4

Eigenvalues of the Fractional p-Laplacian

4.1 Fundamentals

Throughout we suppose that Ω is a bounded open subset of $\mathbb{R}^n$, let $p \in (1, \infty)$ and assume that $s \in (0, 1)$; the spaces involved are assumed to be real. Consider the problem of minimising the fractional Rayleigh quotient

$$R(f, p, s, \Omega) := \frac{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy}{\int_{\mathbb{R}^n} |f(x)|^p \, dx} = \frac{[f]_{s,p,\mathbb{R}^n}^p}{\|f\|_{p,\mathbb{R}^n}^p} \tag{4.1.1}$$

among all $f \in C_0^\infty(\Omega) \backslash \{0\}$. We write

$$\lambda_1 = \lambda_1 (p, s, \Omega) = \inf \left\{ R(f, p, s, \Omega) : f \in C_0^\infty(\Omega) \backslash \{0\} \right\}.$$

Since Ω supports the (s, p)-Friedrichs inequality (see Proposition 3.17), it is clear that $\lambda_1 > 0$; we refer to λ_1 as the *first eigenvalue* of the fractional p-Laplacian $(-\Delta)_p^s$. The attainment of the infimum is discussed in the next theorem, which reinforces the treatment of Section 3.4. We recall that the space $X := \overset{0}{\mathcal{D}}{}_p^s(\Omega)$ that appears there and below is the completion of $C_0^\infty(\Omega)$ with respect to the norm $[\cdot]_{s,p,\mathbb{R}^n}$: see Proposition 3.7 for information about this space. In particular, note that X and X^* are uniformly convex and that X coincides with $\overset{0}{X}{}_p^s(\Omega)$ since Ω is bounded.

Theorem 4.1 *The infimum of R is attained at a non-negative function $f \in X \backslash \{0\}$, with $f = 0$ in $\mathbb{R}^n \backslash \Omega$. This minimising function f satisfies the Euler–Lagrange equation*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^{p-2} (f(x)-f(y)) (\phi(x)-\phi(y))}{|x-y|^{n+sp}} \, dx \, dy = \lambda_1 \int_{\mathbb{R}^n} |f|^{p-2} f \phi \, dx \tag{4.1.2}$$

for all $\phi \in C_0^\infty(\Omega)$.

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Proof When minimising the Rayleigh quotient it is enough to consider non-negative functions since for every $u \in L_p(\Omega)$,

$$||u(x)| - |u(y)||^p \leq |u(x) - u(y)|^p \text{ and } \||u|\|_p = \|u\|_p .$$

Let $\{f_j\}_{j \in \mathbb{N}}$ be a minimising sequence, with $f_j \in C_0^\infty(\Omega)$ and $\|f_j\|_p = 1$ for all j. By Proposition 3.8 there exists $C > 0$ such that for all $j \in \mathbb{N}$,

$$\int_{\mathbb{R}^n} \left|f_j(x+h) - f(x)\right|^p dx \leq C\,|h|^{sp} \to 0 \text{ as } |h| \to 0.$$

Hence by the Riesz–Fréchet–Kolmogorov theorem, there is a subsequence of $\{f_j\}_{j\in\mathbb{N}}$ that converges in $L_p(\Omega)$, to f, say; plainly $\|f\|_{p,\Omega} = 1$. Since X is reflexive, there is a further subsequence $\{g_j\}_{j\in\mathbb{N}}$ that converges weakly in X, to g, say; as this subsequence also converges weakly in $L_p(\Omega)$ we see that $g = f$. As

$$\|f|X\| \leq \liminf_{j\to\infty} \left\|g_j|X\right\| ,$$

it follows that $\|f|X\| = \lambda_1$, showing that the infimum is attained.

It remains to deal with the Euler–Lagrange equation. Let u be a minimising function and consider the competing functions

$$v_t(x) := u(x) + t\phi(x),\ \ \phi \in C_0^\infty(\Omega), t \in \mathbb{R}.$$

Because we have a minimum we must have

$$\frac{d}{dt}\left\{\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v_t(y)-v_t(x)|^p}{|y-x|^{n+sp}}\,dx\,dy}{\int_{\mathbb{R}^n} |v_t(x)|^p\,dx}\right\} = 0 \text{ at } t = 0.$$

This immediately gives the Euler–Lagrange equation. □

Remark 4.2

Note that since the inequality

$$||u(x)| - |u(y)|| \leq |u(x) - u(y)|$$

is strict at almost all points x, y such that $u(x)u(y) < 0$, no minimiser can change sign. It should also be observed that $\lambda_1(p, s, \Omega)$ is the reciprocal of the best constant in the (s, p)-Friedrichs inequality; in fact,

$$\lambda_1\,(p, s, \Omega) \geq 1/C\,(n, s, p, \Omega)\,,$$

where $C\,(n, s, p, \Omega)$ is given in Proposition 3.2.5.

