then the $D_i' := \pi_*^{-1} D_i$ are $\mathbb{Q}$-Cartier.)

To get a good moduli theory out of this, we need to allow certain nonsmall birational maps $X' \to X$. There is a further issue that going freely between X and X' seems to need the Abundance Conjecture to hold (Kollár and Mori, 1998, 3.12). Thus the working definition is more complicated.

8.22 (Canonical contractions and models of nef slc pairs) By Kollár (2011c), there are projective surfaces with normal crossing singularities whose canonical ring is not finitely generated. Thus it is not possible to define the canonical model of a proper slc pair (Y, Δ) in general.

There are problems even if we assume that $K_Y + \Delta$ is semiample. As a typical example, let $S \subset \mathbb{P}^3$ be a surface of degree ≥ 5 with a single singular point $s \in S$ that is simple elliptic (2.21.4.1). Let $S' \to S$ be the minimal resolution with exceptional curve $E' \subset S'$. Next take two copies of (S_i', E_i') and glue them along $E_1' \simeq E_2'$ to get a surface T' with normal crossing singularities. Note that $\omega_{T'}$ is generated by global sections and it maps T' to the surface T obtained by gluing two copies (S_i, s_i) at the points $s_1 \simeq s_2$. Thus T is not S_2. Here the problem is that T' is singular along the exceptional divisor of $T' \to T$. It is easy to see that this is the only obstacle in general.

Claim 8.22.1 Let $g\colon Y \to X$ be a proper, birational morphism of pure dimensional, reduced schemes. Assume that $g_*\mathscr{O}_Y = \mathscr{O}_X$, Y is S_2 and none of the g-exceptional divisors is contained in Sing Y. Then X is S_2. □

Corollary 8.22.2 Let (Y, Δ) be an slc pair such that $K_Y + \Delta$ is semiample, inducing a proper morphism $g\colon Y \to X$. Assume that g is birational and none of the g-exceptional divisors is contained in Sing Y. Then $(X, g_*\Delta)$ is slc, $K_X + g_*\Delta$ is ample and g is a *crepant* contraction. That is, $K_Y + \Delta$ is numerically g-trivial. □

We call $(X, g_*\Delta)$ the *canonical model* of (Y, Δ) and denote it by (Y^c, Δ^c).

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We stress that here we are considering only those cases for which $(Y, \Delta) \to (Y^c, \Delta^c)$ is a crepant contraction.

Lemma 8.23 *Let (Y, Δ) be an slc pair and $g\colon Y \to X$ a proper morphism such that $g_*\mathscr{O}_Y = \mathscr{O}_X$ and $K_Y + \Delta$ is numerically g-trivial. Let Θ_1, Θ_2 be effective divisors such that* $\operatorname{Supp}\Theta_i \subset \operatorname{Supp}\Delta$. *Assume that $-\Theta_1$ is g-ample and $-\Theta_2$ is g-nef. Then the following hold.*

(8.23.1) $(X, g_*\Delta)$ *is slc and g is birational.*

(8.23.2) $g_*\mathscr{O}_Y(mK_Y + \lfloor m\Delta\rfloor) = \mathscr{O}_X(mK_X + \lfloor mg_*\Delta\rfloor)$ *for every* $m \geq 1$.

(8.23.3) $-\Theta_2$ *is g-semiample.*

Proof Since $-\Theta_1$ is g-ample, $\operatorname{Ex}(g) \subset \operatorname{Supp}\Theta_1 \subset \operatorname{Supp}\Delta$. In particular, g is birational and Y is smooth at every generic point of $\operatorname{Ex}(g)$ that has codimension 1 in Y. Thus X is S_2 by (8.22.1), hence demi-normal, so (1) holds by (4.50). Next (2) follows from (11.61).

For (3), assume first that Y is normal. We apply (11.28.2) to $(Y, \Delta - \varepsilon\Theta_2)$. Set $Z = g(\Theta_2)$. Then $(X \setminus Z, g_*\Delta)$ is the canonical model of $(Y \setminus g^{-1}(Z), \Delta - \varepsilon\Theta_2)$. Since $-\Theta_2$ is g-nef, $\operatorname{Supp}\Theta_2 = g^{-1}(Z)$, hence none of the lc centers of $(Y, \Delta - \varepsilon\Theta_2)$ is contained in $g^{-1}(Z)$. Thus $K_Y + \Delta - \varepsilon\Theta_2$ is g-semiample and so is $-\Theta_2$.

In general, we can apply the above to the normalization $\bar{Y} \to Y$, get a canonical model of $(\bar{Y}, \bar{\Delta} - \varepsilon\bar{\Theta}_2)$ and then use (11.38) to conclude. □

8.24 (Stable objects) Alexeev–Filipazzi–Inchiostro stability parametrizes projective, marked, slc pairs

$$(X, \Delta = \textstyle\sum_{j\in J} b_j B^j + \sum_{i\in I} a_i D^i), \tag{8.24.1}$$

where the divisors are indexed by the disjoint union $J \cup I$. We write $\Delta_0 := \sum_{j\in J} b_j B^j$; this divisor will be treated as in KSBA stability. The new aspect is the treatment of the divisors D^i. The index set I is ordered, so we identify it with $\{1, \dots, r\}$.

The sole assumption that we would like to have is the following.

(8.24.2) $K_X + \Delta - \sum_i \varepsilon_i D^i$ is ample for all $0 < \varepsilon_1 \ll \cdots \ll \varepsilon_r \ll 1$.

Since we can choose the $a_i - \varepsilon_i$ to be $\mathbb{Q}$-linearly independent of the b_j, we see that $K_X + \Delta_0$ and the $D^1, \dots, D^r$ are necessarily $\mathbb{R}$-Cartier by (11.43).

Fixing m and $0 < \varepsilon_m \ll \cdots \ll \varepsilon_r$, letting the others go to 0 gives that $K_X + \Delta - \sum_{i=m+1}^r \varepsilon_i D^i$ is nef for $0 \leq m \leq r$. If the Abundance Conjecture holds, then these divisors are semiample, but this is not known. So for now we have to add the assumption:

(8.24.3) $K_X + \Delta$ is semiample.

An slc pair (X, Δ) as in (1) is *AFI-stable* if it satisfies assumptions (2–3).

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If (3) holds, then we have a crepant contraction to the canonical model $\pi\colon (X,\Delta) \to (X^c,\Delta^c)$. Then (2) is equivalent to the following condition.

(8.24.4) There are $\eta_{im} > 0$ such that $-\sum_{i=m}^{r} \eta_{im} D^i$ is π-nef for $1 < m < r$ and π-ample for $m = 1$.

8.25 (Explanation) By (8.23.3), the assumptions (8.24.2–3) imply that

(8.25.1) $K_X + \Delta - \sum_{i=m+1}^{r} \varepsilon_i D^i$ is semiample for $0 \leq m \leq r$.

Thus for each m we get a morphism $\pi_m\colon X \to X_m$. Then (8.23.1) shows that π_m is birational. For the rest of the section, we use a subscript m to denote the image of a divisor on X_m.

Using (8.23), we get that, for $0 < \varepsilon_m \ll \cdots \ll \varepsilon_r \ll 1$,

(8.25.2) (X_m, Δ_m) is slc and $K_{X_m} + \Delta_m - \sum_{i=m+1}^{r} \varepsilon_i D^i_m$ is ample.

In particular, we have

(8.25.3) a tower of morphisms $X =: X_0 \to X_1 \to \cdots \to X_r$ such that

(8.25.4) $K_{X_r} + \Delta_r$ is ample on X_r, and

(8.25.5) $-D^m_{m-1}$ is $(X_{m-1} \to X_m)$-ample for every m.

Repeatedly using Hartshorne (1977, exc.II.7.14) we get that (3–5) are equivalent to (8.24.2–3).

In (8.30), we show how to transform the conditions (8.24.2) involving variable $\varepsilon_i > 0$ into a set of conditions with fixed $\delta_i > 0$. The result is, however, ineffective, and it would be good to find a more constructive approach.

Note that (X_r, Δ_r) is the canonical model (X^c, Δ^c) of (X, Δ), it is thus independent of the ordering of I. By contrast, the intermediate $X \to X_m \to X^c$ do depend on the ordering of I.

8.26 (Stable families) A family $f\colon (X, \Delta = \sum_{j\in J} b_j B^j + \sum_i a_i D^i) \to S$ is *AFI-stable* if the following hold:

(8.26.1) $f\colon (X,\Delta) \to S$ is locally stable.

(8.26.2) $K_{X/S} + \sum_{j\in J} b_j B^j$ and the $D^1, \dots, D^r$ are relatively $\mathbb{R}$-Cartier.

(8.26.3) The fibers (X_s, Δ_s) are AFI-stable as in (8.24).

(8.26.4) $\omega^{[m_0]}_{X/S}(\sum_{j\in J} m_j B^j + \sum_{i\in I} n_i D^i)$ is a flat family of divisorial sheaves if $n_i \in \mathbb{Z}$ and $(m_j\colon\ j \in \{0\} \cup J) \in \mathrm{LEnv}_{\mathbb{Z}}(K_X + \sum_{j\in J} b_j B^j)$.

The next assumption may be redundant; we discuss it in (8.27) and (8.34).

(8.26.5) $f\colon (X,\Delta) \to S$ has a crepant contraction to its simultaneous canonical model. That is, to a stable morphism $f^c\colon (X^c, \Delta^c) \to S$ whose fibers are the canonical models $((X_s)^c, (\Delta_s)^c)$ of the fibers.

8.27 (Explanation) Assumptions (8.26.1–3) closely follow (8.7). Assumption (8.26.4) is modeled on (8.11) and (8.13).

The D^i are $\mathbb{Q}$-Cartier by (11.43.2), hence K-flat by (7.4.6), at least in characteristic 0. Thus if $\Delta_0 := \sum_{j\in J} b_j B^j$ is the 0 divisor, then we avoid using K-flatness entirely.

Since ampleness is an open condition (11.54), we see using (8.24.4) that, if S is Noetherian, then $K_X + \Delta - \sum_i \varepsilon_i D^i$ is f-ample for all $0 < \varepsilon_1 \ll \cdots \ll \varepsilon_r \ll 1$. Thus one can view (8.26) as picking one of the chambers discussed in (6.39).

By (8.25), each fiber (X_s, Δ_s) has a canonical model $((X_s)^c, (\Delta_s)^c)$. More generally, we prove in (8.35) that, if S is reduced, then (8.26.1–4) imply (8.26.5). This implication is not known over arbitrary bases, but we prove in (8.33) that $(X, \Delta) \to S$ uniquely determines its simultaneous canonical model.

Assumption (8.26.5) guarantees that taking the relative canonical model is a natural transformation from AFI-stable families to KSBA-stable families.

A stronger variant of (8.26.5) would be to require that the towers (8.25.3) of the fibers form a flat family. The latter might be equivalent to (8.26.5).

Theorem 8.28 *AFI-stability, as defined in (8.24–8.26) is a good moduli theory (6.10).*

We start with a general discussion on ample perturbations, followed by results on simultaneous canonical models and boundedness. With these preliminaries in place, the proof of (8.28) given in (8.37) is quite short.

Definition 8.29 A pair $(X, \Delta = \sum_{j\in J} b_j B^j + \sum_{i=1}^r a_i D^i)$ as in (8.24) has an intersection form

$$\mathcal{I}(t_0, \ldots, t_r) := (t_0(K_X + \Delta) + t_1 D^1 + \cdots + t_r D^r)^n. \tag{8.29.1}$$

For a family as in (8.26), the intersection form $\mathcal{I}(t_0, \ldots, t_r)$ is a locally constant function on the base by (8.26.2). We can thus decompose the functor of AFI-stable pairs into open and closed subfunctors

$$\mathcal{AFI}(\mathbf{b}, \mathbf{a}, n, \mathcal{I}(t_0, \ldots, t_r)). \tag{8.29.2}$$

Next we see that, for each of these subfunctors, there is a uniform choice of ample divisors.