As in Section 3.4, given $\lambda \in \mathbb{R}$ we say that $u \in X\backslash\{0\}$ is a weak solution of the eigenvalue problem

$$(-\Delta)_p^s\, u = \lambda\,|u|^{p-2}\,u \text{ in } \Omega, u = 0 \text{ in } \mathbb{R}^n\backslash\Omega, \tag{4.1.3}$$

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if

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\phi(x)-\phi(y))}{|x-y|^{n+sp}}\,dx\,dy = \lambda \int_{\mathbb{R}^n} |u|^{p-2}\,u\phi\,dx \tag{4.1.4}$$

for all $\phi \in X$; if such a function u exists, the corresponding λ is an *eigenvalue* and u is a λ*-eigenfunction*.

Every solution of the Euler–Lagrange equation is bounded. When $sp > n$ the Euler–Lagrange equation is not needed: in fact since, by Proposition 3.8, X is embedded in $C^{\alpha}(\mathbb{R}^n)$, where $\alpha = s - n/p$, the boundedness is clear. Much more effort is needed when $sp \le n$.

Theorem 4.3 *Suppose that $sp \le n$ and let u be a minimiser of the Rayleigh quotient. Then $u \in L_{\infty}(\mathbb{R}^n)$; and if $sp < n$,*

$$\|u\|_{\infty} \le C(n,p,s)\lambda_1^{n/\left(sp^2\right)}\,\|u\|_p\,.$$

For a proof of this result we refer to [28], Theorem 3.3 and [84], Theorem 3.2.

Corollary 4.4 *Let $s \in (0,1)$, $p \in (1,\infty)$. Then every (s,p)-eigenfunction is continuous.*

Proof If $sp > n$ there is nothing to prove because of Proposition 3.8. If $sp \le n$, we know from Theorem 4.3 that eigenfunctions are bounded. The continuity then follows from Theorem 1.5 of [111] (see Corollary 3.14 of [29]). □

We now turn to further basic properties of eigenfunctions. The next theorem was established in [27] when Ω is connected, the general case being proved in [29]. A crucial step in the argument is the following logarithmic estimate, given in [50]. It concerns functions $u \in X = \overset{0}{X}{}^s_p(\Omega)$ that are supersolutions (of the problem $(-\Delta)^s u = 0$ in Ω, $u = 0$ in $\mathbb{R}^n \backslash \Omega$) in the sense that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{n+sp}}\,(\phi(x)-\phi(y))\,dx\,dy \ge 0$$

for all $\phi \in C_0^{\infty}(\Omega)$, $\phi \ge 0$.

Lemma 4.5 *Let $s \in (0,1)$, $p \in (1,\infty)$, and let $u \in X = \overset{0}{X}{}^s_p(\Omega)$ be a supersolution such that $u \ge 0$ in $B(x_0, 2r)$ for some $r > 0$ and $x_0 \in \Omega$ with $\overline{B(x_0, 2r)} \subset \Omega$. Then there is a constant $C = C(n,p,s)$ such that for all $\delta > 0$,*

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$$\int_{B(x_0,r)}\int_{B(x_0,r)}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p\frac{1}{|x-y|^{n+sp}}\,dx\,dy$$
$$\leq Cr^{n-sp}\left\{\delta^{1-p}r^{sp}\int_{\mathbb{R}^n\backslash B(x_0,2r)}\frac{u_-(y)^{p-1}}{|y-x_0|^{n-sp}}\,dy+1\right\},$$

where $u_-=\max\{-u,0\}$.

Theorem 4.6 *Let* $u\in X=\overset{0}{X}{}^s_p(\Omega)$ *be a non-negative* (s,p)*-eigenvector with corresponding eigenvalue* λ*. Then* $u>0$ *a.e. in* Ω*.*

Proof First assume that Ω is connected and let K be a compact connected subset of Ω, so that $K\subset\{x\in\Omega:\ \operatorname{dist}(x,\partial\Omega)>2r\}$ for some $r>0$. Then K can be covered by balls $B(x_i,r/2)$ $(i=1,\dots,k)$ with each $x_i\in K$ and

$$|B(x_i,r/2)\cap B(x_{i+1},r/2)|>0\ (i=1,\dots,k-1). \tag{4.1.5}$$

Suppose that $u=0$ on a subset of K with positive measure. Then there exists $i\in\{1,\dots,k-1\}$ such that

$$Z:=\{x\in B(x_i,r/2):u(x)=0\}$$

has positive measure. For each $\delta>0$ set

$$F_\delta(x)=\log\left(1+\frac{u(x)}{\delta}\right),\ x\in B(x_i,r/2).$$

Then $F_\delta(x)=0$ for all $x\in Z$, and so for all $x\in B(x_i,r/2)$ and $y\in Z\backslash\{x\}$,

$$|F_\delta(x)|^p=\frac{|F_\delta(x)-F_\delta(y)|^p}{|x-y|^{n+sp}}|x-y|^{n+sp},$$

from which we have, on integrating with respect to $y\in Z$ and $x\in B(x_i,r/2)$,

$$\int_{B(x_i,r/2)}|F_\delta(x)|^p\,dx\leq\frac{r^{n+sp}}{|Z|}\int_{B(x_i,r/2)}\int_{B(x_i,r/2)}\frac{|F_\delta(x)-F_\delta(y)|^p}{|x-y|^{n+sp}}\,dx\,dy. \tag{4.1.6}$$

Together with Lemma 4.5, noting that $u_-=0$, this shows that

$$\int_{B(x_i,r/2)}\left|\log\left(1+\frac{u(x)}{\delta}\right)\right|^p dx\leq Cr^{2n}/|Z|,$$

where C is independent of δ. As this holds for arbitrarily small $\delta>0$, it follows that $u=0$ a.e. in $B(x_i,r/2)$. In view of (4.1.5) this argument can be repeated for $B(x_{i\pm1},r/2)$, from which we see that $u=0$ a.e. in K. This contradicts our original assumption and we conclude that $u>0$ a.e. on K.

Now recall that u is an eigenvector and so is not identically zero in Ω. Since Ω is connected, there is a sequence $\{K_m\}$ of connected compact subsets of Ω such that for each m, $|\Omega\backslash K_m|<1/m$ and u is not identically zero in K_m. By what we have proved, $u>0$ a.e. on each K_m. It follows that $u>0$ a.e. on Ω.

To complete the proof it remains to deal with the case in which Ω is not connected. We know that $u > 0$ a.e. on each connected component of Ω on which it is not identically zero. Put $\Omega_1 = \{x \in \Omega : u(x) > 0\}$ and suppose there is a connected component Ω_2 of Ω on which u is identically zero. Let $\phi \in C_0^\infty(\Omega_2)$ be a non-negative test function that is not identically zero. Then

$$0 = \lambda \int_\Omega u^{p-1}\phi\, dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} (\phi(x)-\phi(y))\, dx\, dy$$
$$= -2\int_{\Omega_1}\int_{\Omega_2} \frac{u(x)^{p-1}}{|x-y|^{n+sp}}\phi(y)\, dx\, dy.$$

Thus u is identically zero in Ω_1 and we have a contradiction that establishes the theorem. □

All eigenfunctions except those corresponding to λ_1 change sign. Formally,

Theorem 4.7 *Let $v \in X = \overset{0}{X}{}_p^s(\Omega)$ be a solution of (4.1.4), with corresponding eigenvalue λ, such that $v > 0$ in Ω. Then $\lambda = \lambda_1(p, s, \Omega)$.*

The ingenious proof is given in [84], Theorem 4.1. Theorem 4.2 of the same paper shows that λ_1 is simple:

Theorem 4.8 *Let $s \in (0, 1)$ and $p \in (1, \infty)$. All positive eigenfunctions corresponding to $\lambda_1(p, s, \Omega)$ are proportional.*

For later convenience we now summarise these properties of the first eigenvalue.