Proposition 8.30 *Fix* $\mathbf{b}, \mathbf{a}, n$ *and* $\mathcal{I}(t_0, \ldots, t_r)$. *Then there are* $\delta_i > 0$ *such that* $(X, \Delta) \in \mathcal{AFI}(\mathbf{b}, \mathbf{a}, n, \mathcal{I}(t_0, \ldots, t_r))$ *iff the following hold.*

(8.30.1) (X, Δ) *is slc and the* D^i *are* $\mathbb{R}$*-Cartier,*

(8.30.2) $(t_0(K_X + \Delta) + t_1 D^1 + \cdots + t_r D^r)^n = \mathcal{I}(t_0, \ldots, t_r)$,

(8.30.3) $K_X + \Delta - \sum_{i=1}^r \delta_i D^i$ *is ample, and*

(8.30.4) $K_X + \Delta - \sum_{i=m+1}^r \delta_i D^i$ *is semiample for* $m = 1, \ldots, r$.

Proof If $(X, \Delta) \in \mathcal{AFI}(\mathbf{b}, \mathbf{a}, n, \mathcal{I}(t_0, \dots, t_r))$ then (1–2) hold by assumption. The key point is that one can find δ_i that do not depend on (X, Δ).

Start with the canonical model $(X_r, \Delta_r) = (X^c, \Delta^c)$. Since

$$(K_{X^c} + \Delta^c)^n = (K_X + \Delta)^n = \mathcal{I}(1, 0, \dots, 0),$$

the canonical models form a bounded family by (6.8.1). In particular, there is an $\eta > 0$ such that $((K_{X_r} + \Delta_r) \cdot C) \geq \eta$ for every nonzero effective curve $C \subset X_r$ for every X_r. Thus $((K_X + \Delta) \cdot C)$ is either 0 or $\geq \eta$ for every nonzero effective curve $C \subset X$.

Choose $0 < c_1 \ll \cdots \ll c_{r-1} \ll c_r = a_r$ such that $K_X + \Delta - \varepsilon_r \sum c_i D^i$ is ample for some $\varepsilon_r > 0$. Applying (8.31) with $\Delta_2 := \sum c_i D^i$ and $\Delta_1 := \Delta - \Delta_2$, we get a fixed δ_r such that

$$K_X + \Delta_1 + (1 - \delta_r)\Delta_2 = K_X + (\Delta - \delta_r D^r) - \textstyle\sum_{i=1}^{r-1} c_i \delta_r D^i$$

is ample. This holds for all $0 < c_1\delta_r \ll \cdots \ll c_{r-1}\delta_r \ll 1$, so $K_X + \Delta - \delta_r D^r$ is nef, so semiample by (8.23.3). By induction on r, we get the other δ_i and the divisors in (4) are nef. $K_X + \Delta$ is semiample by assumption (8.24.3). This implies the rest of (4) by (8.23.3).

Conversely, convex linear combinations of the divisors $K_X + \Delta - \sum_{i=m+1}^{r} \delta_i D^i$ for $m = 0, \dots, r$ show that (8.24.2) holds. □

Lemma 8.31 (Filipazzi and Inchiostro, 2021, 2.15) *Let* $(X, \Delta_1 + \Delta_2)$ *be a proper slc pair of dimension n. Assume that there is an* $\eta > 0$ *such that*

(8.31.1) $((K_X + \Delta_1 + \Delta_2) \cdot C)$ *is either 0, or* $\geq \eta$ *for every nonzero effective curve C, and*

(8.31.2) $K_X + \Delta_1 + (1 - \varepsilon_0)\Delta_2$ *is ample for some* $\varepsilon_0 > 0$.

Then $K_X + \Delta_1 + (1 - \varepsilon)\Delta_2$ *is ample for every* $\eta/(2n + \eta) > \varepsilon > 0$.

Proof We may assume that X is normal. If $\varepsilon_0 > \varepsilon > 0$, then $K_X + \Delta_1 + (1-\varepsilon)\Delta_2$ is a convex linear combination of $K_X + \Delta_1 + (1 - \varepsilon_0)\Delta_2$ and of $K_X + \Delta_1 + \Delta_2$, hence ample.

Thus consider the case when $\varepsilon_0 < \varepsilon$. We check Kleiman's ampleness criterion. For $Z \in \overline{NE}(X)$ we have that

$$\begin{aligned} &((K_X + \Delta_1 + (1 - \varepsilon)\Delta_2) \cdot Z) \\ &\quad = \tfrac{\varepsilon - \varepsilon_0}{1 - \varepsilon_0}((K_X + \Delta_1) \cdot Z) + \tfrac{1 - \varepsilon}{1 - \varepsilon_0}((K_X + \Delta_1 + (1 - \varepsilon_0)\Delta_2) \cdot Z). \end{aligned} \tag{8.31.3}$$

So the criterion holds on the part where $((K_X + \Delta_1) \cdot Z) \geq 0$.

By the Cone theorem of Fujino (2017, 4.6.2), the rest of $\overline{NE}(X)$ is generated by curves C_i for which $-2n \leq ((K_X + \Delta_1) \cdot C_i) < 0$. If C_i is such a curve, then, applying (8.31.3) with $\varepsilon_0 = 0$, we get that

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$$((K_X+\Delta_1+(1-\varepsilon)\Delta_2)\cdot C_i) = \varepsilon((K_X+\Delta_1)\cdot C_i)+(1-\varepsilon)((K_X+\Delta_1+\Delta_2)\cdot C_i). \quad (8.31.4)$$

Now set $\varepsilon = \varepsilon_0$. Then the left-hand side is positive, hence so is $((K_X + \Delta_1 + \Delta_2) \cdot C_i)$. Thus $((K_X + \Delta_1 + \Delta_2) \cdot C_i) \geq \eta$ by assumption. Thus (8.31.4) gives that, for every $\varepsilon > 0$, $((K_X + \Delta_1 + (1-\varepsilon)\Delta_2) \cdot C_i) \geq -2n\varepsilon + (1-\varepsilon)\eta$. The latter is positive if $\varepsilon < \eta/(2n+\eta)$. □

Definition 8.32 Let $\pi_X : X \to S$ be a flat, proper morphism with S_2 fibers. A *simultaneous contraction* is a factorization $\pi_X : X \xrightarrow{\tau} Y \to S$ where

(8.32.1) $\pi_Y : Y \to S$ is flat, proper with S_2 fibers, and

(8.32.2) $\tau_* \mathscr{O}_X = \mathscr{O}_Y$.

This implies that $(\tau_s)_* \mathscr{O}_{X_s} = \mathscr{O}_{Y_s}$ for every $s \in S$.

If the τ_s are birational, then $X \to S$ and the τ_s uniquely determine Y. When S is Artinian, this is (8.33), which in turn implies the general case.

Lemma 8.33 *Let A be a local, Artinian ring with residue field k. Let $g_k : X_k \to Y_k$ be a birational morphism between proper, pure dimensional, S_2-schemes such that $(g_k)_* \mathscr{O}_{X_k} = \mathscr{O}_{Y_k}$. Let $X_A \to \operatorname{Spec} A$ be a flat, proper morphism.*

(8.33.1) *There is at most one flat $Y_A \to \operatorname{Spec} A$ such that g_k lifts to $g_A : X_A \to Y_A$.*

(8.33.2) *If Y_A exists, then $\mathscr{O}_{Y_A} = (g_A)_* \mathscr{O}_{X_A}$.*

Proof Note that X_A, Y_A are S_2 since $X_A \to \operatorname{Spec} A$ and $Y_A \to \operatorname{Spec} A$ are flat.

Let $U_k \subset Y_k$ be the largest open set over which g_k is an isomorphism. Thus we get open sets $V_k \subset X_k$, $V_A \subset X_A$ and $U_A \subset Y_A$. Note that $Y_A \setminus U_A$ has codimension ≥ 2. Thus $\mathscr{O}_{Y_A}$ is the push-forward of $\mathscr{O}_{U_A}$ by the injection $j_A : U_A \hookrightarrow Y_A$ (10.6).

Since $g_A : V_A \to U_A$ is an isomorphism, we see that $\mathscr{O}_{Y_A} = (j_A)_*(g_A)_* \mathscr{O}_{V_A}$ is determined by X_A and g_k. This also implies that $(g_A)_* \mathscr{O}_{X_A} = \mathscr{O}_{Y_A}$. □

Definition 8.34 Let $h : (X, \Delta) \to S$ be proper and locally stable such that $K_{X/S} + \Delta$ is h-nef. A *simultaneous, canonical, crepant, birational contraction* is a simultaneous contraction $\pi_X : X \xrightarrow{\tau} Y \to S$ such that

(8.34.1) $\pi_Y : (Y, \tau_* \Delta) \to S$ is stable, and

(8.34.2) $\tau_s : (X_s, \Delta_s) \to (Y_s, g_* \Delta_s)$ is the crepant, birational contraction to its canonical model as in (8.22) for every $s \in S$.

By (8.32), $\pi_Y : (Y, \tau_* \Delta) \to S$ is uniquely determined by $\pi_X : (X, \Delta) \to S$, even when S is nonreduced.

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Using the rational approximations $\Delta_j^n \to \Delta$ as in (11.47), we see that

(8.34.3) $X \to Y \to S$ is a simultaneous canonical contraction for Δ iff it is a simultaneous canonical contraction for Δ_j^n for $n \gg 1$ for every j.

Proposition 8.35 *Let $g\colon (X,\Delta) \to S$ be proper and locally stable, S reduced. Assume that (X_s, Δ_s) has a crepant, birational contraction to its canonical model for every $s \in S$. Then $g\colon (X,\Delta) \to S$ has a simultaneous, canonical, crepant, birational contraction.*

Proof By (8.34.3) it is enough to deal with the case when Δ is a $\mathbb{Q}$-divisor.

Next we prove that $g_*(\mathscr{O}_X(mK_{X/S} + m\Delta))$ is locally free and commutes with base change for m sufficiently divisible. By Grauert's theorem (as stated in Hartshorne (1977, III.12.9)) it is enough to prove this when S is a smooth curve. In this case (11.28) and (11.38) show that the relative canonical model exists, τ_s is an isomorphism for the generic point $s \in S$ and a finite, universal homeomorphism (10.78) for closed points $s \in S$. However, (5.4) then implies that in fact τ_s is an isomorphism for every $s \in S$. Thus

$$\begin{aligned} h^0(X_s, \mathscr{O}_{X_s}(mK_{X_s} + m\Delta_s)) &= h^0((X_s)^{\mathrm{c}}, \mathscr{O}_{X_s^{\mathrm{c}}}(mK_{X_s^{\mathrm{c}}} + m\Delta_s^{\mathrm{c}})) \\ &= h^0((X^{\mathrm{c}})_s, \mathscr{O}_{X_s^{\mathrm{c}}}(m(K_{X^{\mathrm{c}}/S} + \Delta^{\mathrm{c}})_s) \end{aligned}$$

is independent of $s \in S$ for m sufficiently divisible, since $X^{\mathrm{c}} \to S$ is flat and $K_{X^{\mathrm{c}}/S} + \Delta^{\mathrm{c}}$ is relatively ample.

With arbitrary S, we get the simultaneous canonical model

$$X^{\mathrm{c}} = \mathrm{Proj}_S \oplus_{r\in\mathbb{N}} g_*(\mathscr{O}_X(rmK_{X/S} + rm\Delta)). \qquad \square$$

For representability, the key step is the following.

Proposition 8.36 *Let $g\colon (X,\Delta) \to S$ be proper and locally stable. Then there is a locally closed partial decomposition $S^{\mathrm{sccc}} \to S$ such that, for any $T \to S$, the base change $g_T\colon (X_T, \Delta_T) \to T$ has a simultaneous, canonical, crepant, birational contraction iff T factors through S^{sccc}.*

Proof As before, using (8.34.3), it is enough to deal with the case when Δ is a $\mathbb{Q}$-divisor. We may assume that S is connected.

Assume that (X_s, Δ_s) has a crepant, birational contraction to its canonical model $(X_s^{\mathrm{c}}, \Delta_s^{\mathrm{c}})$. The self-intersection of $K_{X_s^{\mathrm{c}}} + \Delta_s^{\mathrm{c}}$ equals the self-intersection of $K_{X_s} + \Delta_s$, which is independent of $s \in S$. Thus the pairs $(X_s^{\mathrm{c}}, \Delta_s^{\mathrm{c}})$ are in a bounded family by (6.8.1). In particular, there is an $m > 0$, independent of s, such that $rmK_{X_s^{\mathrm{c}}} + rm\Delta_s^{\mathrm{c}}$ is Cartier, very ample, and has no higher cohomologies for $r > 0$. Moreover, we get only finitely many possible Hilbert functions.

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Thus $\mathscr{O}_{X_s}(rmK_{X_s} + rm\Delta_s)$ is locally free, globally generated, and maps (X_s, Δ_s) to its canonical model. This implies that if $\pi_T\colon (X_T, \Delta_T) \to T$ has a simultaneous canonical, crepant contraction, then, for every $r > 0$,

(8.36.1) $\mathscr{O}_{X_T}(rmK_{X_T/T} + rm\Delta_T)$ is relatively globally generated, and

(8.36.2) $(\pi_T)_*\mathscr{O}_{X_T}(rmK_{X_T/T} + rm\Delta_T)$ is locally free and commutes with base change.

For each Hilbert function, these conditions are representable by a locally closed subscheme by (3.21). □

Remark 8.36.3 If the Abundance Conjecture holds, then $\operatorname{red} S^{\text{sccc}}$ is an open subset of $\operatorname{red} S$. The scheme-theoretic situation is not clear.

Example 8.36.4 Being semiample and big is not a constructible condition for families of line bundles. As a simple example, let $E \subset \mathbb{P}^2$ be an elliptic curve, $\pi\colon X := \operatorname{Proj}_E(\mathscr{O}_E + \mathscr{O}_E(3)) \to E$ the resolution of the cone over E. Consider the line bundles $L_X := \mathscr{O}_X(1) \otimes \pi^* L$ where $L \in \mathbf{Pic}^\circ(E)$.

Then L_X is nef and big for every $L \in \mathbf{Pic}^\circ(E)$, but semiample only if L is a torsion point of $\mathbf{Pic}^\circ(E)$. Thus the set $\{L\colon L_X \text{ is big and semiample}\}$ is not constructible.

A much subtler example of Lesieutre (2014) shows that being nef and big is also not a constructible condition in families of smooth surfaces.

8.37 (Proof of 8.28) First, we show that $\mathcal{AFI}(\mathbf{b}, \mathbf{a}, n, \mathcal{I})$ is bounded and representable. By (8.30), there are fixed $\delta_i > 0$ such that $K_X + \Delta - \sum_{i=1}^r \delta_i D^i$ is ample. The self-intersection of this divisor is $\mathcal{I}(1, a_1 - \delta_1, \ldots, a_r - \delta_r)$, hence all such stable pairs form a bounded family by (6.8.1). We can choose the $a_i - \delta_i$ to be $\mathbb{Q}$-linearly independent of the b_i. Then the D^i are $\mathbb{R}$-Cartier by (11.43).

Note that (6.8.1) gives boundedness for pairs such that $(X, \Delta - \sum_{i=1}^r \delta_i D^i)$ is slc, but we want (X, Δ) to be slc. By (7.65), local stability of (X, Δ) is a representable condition. (Actually, (11.48) shows that, for a suitable choice of the δ_i, the (X, Δ) are in fact slc.)

Semiampleness of the divisors $K_X + \Delta - \sum_{i=m+1}^r \delta_i D^i$ is a representable condition by (8.36). Representability of (8.26.4) is handled as in (6.40). (8.26.5) was treated in (8.36).

Separatedness follows from (11.40) as usual, applied to $(X, \Delta - \sum_{i=m+1}^r \delta_i D^i)$.

To see valuative-properness, assume that we have $f^\circ\colon (X^\circ, \Delta^\circ = \sum_{j\in J} b_j B^{j\circ} + \sum_{i=1}^r a_i D^{i\circ}) \to C^\circ$ over an open subset of a smooth curve $C^\circ \subset C$.

Applying (8.35) to the divisors $K_X + \Delta - \sum_{i=m+1}^r \delta_i D^i$ we also have a tower

$$f^\circ\colon (X^\circ, \Delta^\circ) \to (X_1^\circ, \Delta_1^\circ) \to \cdots \to (X_r^\circ, \Delta_r^\circ) \to C^\circ,$$

where $(X_r^\circ, \Delta_r^\circ)$ is the relative canonical model of (X°, Δ°).

First, we use (2.51) to get that, after a base change (which we suppress in the notation), $(X_r^\circ, \Delta_r^\circ) \to C^\circ$ extends to a stable morphism $(X_r, \Delta_r) \to C$.

Next we extend $(X_{r-1}^\circ, \Delta_{r-1}^\circ) \to C^\circ$. By construction, $K_{X_{r-1}^\circ} + \Delta_{r-1}^\circ - \varepsilon_r D_{r-1}^{r,\circ}$ is relatively ample on $X_{r-1}^\circ \to X_r^\circ$ for $0 < \varepsilon_r \ll 1$ and relatively semiample for $\varepsilon_r = 0$. Thus, by Hacon and Xu (2013, 1.5), after a base change (which we again suppress in the notation), it extends to a model $(X_{r-1}, \Delta_{r-1} - \varepsilon_r D_{r-1}^r) \to C$ with the same properties (with a possibly smaller upper bound for ε_r). This gives $(X_{r-1}, \Delta_{r-1}) \to C$. We can continue this until we get the tower

$$f\colon (X, \Delta) \to (X_1, \Delta_1) \to \cdots \to (X_r, \Delta_r) \to C,$$

proving valuative properness. □

8.4 Polarized Varieties

Assumptions In this section, we work with arbitrary schemes. Because of functoriality, the situation over $\operatorname{Spec} \mathbb{Z}$ determines everything.

8.38 (Ampleness conditions) Let X be a proper scheme over a field k and L a line bundle on X. The most important positivity notion is *ampleness,* but in connection with projective geometry the notion of *very ampleness* seems more relevant. If L is ample then L^r is very ample for $r \gg 1$ and there are numerous Matsusaka-type theorems that give effective control over r; see Matsusaka (1972); Lieberman and Mumford (1975); Kollár and Matsusaka (1983). In practice, this will not be a major difficulty for us.

A problem with very ampleness is that it is not open in flat families (X_s, L_s). Thus one needs to consider stronger variants. The two most frequently needed additional conditions are the following.

(8.38.1) $H^i(X, L) = 0$ for $i > 0$.

(8.38.2) $H^0(X, L)$ generates the ring $\sum_{r\geq 0} H^0(X, L^r)$.

These are connected by the notion of Castelnuovo–Mumford regularity; see Lazarsfeld (2004, sec.I.8) for details.

For our purposes the relevant issue is (1). Thus we say that a line bundle L is *strongly ample* if it is very ample and $H^i(X, L^m) = 0$ for $i, m > 0$. By Lazarsfeld (2004, I.8.3), if this holds for all $m \leq \dim X + 1$ then it holds for all m. Thus strong ampleness is an open condition in flat families.

Let $f\colon X \to S$ be a proper, flat morphism and L a line bundle on X. We say that L is *strongly f-ample* or *strongly ample over S,* if L is strongly ample on the fibers. Equivalently, if $R^i f_* L^m = 0$ for $i, m > 0$ and L is f-very ample. Thus $f_* L$ is locally free and we get an embedding $X \hookrightarrow \mathbb{P}_S(f_* L)$.

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The main case for us is when $f\colon (X,\Delta)\to S$ is stable and $L=\omega_{X/S}^{[r]}(r\Delta)$ for some $r>0$. If $r>1$ then $R^i f_*L^m=0$ for $i,m>0$ by (11.34).

Definition 8.39 (Polarization) A *polarized scheme* is a pair (X,L) consisting of a projective scheme X plus an ample line bundle L on X.

In the most basic version of the definition, a *polarized family of schemes* over a scheme S consists of a flat, projective morphism $f\colon X\to S$, plus a relatively ample line bundle L on X. (See (8.40) for other variants.)

We are interested only in the relative behavior of L, thus two families (X,L) and (X,L') are considered equivalent if there is a line bundle M on S such that $L\simeq L'\otimes f^*M$. There are some quite subtle issues with this in general Raynaud (1970), but if S is reduced and $H^0(X_s,\mathcal{O}_{X_s})\simeq k(s)$ for every $s\in S$, then $L\simeq L'\otimes f^*M$ for some M iff $L|_{X_s}\simeq L'|_{X_s}$ for every $s\in S$ by Grauert's theorem, as in Hartshorne (1977, III.12.9). See also (8.40) for further comments on this.

For technical reasons, it is more convenient to deal with the cases when, in addition, L is strongly f-ample (8.38). We call such an L a *strong polarization.* Thus the "naive" functor of strongly polarized schemes

$$S\mapsto \mathcal{P}^s\mathcal{S}ch(n,N)(S) \tag{8.39.1}$$

associates to a scheme S the equivalence classes of all $f\colon (X,L)\to S$ such that

(8.39.2) f is flat, proper, of pure relative dimension n,

(8.39.3) X_s is pure and $H^0(X_s,\mathcal{O}_{X_s})\simeq k(s)$ for every $s\in S$,

(8.39.4) L is strongly f-ample (8.38), and

(8.39.5) f_*L is locally free of rank $N+1$.

Since L is flat over S, strong f-ampleness implies that f_*L is locally free.

(8.39.6) If we fix the whole Hilbert polynomial $\chi(X,r):=\chi(X,L^r)$, we get the functor $S\mapsto \mathcal{P}^s\mathcal{S}ch(\chi)(S)$.

Let $f\colon X\to S$ be a flat, proper morphism and L a line bundle on X. Having pure fibers is an open condition (10.11) and then pure dimensionality is an open condition. Thus there is a maximal open subscheme $S^\circ\subset S$ such that $f^\circ\colon (X^\circ,L^\circ)\to S^\circ$ satisfies the assumptions (2–5).

Definition 8.40 (Pre-polarization) The definition in (8.39) is geometrically clear, but it does not have the sheaf property. In analogy with the notion of a presheaf, we could define a *pre-polarization* of a projective morphism $f\colon X\to S$ to consist of

(8.40.1) an open cover $\cup_i U_i\to S$, and

(8.40.2) relatively ample line bundles L_i on $X_i:=X\times_S U_i$ such that,

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(8.40.3) for every i, j, the restrictions of L_i and L_j to $X_{ij} := X \times_S U_i \times_S U_j$ are identified as in (8.39).

(That is, there are line bundles M_{ij} on $U_i \times_S U_j$ such that $L_i|_{X_{ij}} \simeq L_j|_{X_{ij}} \otimes f_{ij}^* M_{ij}$.)

Pre-polarizations form a presheaf, hence the "right" notion of polarization should be a global section of the corresponding sheaf.