Theorem 4.9 *Let $s \in (0, 1)$ and $p \in (1, \infty)$; suppose that Ω is bounded. Then*

(i) any first (s, p)-eigenfunction must be strictly positive (or strictly negative);
(ii) $\lambda_1(p, s, \Omega)$ is simple;
(iii) any eigenfunction corresponding to an eigenvector $\lambda > \lambda_1(p, s, \Omega)$ must change sign.

4.2 The Spectrum

In contrast to the standard terminology used for linear operators, the set of all eigenvalues of the eigenvalue problem (4.1.3) is called the *spectrum* of (4.1.3) and is denoted by $\sigma(s, p)$; given $\lambda \in \sigma(s, p)$, the set of all λ-eigenfunctions is the λ-*eigenspace*.

Proposition 4.10 *The spectrum $\sigma(s, p)$ is closed.*

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Proof Let $\lambda \in \overline{\sigma(s,p)}$; then there is a sequence $\{\lambda_k\}$ of eigenvalues such that $\lambda_k \to \lambda$. Put $X = \overset{0}{X}{}^s_p(\Omega)$ and define $A\colon X \to X^*$ as in 3.4, so that for all $u, v \in X$,

$$\langle Au, v\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}\,dx\,dy,$$

and

$$\lambda_k = [u_k]^p_{s,p,\mathbb{R}^n} = \|u_k|X\|^p \text{ for some } u_k \in X \text{ with } \|u_k\|_p = 1\ (k \in \mathbb{N}).$$

Since $\{u_k\}$ is bounded in the reflexive space X, there is a subsequence, still denoted by u_k for convenience, and an element u of X, such that $u_k \rightharpoonup u$ in X; as X is compactly embedded in $L_p(\Omega)$, we may and shall suppose that $u_k \to u$ in $L_p(\Omega)$. Moreover,

$$\begin{aligned}|\langle Au_k, u_k - u\rangle| &= \lambda_k\left|\int_{\mathbb{R}^n} |u_k|^{p-2}\,u_k\,(u_k-u)\,dx\right|\\ &\le \lambda_k\,\|u_k-u\|_p\,\|u_k\|_p^{p/p'} \to 0\end{aligned}$$

as $k \to \infty$. Thus

$$\lim_{k\to\infty} |\langle Au_k - Au, u_k - u\rangle| = 0,$$

and as by Lemma 3.31 the map A is of type $(S)_+$, it follows that $u_k \to u$ in X. By Proposition 1.1.26 of [61], applied to the duality map A and the uniformly smooth space X^*, we see that $Au_k \to Au$ in X^*. Let $J\colon L_p(\Omega) \to L_{p'}(\Omega)$ be the duality map with gauge function $t \longmapsto t^{p-1}$, so that $Jf = |f|^{p-2}f\ \ (f \in L_p(\Omega))$. Then for all $v \in X$,

$$\begin{aligned}\langle Au, v\rangle &= \lim_{k\to\infty}\langle Au_k, v\rangle = \lim_{k\to\infty}\lambda_k\int_{\mathbb{R}^n}|u_k|^{p-2}\,u_k v\,dx = \lim_{k\to\infty}\lambda_k\,\langle Ju_k, v\rangle\\ &= \lambda\,\langle Ju, v\rangle = \lambda\int_{\mathbb{R}^n}|u|^{p-2}\,uv\,dx,\end{aligned}$$

and $\|u\|_{p,\Omega} = 1$, so that $\lambda \in \sigma(s,p)$. □

Theorem 4.11 *If $\{\Omega_j\}$ is a non-decreasing sequence of domains such that $\Omega = \cup_{j=1}^{\infty}\Omega_j$, then $\lambda_1\left(p,s,\Omega_j\right) \downarrow \lambda_1(p,s,\Omega)$.*

Proof Evidently $\lambda_1\left(p,s,\Omega_j\right)$ decreases as j increases; hence the limit exists. Given $\varepsilon > 0$, there is a function $\phi \in C_0^\infty(\Omega)$ such that

$$[\phi]^p_{s,p,\mathbb{R}^n} \,/\, \|\phi\|^p_{p,\mathbb{R}^n} < \lambda_1\,(p,s,\Omega) + \epsilon, \tag{4.2.1}$$

since $\lambda_1\,(p,s,\Omega)$ is the infimum. However, as $\operatorname{supp}\phi \subset \Omega_j$ for large enough j, the function ϕ can be used as a test function in the Rayleigh quotient for Ω_j, and so