If $\cup_i U_i \to S$ is a cover by Zariski open subsets, the resulting notion is very similar to what we have in (8.39). The only difference is in property (8.39.5) since f_*L need not exists globally. However, $\mathbb{P}_S(f_*L)$ does exist as a Zariski locally trivial $\mathbb{P}^N$-bundle over S and we usually use $\mathbb{P}_S(f_*L)$ anyhow.

If the $U_i \to S$ are étale, then we still get an object $\mathbb{P}_S(f_*L) \to S$, but this is a Severi–Brauer scheme, that is, an étale locally trivial $\mathbb{P}^N$-bundle over S. (See (8.40.5) for an example with $N = 1$.) From the theoretical point of view, it is most natural to use the étale topology for the moduli theory of varieties. Pre-polarizations define a pre-sheaf in the étale topology and sheafifying gives the functors

$$S \mapsto \mathcal{P}^s\mathcal{S}ch^{\mathrm{et}}(n, N)(S) \quad \text{and} \quad S \mapsto \mathcal{P}^s\mathcal{S}ch^{\mathrm{et}}(\chi)(S). \tag{8.40.4}$$

(For arbitrary schemes one needs finer topologies; see Raynaud (1970).)

For the difference between $\mathcal{P}^s\mathcal{S}ch^{\mathrm{et}}$ and $\mathcal{P}^s\mathcal{S}ch$, a simple example to keep in mind is the following. Consider

$$X := (x^2 + sy^2 + tz^2 = 0) \subset \mathbb{P}^2_{xyz} \times (\mathbb{A}^2_{st} \setminus (st = 0)), \tag{8.40.5}$$

with coordinate projection to $S := \mathbb{A}^2_{st} \setminus (st = 0)$. The fibers are all smooth conics. In the analytic or étale topology, there is a pre-polarization whose restriction to each fiber is a degree 1 line bundle, but there is no such line bundle on X. However, $\mathcal{O}_{\mathbb{P}^2}(1)$ gives a line bundle on X whose restriction to each fiber has degree 2.

We will, however, stick to the naive versions for several reasons.

- Stable families come with preferred polarizing line bundles $\omega_{X/S}^{[m]}(m\Delta)$.
- $\mathcal{P}^s\mathcal{S}ch^{\mathrm{et}}$ and $\mathcal{P}^s\mathcal{S}ch$ have the same coarse moduli spaces (8.56.1).
- A suitable power of any pre-polarization naturally gives an actual polarization using (8.66.6).

So, at the end, the distinction between the functors $\mathcal{P}^s\mathcal{S}ch^{\mathrm{et}}$ and $\mathcal{P}^s\mathcal{S}ch$ does not matter much for us. There is, however, another related notion that does lead to different coarse moduli spaces.

8.40.7 (*Numerical polarization*) Given $f: X \to S$, two relatively ample line bundles L and L' on X are considered equivalent if $L_s \equiv L'_s$ (p.xv) for every geometric point $s \to S$. This is the original definition used by Matsusaka

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(1972) and it may be the most natural notion for general polarized pairs. Stable varieties come with an ample divisor, not just with an ample numerical equivalence class, which simplifies our task.

8.41 (Strongly embedded schemes) Fix a projective space $\mathbb{P}^N_{\mathbb{Z}}$. Over the Hilbert scheme there is a universal family, hence we get

$$\mathrm{Univ}(\mathbb{P}^N_{\mathbb{Z}}) \subset \mathbb{P}^N_{\mathbb{Z}} \times \mathrm{Hilb}(\mathbb{P}^N_{\mathbb{Z}}), \tag{8.41.1}$$

and $\mathscr{O}_{\mathbb{P}^N}(1)$ gives a polarization of $\mathrm{Univ}(\mathbb{P}^N_{\mathbb{Z}}) \to \mathrm{Hilb}(\mathbb{P}^N_{\mathbb{Z}})$. As in (8.39) there is a largest open subset

$$\mathrm{Hilb}^{\mathrm{str}}_n(\mathbb{P}^N_{\mathbb{Z}}) \subset \mathrm{Hilb}_n(\mathbb{P}^N_{\mathbb{Z}}), \tag{8.41.2}$$

over which the polarization is strong (8.39.2–5). One should think of this as pairs (X, L) that "naturally live" in $\mathbb{P}^N$. The universal family restricts to

$$\mathrm{Univ}^{\mathrm{str}}_n(\mathbb{P}^N_{\mathbb{Z}}) \to \mathrm{Hilb}^{\mathrm{str}}_n(\mathbb{P}^N_{\mathbb{Z}}). \tag{8.41.3}$$

The corresponding functor associates to a scheme S the set of all flat families of closed subschemes of pure dimension n of $\mathbb{P}^N_S$

$$f\colon (X \subset \mathbb{P}^N_S; \mathscr{O}_X(1)) \to S, \tag{8.41.4}$$

where $\mathscr{O}_X(1)$ is strongly f-ample. Equivalently, we parametrize objects

$$\big(f\colon (X; L) \to S; \phi \in \mathrm{Isom}_S(\mathbb{P}_S(f_*L), \mathbb{P}^N_S)\big), \tag{8.41.5}$$

consisting of strongly polarized, flat families of purely n-dimensional schemes, plus an isomorphism $\phi\colon \mathbb{P}_S(f_*L) \simeq \mathbb{P}^N_S$. We call the latter a *projective framing* of f_*L or of L. We can also fix the Hilbert polynomial χ of X and, for $N := \chi(1) - 1$ consider the subschemes

$$\mathrm{Univ}^{\mathrm{str}}_\chi(\mathbb{P}^N_{\mathbb{Z}}) \to \mathrm{Hilb}^{\mathrm{str}}_\chi(\mathbb{P}^N_{\mathbb{Z}}) \subset \mathrm{Hilb}^{\mathrm{str}}_n(\mathbb{P}^N_{\mathbb{Z}}). \tag{8.41.6}$$

By the theory of Hilbert schemes, the spaces $\mathrm{Hilb}^{\mathrm{str}}_\chi(\mathbb{P}^N_{\mathbb{Z}})$ are quasi-projective, though usually non-projective, reducible and disconnected; see Grothendieck (1962), Kollár (1996, chap.I), or Sernesi (2006).

We can summarize these discussions as follows.

Proposition 8.42 *Fix a polynomial $\chi(t)$. Then*

$$\mathrm{Univ}^{str}_\chi(\mathbb{P}^N_{\mathbb{Z}}) \to \mathrm{Hilb}^{str}_\chi(\mathbb{P}^N_{\mathbb{Z}})$$

constructed in (8.41) represents the functor of strongly polarized schemes with Hilbert polynomial χ and a projective framing. That is, for every scheme S, pull-back gives a one-to-one correspondence between

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(8.42.1) $\mathrm{Mor}_{\mathbb{Z}}(S, \mathrm{Hilb}^{str}_{\chi}(\mathbb{P}^N_{\mathbb{Z}}))$, *as in (8.63), and*

(8.42.2) *flat, projective families of purely n-dimensional schemes* $f\colon X \to S$ *with a strong polarization L of Hilbert polynomial* χ, *plus an isomorphism* $\mathbb{P}_S(f_*L) \simeq \mathbb{P}^N_S$, *where* $N+1 = \chi(1)$. □

The general correspondence between the moduli of polarized varieties and the moduli of embedded varieties (8.56.1) gives now the following.

Corollary 8.43 *Fix a Hilbert polynomial* χ *with* $N+1=\chi(1)$. *Then the stack* $[\mathrm{Hilb}^{str}_{\chi}(\mathbb{P}^N)/\mathrm{PGL}_{N+1}]$ *represents the functor* $\mathcal{P}^s\mathcal{S}ch^{et}(\chi)$ *defined in (8.40.3).* □

8.44 (Marking points) So far we have studied varieties with marked divisors on them. It is sometimes useful to also mark some points. For curves, the points are also divisors and they interact with the log canonical structure. By contrast, in dimension ≥ 2, the points and the log canonical structure are independent of each other. This makes the resulting notion much less interesting theoretically, but it gives a quick way to rigidify slc pairs, which was quite useful in Section 5.9.

A flat family of *r-pointed* schemes is a flat morphism $f\colon X \to S$ plus r sections $\sigma_i\colon S \to X$. This gives a functor of r-pointed schemes.

Consider the Hilbert scheme with its universal family $\mathrm{Univ}(\mathbb{P}^N) \to \mathrm{Hilb}(\mathbb{P}^N)$. Then the r-fold fiber product

$$\mathrm{Univ}(\mathbb{P}^N) \times_{\mathrm{Hilb}(\mathbb{P}^N)} \mathrm{Univ}(\mathbb{P}^N) \cdots \times_{\mathrm{Hilb}(\mathbb{P}^N)} \mathrm{Univ}(\mathbb{P}^N)$$

represents the functor of r-pointed subschemes of $\mathbb{P}^N$. More generally, for any functor that is representable by a flat universal family $\mathrm{Univ}_M \to M$, its r-pointed version is representable by the r-fold fiber product of Univ_M over M.

In particular, we get MpSP, the moduli of pointed stable pairs.

8.5 Canonically Embedded Pairs

Assumptions In this section, we work with arbitrary schemes. As before, the situation over Spec $\mathbb{Z}$ determines everything.

Definition 8.45 A *strongly polarized family of schemes marked with K-flat divisors* is written as

(8.45.1) $f\colon (X; D^1, \ldots, D^r; L) \to S$, where

(8.45.2) $f\colon X \to S$ satisfies (8.39.2–5),

(8.45.3) the D^i are K-flat families of relative Mumford divisors (7.1), and

(8.45.4) L is strongly f-ample (8.39).

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If we fix the relative dimension and the rank of f_*L, then, as in (8.39.6), we get the functor

$$\mathcal{P}^s\mathcal{MS}ch(r, n, N). \tag{8.45.5}$$

We write $\mathcal{P}^s\mathcal{MS}ch(r, \chi)$ if the Hilbert polynomial $\chi = \chi(X_s, L_s^m)$ of L is also fixed. These can also be sheafified in the étale topology as in (8.40.3). (The notation does not indicate K-flatness; but it has enough letters in it already.)

The embedded version is denoted by

$$\mathcal{E}^s\mathcal{MS}ch(r, n, \mathbb{P}^N). \tag{8.45.6}$$

These functors associate to a scheme S the set of all families of closed subschemes of a given $\mathbb{P}^N_S$ (where $N = \chi(1) - 1$) marked with K-flat divisors

$$f\colon (X \subset \mathbb{P}^N_S; D^1, \dots, D^r; \mathscr{O}_X(1)) \to S, \tag{8.45.7}$$

where $\mathscr{O}_X(1)$ is strongly ample.

Equivalently, we can view $\mathcal{E}^s\mathcal{MS}ch(r, n, \mathbb{P}^N)$ as parametrizing objects

$$\big(f\colon (X; D^1, \dots, D^m; L) \to S\,;\phi \in \operatorname{Isom}_S(\mathbb{P}_S(f_*L), \mathbb{P}^N_S)\big) \tag{8.45.8}$$

consisting of a strongly polarized family of schemes marked with K-flat divisors, plus a projective framing $\phi\colon \mathbb{P}_S(f_*L) \simeq \mathbb{P}^N_S$ as in (8.41.5).