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$$\lambda_1\left(p, s, \Omega_j\right) < \lambda_1\left(p, s, \Omega\right) + \epsilon$$

for all large enough j. The result follows. $\square$

Let $S_p(\Omega) = \left\{u \in X\colon \ \|u\|_p = 1\right\}$ and define

$$\lambda_2(s, p, \Omega) = \inf_{f \in \mathcal{C}_1(\Omega)} \max_{u \in im(f)} \|u\|^p_{\overset{0}{X}{}^s_p(\Omega)},$$

where

$$\mathcal{C}_1(\Omega) = \left\{f\colon \mathcal{S}^1 \to S_p(\Omega)\colon f \text{ odd and continuous}\right\}.$$

Theorem 4.12 *Let $s \in (0,1)$ and $p \in (1,\infty)$; suppose that Ω is bounded. Then $\lambda_2(s,p,\Omega)$ is an (s,p)-eigenvalue, $\lambda_2(s,p,\Omega) > \lambda_1(s,p,\Omega)$, and for every (s,p)-eigenvalue $\lambda > \lambda_1(p,s,\Omega)$ we have $\lambda \geq \lambda_2(s,p,\Omega)$.*

This is given in detail in [29], Theorem 4.1 and Proposition 4.2. Here we simply sketch some of the main ideas used to establish the result.

To prove that $\lambda_2(s,p,\Omega)$ is an (s,p)-eigenvalue amounts to showing that it is a critical point of the functional

$$\Phi_{s,p}(u) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy$$

defined on the manifold

$$S_p(\Omega) := \left\{u \in X(\Omega)\colon \ \|u\|_{p,\Omega} = 1\right\}.$$

The strategy is to show that $\Phi_{s,p}$ satisfies the Palais–Smale condition: once this is done, the result will follow from Theorem 1.4.1. To do this, let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in $S_p(\Omega)$ such that there exists $C > 0$ with

$$\Phi_{s,p}(u_k) \leq C \ \ (k \in \mathbb{N}) \ \text{ and } \lim_{k\to\infty} \left\|\Phi'_{s,p}(u_k)|T_{u_k}S_p(\Omega)\right\| = 0. \tag{4.2.2}$$

In this context the tangent space to $S_p(\Omega)$ at u_k is given by

$$T_{u_k}S_p(\Omega) = \left\{\phi \in X\colon \ \int_\Omega |u_k|^{p-2}\, u_k\phi\,dx = 0\right\}.$$

By the second part of (4.2.2), there exists $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\varepsilon_k > 0$, $\varepsilon_k \to 0$ such that for all $k \in \mathbb{N}$,

$$\left|\Phi'_{s,p}(u_k)(\phi)\right| \leq \varepsilon_k \|\phi\|_X \ \text{ for all } \sigma \in T_{u_k}S_p(\Omega).$$

By the first part of (4.2.2), there is a subsequence of $\{u_k\}_{k\in\mathbb{N}}$, still denoted by $\{u_k\}_{k\in\mathbb{N}}$ for convenience, and a function $u \in X$ such that $u_k \to u$ in $L_p(\Omega)$ and $u_k \rightharpoonup u$ in X. Clearly $u \in S_p(\Omega)$. It remains to prove that a further subsequence of $\{u_k\}_{k\in\mathbb{N}}$ converges to u in X: details of the technical argument needed to establish this are given in [29].

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To prove that $\lambda_2(s,p,\Omega) > \lambda_1(s,p,\Omega)$, suppose that this is false, so that

$$\lambda_2(s,p,\Omega) = \inf_{f\in\mathcal{C}_1(\Omega)} \max_{u\in im(f)} \|u\|_X = \lambda_1(s,p,\Omega).$$

Hence given any $k \in \mathbb{N}$, there is an odd continuous map $f_k\colon \mathbb{S}^1 \to S_p(\Omega)$ such that

$$\max_{u\in f_k(\mathbb{S}^1)} \|u\|_X \le \lambda_1(s,p,\Omega) + k^{-1}.$$

Let $u_1 \in S_p(\Omega)$ be the unique (modulo the choice of sign) global minimiser, and for small enough positive ε consider the disjoint neighbourhoods

$$U_+ := \left\{u \in S_p(\Omega) : \|u - u_1\|_p < \varepsilon, \right\},$$
$$U_- := \left\{u \in S_p(\Omega) : \|u - (-u_1)\|_p < \varepsilon, \right\};$$