8.46 (Universal family of strongly embedded, marked schemes) Fix a projective space $\mathbb{P}^N_{\mathbb{Z}}$ and integers $n \geq 1$ and $r \geq 0$. By (8.41) we have a universal family of strongly embedded schemes

$$\operatorname{Univ}^{\rm str}_n(\mathbb{P}^N_{\mathbb{Z}}) \to \operatorname{Hilb}^{\rm str}_n(\mathbb{P}^N_{\mathbb{Z}}) \tag{8.46.1}$$

satisfying (8.39.2–5). The universal family of K-flat, Mumford divisors

$$\operatorname{KDiv}(\operatorname{Univ}^{\rm str}_n(\mathbb{P}^N_{\mathbb{Z}})/\operatorname{Hilb}^{\rm str}_n(\mathbb{P}^N_{\mathbb{Z}})) \to \operatorname{Hilb}^{\rm str}_n(\mathbb{P}^N_{\mathbb{Z}})$$

was constructed in (7.3). If we need r such divisors, the base of the universal family we want is the r-fold fiber product

$$\mathrm{E^sMSch}(r, n, \mathbb{P}^N_{\mathbb{Z}}) := \times^r_{\operatorname{Hilb}_n(\mathbb{P}^N_{\mathbb{Z}})} \operatorname{KDiv}(\operatorname{Univ}^{\rm str}_n(\mathbb{P}^N_{\mathbb{Z}})/\operatorname{Hilb}^{\rm str}_n(\mathbb{P}^N_{\mathbb{Z}})). \tag{8.46.2}$$

We denote the universal family by

$$\mathbf{F}\colon (\mathbf{X}, \mathbf{D}^1, \dots, \mathbf{D}^r; \mathbf{L}) \to \mathrm{E^sMSch}(r, n, \mathbb{P}^N_{\mathbb{Z}}), \tag{8.46.3}$$

where we really should have written the rather cumbersome

$$\big(\mathbf{X}(r, n, \mathbb{P}^N_{\mathbb{Z}}), \mathbf{D}^1(r, n, \mathbb{P}^N_{\mathbb{Z}}), \dots, \mathbf{D}^r(r, n, \mathbb{P}^N_{\mathbb{Z}}); \mathbf{L}(r, n, \mathbb{P}^N_{\mathbb{Z}})\big).$$

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It is clear from the construction that the spaces $\mathrm{E^sMSch}(r, n, \mathbb{P}^N_{\mathbb{Z}})$ parametrize polarized families of schemes marked with divisors, equipped with an extra framing.

Proposition 8.47 *Fix r, n, N. Then the scheme of embedded, marked schemes $\mathrm{E^sMSch}(r, n, \mathbb{P}^N_{\mathbb{Z}})$ constructed in (8.46.3) represents $\mathcal{E}^s\mathcal{MS}ch(r, n, \mathbb{P}^N_{\mathbb{Z}})$, defined in (8.45). That is, for every $\mathbb{Z}$-scheme S, pulling back the family (8.46.3) gives a one-to-one correspondence between*

(8.47.1) $\mathrm{Mor}_{\mathbb{Z}}(S, \mathrm{E^sMSch}(r, n, \mathbb{P}^N_{\mathbb{Z}}))$, *and*

(8.47.2) *families $f\colon (X; D^1, \dots, D^r; L) \to S$ of n-dimensional schemes, with a strong polarization and marked with K-flat Mumford divisors, plus a projective framing $\mathbb{P}_S(f_*L) \simeq \mathbb{P}^N_S$.* □

As in (8.43) and (8.56.1), this implies the following.

Corollary 8.48 *Fix n, m, N. Then the stack $[\mathrm{E^sMSch}(r, n, \mathbb{P}^N_{\mathbb{Z}})/\mathrm{PGL}_{N+1}]$ represents the functor $\mathcal{P}^s\mathcal{MS}ch(r, n, N)$, defined in (8.45).* □

8.49 (Boundedness conditions) The schemes $\mathrm{E^sMSch}(r, n, \mathbb{P}^N)$ have infinitely many irreducible components since we have not fixed the degrees of X and of the divisors D^i. Set

$$\deg_L(X; D^1, \dots, D^r) := (\deg_L X, \deg_L D^1, \dots, \deg_L D^r) \in \mathbb{N}^{r+1}. \tag{8.49.1}$$

This multidegree is a locally constant function on $\mathrm{E^sMSch}(r, n, \mathbb{P}^N)$, hence its level sets give a decomposition

$$\mathrm{E^sMSch}(r, n, \mathbb{P}^N) = \amalg_{\mathbf{d}\in\mathbb{N}^{r+1}} \mathrm{E^sMSch}(r, n, \mathbf{d}, \mathbb{P}^N). \tag{8.49.2}$$

The schemes $\mathrm{E^sMSch}(r, n, \mathbf{d}, \mathbb{P}^N)$ are still not of finite type since the fibers are allowed to be nonreduced. However, the subscheme

$$\mathrm{E^sMV}(r, n, \mathbf{d}, \mathbb{P}^N) \subset \mathrm{E^sMSch}(r, n, \mathbf{d}, \mathbb{P}^N), \tag{8.49.3}$$

which parametrizes geometrically reduced fibers, is quasi-projective, though usually non-projective, reducible, and disconnected.

Definition 8.50 A family of marked pairs $f\colon (X, \Delta) \to S$ as in (8.4) is *m-canonically strongly polarized* if

(8.50.1) $\omega_{X/S}$ is locally free outside a codimension ≥ 2 subset of each fiber,

(8.50.2) $\omega^{[m]}_{X/S}(m\Delta)$ is a line bundle, and

(8.50.3) $\omega^{[m]}_{X/S}(m\Delta)$ is strongly f-ample.

If $X \subset \mathbb{P}^N_S$ then $f\colon (X,\Delta) \to S$ is *m-canonically strongly embedded* if, in addition,

(8.50.4) $\omega^{[m]}_{X/S}(m\Delta) \simeq \mathscr{O}_{\mathbb{P}^N}(1) \otimes f^* M_S$ for some line bundle M_S on S.

These define the functors $\mathcal{C}^m\mathcal{P}^s\mathcal{M}\mathcal{S}ch$ and $\mathcal{C}^m\mathcal{E}^s\mathcal{M}\mathcal{S}ch$.

Theorem 8.51 *Fix $m, n, N \in \mathbb{N}$ and a rational vector $\mathbf{a} = (a_1, \dots, a_r)$. Then the functor $\mathcal{C}^m\mathcal{E}^s\mathcal{M}\mathcal{S}ch(\mathbf{a}, n, \mathbb{P}^N)$ is represented by a monomorphism*

$$\mathrm{C^mE^sMSch}(\mathbf{a}, n, \mathbb{P}^N_{\mathbb{Z}}) \to \mathrm{E^sMSch}(r, n, \mathbb{P}^N_{\mathbb{Z}})$$

Proof Start with the universal family, as in (8.46.3),

$$\mathbf{F}\colon (\mathbf{X}, \mathbf{D}^1, \dots, \mathbf{D}^r; \mathbf{L}) \to \mathrm{E^sMSch}(r, n, \mathbb{P}^N_{\mathbb{Z}}).$$

Note that (8.50.1) is an open condition and it holds iff ω_{X_s} is locally free outside a closed subset of codimension ≥ 2 of X_s for every $s \in S$. Being a line bundle is representable by (3.30) and, once it holds, being a strong polarization is an open condition. Applying (3.22) to $\omega^{[m]}_{X/S}(m\Delta)(-1)$ shows that condition (8.50.4) is representable. □

By (4.45), if $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier, then the stable fibers are parametrized by an open subset, at least in characteristic 0. Thus we get the following.

Corollary 8.52 *Fix $m, n, N \in \mathbb{N}$ and a rational vector $\mathbf{a} = (a_1, \dots, a_r)$. Then, over* Spec $\mathbb{Q}$*, there is an open subscheme*

$$\mathrm{C^mESP}(\mathbf{a}, n, \mathbb{P}^N_{\mathbb{Q}}) \subset \mathrm{C^mE^sMSch}(\mathbf{a}, n, \mathbb{P}^N_{\mathbb{Q}}),$$

representing the functor of m-canonically, strongly embedded, stable families.

Warning 8.52.1 The reduced subspace of $\mathrm{C^mESP}$ is the correct one, but its scheme structure is still a little too large. The reason is that (8.7.3) imposes restrictions on $\omega^{[r]}_{X/S}(r\Delta)$ for various values of r, but we took care only of our chosen m (and its multiples).

We dropped the superscript from E^s since, as we noted in (8.38), an m-canonical polarization is automatically strong.

8.6 Moduli Spaces as Quotients by Group Actions

Notation 8.53 For a scheme S, we use $\mathrm{PGL}_n(S)$ to denote the group scheme PGL_n over S. We will formulate definitions and results for general algebraic group schemes whenever possible, but in the applications we use only PGL_n, which is smooth and geometrically reductive.

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Keep in mind that, if k is a field, then, in the literature, $\mathrm{PGL}_n(k)$ usually denotes the k-points of the group scheme PGL_n, not $\mathrm{PGL}_n(\operatorname{Spec} k)$. It is customary to use PGL_n to denote $\mathrm{PGL}_n(\operatorname{Spec}\mathbb{Z})$ if we work with arbitrary schemes and $\mathrm{PGL}_n(\operatorname{Spec}\mathbb{Q})$ if we work in characteristic 0.

8.54 (Comment on algebraic spaces) We will consider quotients of schemes by algebraic groups, primarily PGL_n. It turns out that in many cases such quotients are not schemes, but algebraic spaces. For this reason, it is natural to formulate the basic definitions using algebraic spaces.

In our cases, these quotients turn out to be schemes, even projective, but this is not easy to prove.

In any case, this means that the reader can substitute "scheme" for "algebraic space" in the sequel, without affecting the final theorems.

Definition 8.55 An action of an algebraic group scheme G on an algebraic space X is a morphism $\sigma\colon G\times X\to X$ that satisfies the scheme-theoretic version of the condition $g_1(g_2(x)) = (g_1g_2)(x)$. That is, the diagram

$$\begin{array}{ccc} G\times G\times X & \xrightarrow{1_G\times\sigma} & G\times X \\ \downarrow{\scriptstyle m\times 1_X} & & \downarrow{\scriptstyle\sigma} \\ G\times X & \xrightarrow{\sigma} & X \end{array}$$

commutes. If G acts on X_1, X_2 then $\pi\colon X_1\to X_2$ is a G-morphism if the following diagram commutes:

$$\begin{array}{ccc} G\times X_1 & \xrightarrow{\sigma_1} & X_1 \\ \downarrow{\scriptstyle 1_G\times\pi} & & \downarrow{\scriptstyle\pi} \\ G\times X_2 & \xrightarrow{\sigma_2} & X_2. \end{array}$$

The *categorical quotient* is a G-morphism $q\colon X\to Y$ such that the G-action is trivial on Y and q is universal among such.

Fix N and consider the functor $\mathcal{P}^s\mathcal{S}ch(N)$ of strongly polarized schemes of embedding dimension N. By (8.42), its embedded version has a moduli space with a universal family $\mathrm{Univ}^{\mathrm{str}}(\mathbb{P}^N)\to \mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N)$. The connection between the two versions is the following impressive sounding, but quite simple claim.

Theorem 8.56 *The categorical quotient* $\mathrm{Hilb}^{str}(\mathbb{P}^N)/\mathrm{PGL}_{N+1}$ *is also the categorical moduli space* $\mathrm{P^sSch}(n, N)$.

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Proof We have a universal family over $\mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N)$, so we get $\mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N) \to \mathrm{P^sSch}(n, N)$ which is PGL_{N+1}-equivariant.

Conversely, let $f\colon (X, L) \to S$ be a family in $\mathcal{P}^s\mathcal{S}ch(n, N)$. Then f_*L is locally free of rank $N + 1$ on S, hence S has an open cover $S = \cup S_i$ such that each $f_*L|_{S_i}$ is free. Choosing a trivialization gives embedded families, hence morphisms $\phi_i\colon S_i \to \mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N)$. Over $S_i \cap S_j$ we have two different trivializations, these differ by a section of $g_{ij} \in H^0(S_i \cap S_j, \mathrm{PGL}_{N+1})$. Thus, composing with the quotient map $q\colon \mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N) \to \mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N)/\mathrm{PGL}_{N+1}$ we get that $q \circ (\phi_i|_{S_i \cap S_j}) = q \circ (g_{ij}(\phi_j|_{S_i \cap S_j})) = q \circ (\phi_j|_{S_i \cap S_j})$, since q is PGL_{N+1}-equivariant. Thus the $q \circ \phi_i$ glue to a morphism $\phi\colon S \to \mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N)/\mathrm{PGL}_{N+1}$. □

Remark 8.56.1 Since one can glue a morphism from étale charts, we see that $\mathcal{P}^s\mathcal{S}ch^{\mathrm{et}}$ and $\mathcal{P}^s\mathcal{S}ch$ have the same categorical moduli spaces (8.40.5). For those conversant with stacks, this argument proves (8.43) and (8.48).

The same proof applies to pairs and we get the following.

Corollary 8.57 *Fix $m, n, N \in \mathbb{N}$ and a rational vector $\mathbf{a} = (a_1, \ldots, a_r)$. Then the categorical quotient $\mathrm{C^mESP}(\mathbf{a}, n, \mathbb{P}^N_{\mathbb{Q}})/\mathrm{PGL}_{N+1}$ is also the categorical moduli space of $\mathcal{SP}(\mathbf{a}, n, *, m, N)$, the functor of stable families that have an m-canonical, strong embedding into $\mathbb{P}^N$.* □

Existence of Quotients

Let G be an algebraic group acting on an algebraic space X. Under very mild conditions, the categorical quotient X/G exists, but it may be very degenerate. For example, consider $\mathbb{A}^n_k$ with the scalar $\mathbb{G}_m$-action $x_i \mapsto \lambda x_i$. Then $\mathbb{A}^n/\mathbb{G}_m = \operatorname{Spec} k$, but $(\mathbb{A}^n \setminus \{\mathbf{0}\})/\mathbb{G}_m = \mathbb{P}^{n-1}$. Note that here the stabilizer is $\mathbb{G}_m$ for the origin, but trivial for every other point. This and many other examples suggest that points with infinite stabilizers cause problems.

With PGL_{N+1} acting on the Hilbert scheme, the stabilizer of the point $[X]$ corresponding to a strongly embedded $X \subset \mathbb{P}^N$ is the automorphism group of the polarized scheme $(X, \mathscr{O}_X(1))$. As we saw in Section 1.8, infinite automorphism groups cause many problems.

We get the best results if all automorphism groups are trivial; we discuss these in Section 8.7. For stable pairs the automorphism groups are finite, but we need a scheme-theoretic version of this.

Definition 8.58 (Proper action) Let $\sigma\colon G \times X \to X$ be an algebraic group scheme acting on an algebraic space X. Combining σ with the coordinate projection to X gives $(\sigma, \pi_X)\colon G \times X \longrightarrow X \times X$. The action is called *proper* if (σ, π_X) is proper and called *free* if (σ, π_X) is a closed embedding. Note that

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the preimage of a diagonal point (x, x) is the stabilizer of x. Thus free implies that all stabilizers are trivial and, if G is affine (for example PGL), then proper implies that all stabilizers are finite. (The converses are, however, not true; see Mumford (1965, p.11).)

Assume that $X \subset \mathrm{C^mESP}(\mathbf{a}, n, \mathbb{P}^N)$ parametrizes pluricanonically embedded stable subvarieties in $\mathbb{P}^N$ and $G = \mathrm{PGL}_{N+1}$. We claim that the properness of the PGL_{N+1}-action is equivalent to the uniqueness of stable extensions considered in (2.50) (and called separatedness there).[1]

Over X, we have a universal family $Y \to X$. Let T be the spectrum of a DVR with generic point η and $(q_1, q_2)\colon T \to X \times X$ a morphism. Thus the $q_i^*Y \to T$ give families of stable varieties over T. The generic point η lifts to $G \times T$ iff there is a $g(\eta) \in G_\eta$ such that $q_1(\eta) = \sigma(g(\eta), q_2(\eta))$. Equivalently, if the generic fibers $(q_1^*Y)_\eta$ and $(q_2^*Y)_\eta$ are isomorphic, (2.50) then says that the families q_1^*Y and q_2^*Y are isomorphic. This isomorphism gives $q_G\colon T \to G$ and $(q_G, q_2)\colon T \to G \times X$ shows that the valuative criterion of properness holds for $G \times X \to X \times X$.

Now we come to the definition of the right class of quotients.

Definition 8.59 (Mumford, 1965, p.4) Let G be an algebraic group scheme acting on an algebraic space X with categorical quotient $q\colon X \to X/G$ (8.55). It is called a *geometric quotient* if

(8.59.1) $q(K)\colon X(K)/G(K) \to (X/G)(K)$ is a bijection of sets, whenever K is algebraically closed,

(8.59.2) q is of finite type and universally surjective, and

(8.59.3) $\mathscr{O}_{X/G} = (q_*\mathscr{O}_X)^G$.

The geometric quotient is denoted by $X/\!/G$.

The fundamental theorem for the existence of geometric quotients is the following. Seshadri (1962/1963, 1972) came close to proving it. His ideas were developed in Kollár (1997) to settle many cases, including PGL that we need. The general case was treated in Keel and Mori (1997); see Olsson (2016) for a thorough treatment.

Theorem 8.60 *Let G be a flat group scheme acting properly on an algebraic space X. Then the geometric quotient X//G exists.* □

For free actions, the quotient map is especially simple. Over fields, this is proved in Mumford (1965, prop.0.9). The general case follows from Stacks (2022, tag 0CQJ).

[1] This clash of terminologies is, unfortunately, well entrenched.

Complement 8.61 *Assume in addition that the G-action is free on X. Then $X \to X//G$ is a principal G-bundle.* □

For us the main application is the following.

Theorem 8.62 *Fix $m, n, N \in \mathbb{N}$ and a rational vector $\mathbf{a} = (a_1, \dots, a_r)$. Then the PGL_{N+1}-action on $\mathrm{C^mESP}(\mathbf{a}, n, \mathbb{P}^N_{\mathbb{Q}})$ (8.52) is proper.*

Thus the geometric quotient $\mathrm{C^mESP}(\mathbf{a}, n, \mathbb{P}^N_{\mathbb{Q}})//\mathrm{PGL}_{N+1}$ exists and it is the coarse moduli space of $\mathcal{SP}(\mathbf{a}, n, *, m, N)$, the functor of stable families that have an m-canonical, strong embedding into $\mathbb{P}^N$.

Proof For simplicity, write Univ → $\mathrm{C^mESP}$ for the universal family over $\mathrm{C^mESP}(\mathbf{a}, n, \mathbb{P}^N_{\mathbb{Q}})$. Following (8.58), we need to show that

$$\mathrm{PGL}_{N+1} \times \mathrm{C^mESP} \longrightarrow \mathrm{C^mESP} \times \mathrm{C^mESP} \tag{8.62.1}$$

is proper. First, we claim that (8.62.1) is isomorphic to

$$\mathbf{Isom}(\pi_1^* \,\mathrm{Univ}, \pi_2^* \,\mathrm{Univ}) \longrightarrow \mathrm{C^mESP} \times \mathrm{C^mESP}, \tag{8.62.2}$$

where the $\pi_i\colon \mathrm{C^mESP} \to \mathrm{C^mESP} \times \mathrm{C^mESP}$ are the coordinate projections. This is simply the statement that giving a stable pair (X, Δ) plus two m-canonical embeddings into $\mathbb{P}^N$ is the same as giving one m-canonical embedding into $\mathbb{P}^N$ plus an element of PGL_{N+1}.

The properness of (8.62.2) follows from (8.64). The rest then follow from (8.60) and (8.57). □

8.63 (Morphism schemes) For S-schemes X, Y let $\mathrm{Mor}_S(X, Y)$ be the set of morphisms that commute with projections to S. We get the functor of morphisms on S-schemes $T \mapsto \mathrm{Mor}_T(X_T, Y_T)$.

Claim 8.63.1 Assume that $X \to S$ is flat, proper and $Y \to S$ is of finite type. Then the functor of morphisms is representable by a scheme $\mathbf{Mor}_S(X, Y)$.

Proof We can identify a morphism with its graph, which is in $\mathrm{Hilb}_S(X \times_S Y)$ since $X \to S$ is flat. Conversely, a subscheme $Z \subset X \times_S Y$ is the graph of a morphism iff the first projection $\pi_X\colon Z \to X$ is finite and $(\pi_X)_* \mathscr{O}_Z \simeq \mathscr{O}_X$. The first of these is always an open condition, for the second we need the flatness of $Z \to S$ (10.54). □

We also get sets $\mathrm{Isom}_S(X, Y)$, $\mathrm{Aut}_S(X)$ and schemes $\mathbf{Isom}_S(X, Y)$ $\mathbf{Aut}_S(X)$. that represent the functor of isomorphisms (resp. automorphisms). The identity is always in automorphism, thus we have the identity section $S \subset \mathbf{Aut}_S(X)$. We say that X is *rigid* (over S) if $S = \mathbf{Aut}_S(X)$.

The definitions of Mor, Isom, Aut and **Mor**, **Isom**, **Aut** also apply to pairs.

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With the definition of stable families in place, we get the following consequence of (11.40) about isomorphism schemes.

Proposition 8.64 *Let $f_i\colon (X_i, \Delta_i) \to S$ be stable morphisms. Then the structure map* $\mathbf{Isom}_S((X_1, \Delta_1), (X_2, \Delta_2)) \to S$ *is finite.*

Proof Choose m such that the divisors $m(K_{X_i/S} + \Delta_i)$ are very f_i-ample. Set $F_i := (f_i)_*\mathscr{O}_{X_i}(mK_{X_i/S} + m\Delta_i)$. Then

$$\mathbf{Isom}_S((X_1, \Delta_1), (X_2, \Delta_2)) \subset \mathbf{Isom}_S(\mathbb{P}_S(F_1), \mathbb{P}_S(F_2))$$

is closed, hence affine over S.

Let T be the spectrum of a DVR over k with generic point t_g and $\phi_g\colon t_g \to \mathbf{Isom}_S((X_1, \Delta_1), (X_2, \Delta_2))$ a morphism. We can view it as an isomorphism of the generic fibers $\phi_g\colon (X_1, \Delta_1) \times_S \{t_g\} \simeq (X_2, \Delta_2) \times_S \{t_g\}$. By (2.50), ϕ_g extends uniquely to an isomorphism $\Phi\colon (X_1, \Delta_1) \times_S T \simeq (X_2, \Delta_2) \times_S T$. This is the valuative criterion of properness for $\mathbf{Isom}_S((X_1, \Delta_1), (X_2, \Delta_2))$, which is thus both affine and proper, hence finite over S. □

Next we verify (1.77.1) for stable pairs.

Corollary 8.65 *Let $f\colon (X, \Delta) \to S$ be a stable morphism. Then the structure map $\pi\colon$ $\mathbf{Aut}_S(X, \Delta) \to S$ is finite, the subset $S^\circ \subset S$ of rigid fibers is open and* $\mathbf{Aut}_S(X, \Delta) = S$ *iff* $\mathbf{Aut}(X_s, \Delta_s)$ *is trivial for every geometric point $s \to S$.*

Proof Finiteness follows from (8.64). The identity section gives that $\mathscr{O}_S$ is a direct summand of $\pi_*\mathscr{O}_{\mathbf{Aut}_S(X,\Delta)}$. Thus S° is the complement of the support of $\pi_*\mathscr{O}_{\mathbf{Aut}_S(X,\Delta)}/\mathscr{O}_S$. The fibers of $\mathbf{Aut}_S(X, \Delta) \to S$ are the $\mathbf{Aut}(X_s, \Delta_s)$. □

8.7 Descent

Let $q\colon S' \to S$ be a morphism of schemes and assume that we have an object over S'. We say that the object *descends* to S if it is isomorphic to the pull-back of an object on S. Typical examples are

• a (quasi)coherent sheaf F', in which case we want to get a (quasi)coherent sheaf F on S such that $F' \simeq q^*F$, or

• a morphism $X' \to S'$, in which case we want to get a morphism $X \to S$ such that $X' \simeq X \times_S S'$.