note that $U_+ \cup U_-$ is symmetric and disconnected. For every $k \in \mathbb{N}$, the image $f_k\left(\mathbb{S}^1\right)$ of $\mathbb{S}^1$ under the odd continuous map f_k is symmetric and connected: it follows that there exists $v_k \in f_k\left(\mathbb{S}^1\right) \setminus (U_+ \cup U_-)$. The sequence $\{v_k\}_{k\in\mathbb{N}}$ is contained in $S_p(\Omega)$ and is bounded in X: by passage to a subsequence if necessary we see that there exists $v \in S_p(\Omega)$ such that $v_k \rightharpoonup v$ in X and $v_k \to v$ in $L_p(\Omega)$. Hence

$$\|v\|_X \le \lim_{k\to\infty} \inf \|v_k\|_X = \lambda_1(s,p,\Omega).$$

Thus $v \in S_p(\Omega)$ is a global minimiser, so that either $v = u_1$ or $v = -u_1$. But $v \in S_p(\Omega) \setminus (U_+ \cup U_-)$ and we have a contradiction. Hence $\lambda_2(s,p,\Omega) > \lambda_1(s,p,\Omega)$.

For the ingenious and technical proof of the last part of the theorem we refer to Proposition 4.2 of [29].

This result shows that $\lambda_2(s,p,\Omega)$ may properly be called the second (s,p)-eigenvalue. It also shows that $\lambda_1(s,p,\Omega)$ is isolated.

4.3 Inequalities of Faber–Krahn Type

In this section we deal with a fractional version of the celebrated Faber–Krahn inequality concerning the first eigenvalue $\lambda_1(p,\Omega)$ of the Dirichlet p-Laplacian. This asserts that balls minimise the first eigenvalue among open sets with given volume; more precisely,

$$\lambda_1(p,B) \le \lambda_1(p,\Omega),$$

where

$$\lambda_1(p,\Omega) = \min\left\{\int_\Omega |\nabla u|^p\,dx\colon u \in \overset{0}{W}{}^1_p(\Omega),\ \|u\|_p = 1\right\}$$

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and B is a ball with the same measure as Ω. A crucial component of the proof is the Pólya–Szegö inequality, which involves the notion of the symmetric rearrangement of a function and which we now explain. Given $u\colon \mathbb{R}^n \to \mathbb{R}^+ \cup \{\infty\}$, its symmetric rearrangement is defined to be the unique function $u^\bigstar\colon \mathbb{R}^n \to \mathbb{R}^+ \cup \{\infty\}$ such that for all $\lambda \geq 0$, there exists $R \geq 0$ with

$$B_R = \left\{x \in \mathbb{R}^n \colon u^\bigstar(x) > \lambda\right\} \text{ and } |B_R| = |\{x \in \mathbb{R}^n \colon u(x) > \lambda\}|\,.$$

The function $u^\bigstar$ is radial and radially decreasing. It is easy to see that if u belongs to $L_p(\mathbb{R}^n)$ then so does $u^\bigstar$, which has the same L_p norm. The famous Pólya–Szegö inequality asserts that if $u \in W_p^1(\mathbb{R}^n)$ is non-negative, then $u^\bigstar \in W_p^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \left|\nabla u^\bigstar\right|^p dx \leq \int_{\mathbb{R}^n} |\nabla u|^p\, dx.$$

This was extended to fractional Sobolev spaces in [9], Theorem 9.2 (see also [70], Theorem A1); more precisely,

Theorem 4.13 *Let $s \in (0,1)$ and $p \in (1,\infty)$. Then for all $u \in W_p^s(\mathbb{R}^n)$,*

$$[u]_{s,p,\mathbb{R}^n} \geq \left[u^\bigstar\right]_{s,p,\mathbb{R}^n}.$$

Armed with this inequality, Brasco, Lindgren and Parini [28] established the following version of the Faber–Krahn inequality.

Theorem 4.14 *Let $s \in (0,1)$ and $p \in (1,\infty)$; suppose that Ω is bounded. Then for every open ball $B \subset \mathbb{R}^n$,*

$$|\Omega|^{sp/n}\, \lambda_1(s,p,\Omega) \geq |B|^{sp/n}\, \lambda_1(s,p,B). \tag{4.3.1}$$

If equality holds, then Ω is a ball.