A systematic theory was developed in Grothendieck (1962, lec.1), treating the case when $S' \to S$ is faithfully flat; see also Grothendieck (1971,

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chap.VIII), Bosch et al. (1990, chap.6) or Stacks (2022, tag 03O6) for more detailed treatments. We explain the basic idea during the proof of (8.69).

Here we discuss the consequences of descent theory for the moduli of stable pairs; the main one is (8.71). We also prove some special cases that are representative of the general theory, yet can be obtained by simpler methods.

8.66 (Functorial polarization) Kollár (1990) Let $\mathcal{F}$ be a subfunctor of $\mathcal{P}^s\mathcal{S}ch$. A *functorial polarization* (of level r) of $\mathcal{F}$ assigns

(8.66.1) to any $(f\colon (X,L) \to S) \in \mathcal{F}(S)$ another $(f\colon (X,\bar{L}) \to S) \in \mathcal{F}(S)$ such that $\bar{L}$ is equivalent to L^r, and

(8.66.2) to every $q\colon S' \to S$ an isomorphism $\sigma(q)\colon q_X^*(\bar{L}) \simeq \overline{(q_X^* L)}$ such that

(8.66.3) $\sigma(q \circ q') = \sigma(q') \circ (q'_X)^* \sigma(q)$ for every $q'\colon S'' \to S'$ and $q\colon S' \to S$.

Note that in (2) we need to fix an isomorphism, it is not enough to say that the two sides are isomorphic.

If the choice of $\bar{L}$ is specified, then we say that $\mathcal{F}$ is *functorially polarized.*

The following are examples of functorial polarizations.

(8.66.4) If $L_s \simeq \omega_{X_s}$ for $s \in S$, then $\bar{L} := \omega_{X/S}$ is a functorial polarization.

(8.66.5) If every family in $\mathcal{F}$ has a natural section $\sigma\colon S \to X$, then we can take $\bar{L} := L \otimes f^*(\sigma^* L)^{-1}$. This applies, for instance, to pointed varieties and (depending on our definition) to polarized abelian varieties.

(8.66.6) Assume that $r := \chi(X_s, L_s)$ is constant and positive for every (X_s, L_s) in $\mathcal{F}$. Then, using the notation of (3.24.3), $\bar{L} := L^r \otimes f^*(\det R^\bullet f_* L)^{-1}$ is a level r functorial polarization.

(8.66.7) $(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1))$ does not have a functorial polarization of level 1, since that would lead to a nontrivial representation of $\operatorname{Aut}(\mathbb{P}^1)$ on $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)) \simeq k^2$. On the other hand, $(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{-1})$ gives a functorial polarization of level 2.

Functorial polarizations also give natural line bundles on the base spaces of families. Let $\mathcal{F}$ be a functorially polarized subfunctor of $\mathcal{P}\mathcal{S}ch$. For any $(f\colon (X,\bar{L}) \to S) \in \mathcal{F}(S)$ we get the line bundle $\det R^\bullet f_*(\bar{L}^{\otimes k})$ as in (3.24.3). For $k \gg 1$ it is given by the simpler formula $\det f_*(\bar{L}^{\otimes k})$.

These line bundles are functorial for base changes, thus they give line bundles on the moduli stack of $\mathcal{F}$.

Uniqueness of descent now follows easily.

Proposition 8.67 *Let $S' \to S$ be a faithfully flat morphism and $X' \to S'$ a flat, proper morphism such that X' is rigid over S'. Then there is at most one scheme $X \to S$ such that $X' \simeq X \times_S S'$.*

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Proof Assume that we have $X_1 \to S$ and $X_2 \to S$. Since the $X_i \times_S S' \simeq X'$ are flat and proper, so are $X_i \to S$. We aim to prove that $\mathbf{Isom}_S(X_1, X_2) \simeq S$. To see this, take any $T \to S'$, and note that

$$\mathrm{Isom}_T(X'_T, X'_T) = \mathrm{Isom}_T(X_1 \times_S T, X_2 \times_S T) = \mathrm{Mor}_S(T, \mathbf{Isom}_S(X_1, X_2)).$$

If X' is rigid over S' then $\mathrm{Isom}_T(X'_T, X'_T)$ has only 1 element, so $\mathrm{Mor}_S(T, S) = \mathrm{Mor}_S(T, \mathbf{Isom}_S(X_1, X_2))$ for every T. Thus $S = \mathbf{Isom}_S(X_1, X_2)$. □

The simplest descent result is the following; see (1.73).

Lemma 8.68 *Let K/k be a finite, separable field extension and (X, L) a rigid, functorially polarized, projective variety defined over K. Then (X, L) descends to k iff $(X, L) \simeq (X^\sigma, L^\sigma)$ for every $\sigma \in \mathrm{Gal}(\bar{k}/k)$.*

Proof We may assume that K/k is Galois. Then only the $\sigma \in \mathrm{Gal}(K/k)$ matter. We get an action of $\mathrm{Gal}(K/k)$ on $H^0(X, L)$ by

$$H^0(X, L) \stackrel{\sigma-lin}{\longrightarrow} H^0(X^\sigma, L^\sigma) \stackrel{K-isom}{\longrightarrow} H^0(X, L).$$

This is well defined since the K-isomorphism is unique, even on L. By the fundamental lemma on quasi-linear maps (see Shafarevich (1974, sec.A.3)) there is a unique k-subspace $V(X, L) \subset H^0(X, L)$ such that $V(X, L) \otimes_k K = H^0(X, L)$. Since $X = \mathrm{Proj}_K \sum H^0(X, L^m)$, we see that $X_k := \mathrm{Proj}_k \sum V(X, L^m)$ defines the descent. □

Theorem 8.69 *Let $S' \to S$ be a faithfully flat morphism and $f' : (X', L') \to S'$ a flat, functorially polarized projective morphism that is rigid over S'. The following are equivalent.*

(8.69.1) *$f' : (X', L') \to S'$ descends to $f : (X, L) \to S$.*

(8.69.2) *For every Artinian scheme $\tau : A \to S$, the pull-back $f'_A : (X'_A, L'_A) \to A$ is independent of the lifting $\tau' : A \to S'$.*

If S is normal and $S' \to S$ is smooth, then it is enough to check (2) for spectra of fields.

Proof We just explain how this fits in the framework of faithfully flat descent, for which we refer to Stacks (2022, tag 03O6).

Let $\pi_i : S' \times_S S' \to S'$ denote the coordinate projections for $i = 1, 2$. Pulling back $f' : (X', L') \to S'$ to $S' \times_S S'$ by the π_i, we get two families

$$f'_i : (X'_i, L'_i) \to S' \times_S S'.$$

If $f : (X, L) \to S$ exists then these are both isomorphic to the pull-back of $f : (X, L) \to S$, hence to each other $\sigma_{12} : (X'_1, L'_1) \simeq (X'_2, L'_2)$. The existence of σ_{12} is a necessary condition for descent. The key observation is that it is not sufficient, one also needs certain compatibility conditions over the triple product $S' \times_S S' \times_S S'$. However, if (X', L') is rigid over S', then σ_{12} is unique and the compatibility conditions are automatic.

To prove that σ_{12} exists, consider

$$\pi\colon \operatorname{Isom}_{S' \times_S S'}((X'_1, L'_1), (X'_2, L'_2)) \to S' \times_S S'.$$

Since (X', L') is rigid over S', π is a monomorphism. Assumption (2) implies that it is scheme-theoretically surjective, hence an isomorphism.

If $S' \to S$ is smooth then $S' \times_S S' \to S$ is also smooth, hence $S' \times_S S'$ is normal if S is normal. In that case, surjectivity is a set-theoretic question. □

Corollary 8.70 *Let G be a flat group scheme over S and $S' \to S$ a principal G-bundle. Let $f'\colon (X', L') \to S'$ be a flat, functorially polarized projective morphism that is rigid over S'. Assume that the G actions lifts to (X', L').*

Then $f'\colon (X', L') \to S'$ descends to $f\colon (X, L) \to S$.

Proof We need to check assumption (8.69.2). So fix $\tau\colon A \to S$ and liftings $\tau_i\colon A \to S'$. Then S'_A is a principal G-bundle with two sections τ_i. Thus $\tau_2 = g_{12} \circ \tau_1$ for some section g_{12} of G_A. Since the G-action lifts to (X', L'), the corresponding pull-backs are isomorphic. □

Now we come to the main theorem.

Theorem 8.71 *Let $\mathrm{SP}^{rigid} \subset \mathrm{SP}$ be the open subset parametrizing stable pairs without automorphisms. Then there is a universal family over SP^{rigid}.*

Proof First, note that $\mathrm{SP}^{\mathrm{rigid}}$ is indeed open by (8.65).

For rigid families the existence is a local question. We may thus fix the dimension n, the number of marked divisors r, the coefficient vector $(a_1, \dots, a_r)$, the volume v and the intended embedding dimension N.

First, consider the case when the a_i are rational and also fix $m > 1$, a multiple of $\operatorname{lcd}(a_1, \dots, a_r)$. Set $\mathbf{d} := (n, r, a_1, \dots, a_r, m, v, N)$.

Let $\mathcal{SP}(\mathbf{d})(S)$ denote the set of marked families $f\colon (X, \Delta) \to S$ with these numerical data, for which $m(K_{X/S} + \Delta)$ is a Cartier $\mathbb{Z}$-divisor and a strong polarization, and such that $f_*\mathcal{O}_X(m(K_{X/S} + \Delta))$ has rank $N + 1$. Similarly, let $\mathcal{EMSP}(\mathbf{d})(S)$ denote the set of these objects together with a strong embedding into $\mathbb{P}^N_S$.

By (8.52), we have the moduli spaces $\mathrm{EMSP}^{\mathrm{rigid}}(\mathbf{d}) \subset \mathrm{EMSP}(\mathbf{d})$, with universal families. By (8.61), $\mathrm{EMSP}^{\mathrm{rigid}}(\mathbf{d}) \to \mathrm{SP}^{\mathrm{rigid}}(\mathbf{d})$ is a principal PGL_{N+1}-bundle. Hence the universal family over $\mathrm{EMSP}^{\mathrm{rigid}}(\mathbf{d})$ descends to $\mathrm{SP}^{\mathrm{rigid}}(\mathbf{d})$ by (8.70).

The case of irrational coefficients is very similar. We need to work with the rational approximations $(X, \sigma_j^m(\Delta)) \to S$ as in (8.21). □

Complement 8.71.1 The same proof works for other variants of the moduli of stable pairs, in particular we get universal families over the moduli space $\mathrm{MpSP}^{\mathrm{rigid}}$ of rigid, pointed, stable pairs (8.44).

8.8 Positive Characteristic

We discuss, mostly through examples, two types of problems that complicate the moduli theory of pairs in positive characteristic.

The first problem is that, as we already noted in (2.4), the four versions of the definition of local stability in (2.3) are not equivalent in positive characteristic. The first such examples are in Kollár (2022); these are families of 3-folds. In (8.73) we discuss a series of higher dimensional examples that have very mild singularities.

The second is due to p-torsion in local class groups, visible most clearly in (4.39). As we see starting with (8.75), this issue appears already for the moduli of 4 points on $\mathbb{P}^1$. This difficulty can be avoided either by working only over weakly normal bases, or by a strong reliance on markings.

Theorem 8.72 *Kollár (2022) Let k be an algebraically closed field of characteristic* $\neq 0$*. There are flat, projective morphisms* $f\colon (X, \Delta) \to \mathbb{A}^1_k$ *of relative dimension 3 such that*

(8.72.1) $(X, X_t + \Delta)$ *is lc for every* $t \in \mathbb{A}^1$*,*

(8.72.2) $(\bar X_t, \mathrm{Diff}_{\bar X_t} \Delta)$ *is lc for every* $t \in \mathbb{A}^1$*,*

(8.72.3) $(\bar X_0, \mathrm{Diff}_{\bar X_0} \Delta)$ *lifts to characteristic 0, yet*

(8.72.4) X_0 *is not weakly normal,* $\mathrm{Sing}\, X_0$ *is 1-dimensional, and* $\bar X_0 \to X_0$ *is purely inseparable over* $\mathrm{Sing}\, X_0$.

The singularities of the 3-folds in Kollár (2022) are rather complicated. We discuss here instead another series of examples, arising from cones over homogeneous spaces. These are higher dimensional, but similar to the various examples discussed in Section 2.3.

Example 8.73 (Kovács–Totaro–Bernasconi examples) Let $X = G/P$ be a projective, homogeneous space. If P is reduced, then G/P is Fano and Kodaira vanishing holds on X in any characteristic by the Bott–Kempf theorem.

The cases when P is non-reduced were studied in Haboush and Lauritzen (1993). For some of these, $X = G/P$ is Fano, but Kodaira vanishing fails for

https://doi.org/10.1017/9781009346115.010 Published online by Cambridge University Press

a multiple of the canonical class. The first example was identified by Kovács (2018); giving a seven-dimensional canonical singularity in characteristic 2, that is not CM. A large series of examples is exhibited in Totaro (2019), leading to terminal singularities in any characteristic $p > 0$, that are not CM. These were further studied by Bernasconi (2018).

Kollár (2022) observed that they can be used to construct stable degenerations, where the generic fibers are smooth with ample canonical class and the special fibers have isolated, nonnormal singularities.

Assume that $X = G/P$ as above and $-K_X = mH$ for some ample divisor H for some $m \geq 1$. $|H|$ is very ample by Lauritzen (1996), so it gives an embedding $X \hookrightarrow \mathbb{P}^N$, where $N = \dim |H|$. Let $Y := C(X, H) \subset \mathbb{P}^{N+1}$ be the projective cone over X with vertex v.

Let $D \in |H|$ be a smooth divisor and $D_Y \subset Y$ its preimage. Since $K_X + D \sim (m-1)H$, (2.35) shows that (Y, D_Y) is a log canonical pair if $m = 1$, a canonical pair if $m > 1$.

$D_Y \subset Y$ is a Cartier divisor that is smooth outside v. Thus D_Y is normal $\Leftrightarrow$ $\operatorname{depth}_v D_Y \geq 2 \Leftrightarrow \operatorname{depth}_v Y \geq 3$; see (2.36). Since $H^{i+1}_v(Y, \mathscr{O}_Y) \simeq \sum_{m\in\mathbb{Z}} H^i(X, \mathscr{O}_X(mH))$ by (2.35.1), D_Y is normal iff $H^1(X, \mathscr{O}_X(mH)) = 0$ for all $m \in \mathbb{Z}$ by (10.29.5).

Therefore, if $H^1(X, \mathscr{O}_X(H)) \neq 0$, then D_Y is not normal. Intersecting Y with a pencil of hyperplanes with base locus $Z \not\ni v$, we get a locally stable morphism $\pi : B_Z Y \to \mathbb{P}^1$. It has one fiber isomorphic to D_Y, the others are isomorphic to X.

Taking a suitable cyclic cover (2.13), we get a series of examples of stable families, where the generic fibers are smooth varieties with ample canonical class and the special fibers have isolated nonnormal singularities.

The cases described in Totaro (2019) have $m = 2$. Then the normalization of D_Y has canonical singularities, hence these families occur in what is usually considered the "interior" of the moduli space.

Aside 8.73.1 Another class of non-CM, cyclic, quotient singularities is described in Yasuda (2019). These all have depth ≥ 3 by Ellingsrud and Skjelbred (1980), so they do not lead to families as in (8.72).

8.74 (Cartier or $\mathbb{Q}$-Cartier?) One of the early key conceptual steps of the minimal model program was the realization that, starting with dimension 3, minimal models can be singular. Moreover, their canonical class need not be Cartier. It was gradually understood that the more general $\mathbb{Q}$-Cartier condition is the important one.

In moduli theory, we frequently start with pairs (X, B) where X is smooth and B is Cartier, but in compactifying their moduli space we encounter pairs

(X', B') where X' is singular and $K_{X'}$ is only $\mathbb{Q}$-Cartier. Thus the usual approach is to work with pairs (X, B) where $K_X + B$ is $\mathbb{Q}$-Cartier.

Next we discuss various problems that arise when the denominators involve the characteristic.

8.75 (Moduli of points on $\mathbb{P}^1$) We consider the moduli problem of $n = 2r+1 \geq 3$ unordered, distinct points in $\mathbb{P}^1$. Fix an index set I of n elements. There is only one natural way of defining the objects of this theory.

(8.75.1) (Geometric objects) $(\mathbb{P}^1, \sum_{i\in I}[p_i])$ where the p_i are distinct points.

(8.75.2) (Objects over a field) $(\mathbb{P}^1, Z)$ where $Z \subset \mathbb{P}^1$ is a geometrically reduced, 0-dimensional subscheme of degree n.

The question becomes more subtle when families are considered.

(8.75.3) (Families) $(P_S \to S, D)$ where $P_S \to S$ is a locally trivial $\mathbb{P}^1$-bundle and $D \subset P_S$ is a divisor over S of degree n. For ordered points the traditional choice is to take D to be a union of sections of $P_S \to S$, but for unordered points we have two natural choices.

(3.a) (Cartier) D is a relative Cartier divisor over S.

(3.b) ($\mathbb{Q}$-Cartier) D is a relative $\mathbb{Q}$-Cartier divisor over S.

The first is closest to the traditional choice of union of sections, the second is more in the spirit of the higher dimensional theory.

(8.75.4) (Base spaces) Ideally we should work over arbitrary base schemes, but it turns out that unexpected things happen even when the base is quite nice. We consider three classes of base schemes.

(4.a) (Reduced)

(4.b) (Seminormal)

(4.c) (Weakly normal)

The cases (3.a–b) and (4.a–c) are in principle independent, thus we have six different settings for the moduli problem. We might expect that, for all of them, $M_{0,n}/S_n \simeq (\mathrm{Sym}^n \mathbb{P}^1 \setminus (\text{diagonal}))/\mathrm{PGL}_2$ is a fine moduli space.

Theorem 8.76 *Consider the above six settings of the moduli problem of $n \geq 3$ unordered points in $\mathbb{P}^1$ over a field k.*

(8.76.1) *If* $\operatorname{char} k = 0$ *then $M_{0,n}/S_n$ is a fine moduli space in all six settings.*

(8.76.2) *If* $\operatorname{char} k > 0$ *then $M_{0,n}/S_n$ is a fine moduli space, provided either (8.75.3.a) or (8.75.4.c) holds.*

(8.76.3) *If* $\operatorname{char} k > 0$ *and we are in (8.75.3.b+4.a) or (8.75.3.b+4.b), then $M_{0,n}/S_n$ is not even a coarse moduli space. In fact the categorical moduli space (1.9) is* $\operatorname{Spec} k$.

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Proof Let $(P_S \to S, D)$ be as in (8.75.3). If D is flat over S, then choosing an open cover $S = \cup_j U_j$ and isomorphisms $P_{U_j} \simeq \mathbb{P}^1 \times U_j$ gives morphisms $\phi_j \colon U_j \to \mathrm{Hilb}_n(\mathbb{P}^1)$. Changing the local trivialization changes the ϕ_j by an element of $\mathrm{Aut}(\mathbb{P}^1)$. Thus the ϕ_j glue to give a global morphism $\phi \colon S \to M_{0,n}/S_n$.

Since $P_S \to S$ is smooth, a relatively $\mathbb{Q}$-Cartier divisor D is Cartier by (4.39) if char $k = 0$. The same holds in any characteristic if the base is weakly normal by (4.41). In both cases D is flat over S, showing (1) and (2).

The proof of (3) relies on the following construction.

Let k be a field of characteristic $p > 0$, B a smooth projective curve over k and S a k-variety, for example a smooth curve. Let Δ be an effective, relative Cartier divisor on $B \times S \to S$. Any universal homeomorphism $\tau \colon S \to T$ (10.78) factors through a power of the Frobenius (for some $q = p^m$) as

$$F_q \colon S \xrightarrow{\tau} T \xrightarrow{\tau'} S.$$

Taking product with B we get $\tau_B \colon B \times S \to B \times T$ and $\tau'_B \colon B \times T \to B \times S$. Set $\Delta_T := (\tau_B)_* \Delta$ on $B \times T$. If τ is birational, the coefficients of Δ_T are the same as the coefficients of Δ. Also, Δ_T is $\mathbb{Q}$-Cartier since $q\Delta_T = (\tau'_B)^* \Delta$. However, the Cartier index may get multiplied by q. We have thus proved the following.

Claim 8.76.4 If $(B \times S, \Delta) \to S$ is in our moduli problem using (8.75.3.b), then so is $(B \times T, \Delta_T) \to T$. □

A typical example with concrete equations is in (4.12).

Assume now that we work in the settings (8.75.3.b+4.a) or (8.75.3.b+4.b). Let $\mathbf{M}_n$ be the categorical moduli space. If $(\mathbb{P}^1 \times S, D)$ is a family of n points on $\mathbb{P}^1$, then we get a moduli map $\phi \colon S \to \mathbf{M}_n$. By the above construction, for any $\tau \colon S \to T$ we get a factorization $\phi \colon S \xrightarrow{\tau} T \to \mathbf{M}_n$.

Corollary 8.76.5 If the universal push-out of all the above $\tau \colon S \to T$ is $S \to \operatorname{Spec} k$, then the moduli map $\phi \colon S \to \mathbf{M}_n$ is constant.

Instead of proving this in general, we work out some typical examples.

Example 8.76.6 The map $\operatorname{Spec} k[x] \to \operatorname{Spec} k[(x-c)^r, (x-c)^s]$ is a birational, universal homeomorphism for any $(r, s) = 1$ and $c \in k$. The universal push-out of all of them is $\operatorname{Spec} k[x] \to \operatorname{Spec} k$; cf. (10.87).

Indeed, if $f(x) \in k[(x-c)^r, (x-c)^s]$ vanishes at c then it has a zero of multiplicity $\geq \min\{r, s\}$. Thus only the constants are contained in the intersection of all of them.

This settles (8.75.3.b+4.a), but the curves $\operatorname{Spec} k[(x-c)^r, (x-c)^s]$ are not seminormal if $r, s > 1$. Over an algebraically closed field k, there are two-dimensional seminormal examples.

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Example 8.76.7 Set $R_q := k[x] + (y^q - x)k[x, y] \subset k[x, y]$. R_q is seminormal, but not weakly normal and its normalization is $k[x, y]$. The conductor ideal is $(y^q - x)k[x, y]$. It is a principal ideal in $k[x, y]$, but not in R_q.

The map $\operatorname{Spec} k[x, y] \to \operatorname{Spec} R_q$ is birational. It is again easy to check that the universal push-out of all of them is $\operatorname{Spec} k[x, y] \to \operatorname{Spec} k[x]$. Thus if we combine the maps $\operatorname{Spec} k[x, y] \to \operatorname{Spec} R_q$ with all linear coordinate changes, then the universal push-out is $\operatorname{Spec} k[x, y] \to \operatorname{Spec} k$. □