Proof Note that $\lambda_1(s,p,t\Omega) = t^{-sp}\lambda_1(s,p,\Omega)$. Without loss of generality we may suppose that $|\Omega| = |B|$. Then (4.3.1) follows immediately from Theorem 4.3.1. As for equality, if $|\Omega| = |B|$ and $\lambda_1(s,p,\Omega) = \lambda_1(s,p,B)$, then by Theorem A.1 of [82], (4.3.1) holds with equality. However, again by Theorem A1 of [82], any first eigenfunction with respect to Ω must coincide with a translate of a radially symmetric decreasing function, which means that Ω must be a ball. □

A lower bound for the second eigenvalue was obtained in [29]; their proof relies on the following lemma.

Lemma 4.15 *Let $s \in (0,1)$ and $p \in (1,\infty)$; suppose that Ω is bounded and let λ be an eigenvalue with corresponding eigenvector $u \in S_p(\Omega)$ and with $\lambda > \lambda_1(s,p,\Omega)$. Put*

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$$\Omega_+ = \{x \in \Omega \colon u(x) > 0\}, \Omega_- = \{x \in \Omega \colon u(x) < 0\}.$$

Then

$$\lambda > \max\{\lambda_1(s,p,\Omega_+), \lambda_1(s,p,\Omega_-)\}.$$

Proof By Corollary 4.4, u is continuous in Ω; hence $\Omega_\pm$ are open and $\lambda_1(s,p,\Omega_\pm)$ are well defined. Write $u = u_+ - u_-$, where u_+ and u_- are the positive and negative parts of u; recall that u is sign-changing. Use u_+ as the test function in the equation satisfied again by u: this gives

$$\lambda \int_\Omega |u_+|^p \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} (u_+(x) - u_+(y)) \, dx \, dy.$$

Application of Lemma 1.9 with $a = u_+(x) - u_+(y)$ and $b = u_-(x) - u_-(y)$ gives

$$\lambda \int_\Omega |u_+|^p \, dx > \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy.$$

As u_+ is admissible for the variational problem defining $\lambda_1(s,p,\Omega_+)$, it follows that $\lambda > \lambda_1(s,p,\Omega_+)$. In the same way, using Lemma 1.5.1 again, this time with $a = u_-(x) - u_-(y)$ and $b = u_+(x) - u_+(y)$, we find that $\lambda > \lambda_1(s,p,\Omega_-)$, and the proof is complete. □

For the classical Laplacian on Ω it is a familiar fact that the restriction of a higher eigenfunction (with corresponding eigenvalue λ) to one of its nodal domains, Ω_1 say, is a first eigenfunction on Ω_1, with corresponding eigenvalue λ. The inequality of the last lemma illustrates the sharp contrast between the classical result and that of the fractional, nonlocal situation considered here. The following theorem is given in [29].

Theorem 4.16 *Let $s \in (0,1)$ and $p \in (1,\infty)$; suppose that Ω is bounded. Then for every ball $B \subset \mathbb{R}^n$ with $|B| = |\Omega|/2$,*

$$\lambda_2(s,p,\Omega) > \lambda_1(s,p,B). \tag{4.3.2}$$

Equality is never attained, but the estimate is sharp in the sense that given any sequences $\{x_k\}, \{y_k\}$ in $\mathbb{R}^n$ with $\lim_{k\to\infty} |x_k - y_k| = \infty$, and with $\Omega_k := B(x_k,R) \cup B(y_k,R)$, where $R > 0$, then

$$\lim_{k\to\infty} \lambda_2(s,p,\Omega_k) = \lambda_1(s,p,B_R).$$

Proof Let $u \in S_p(\Omega)$ be an eigenfunction with corresponding eigenvalue $\lambda_2(s,p,\Omega)$; define

$$\Omega_+ = \{x \in \Omega \colon u(x) > 0\}, \Omega_- = \{x \in \Omega \colon u(x) < 0\}$$

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(recall that u is sign-changing). By Lemma 4.15 and Theorem 4.14,

$$\lambda_2(s,p,\Omega) > \lambda_1(s,p,\Omega_+) > \lambda_1(s,p,B_{R_1}), \lambda_2(s,p,\Omega) > \lambda_1(s,p,\Omega_-)$$
$$> \lambda_1(s,p,B_{R_2}),$$

where $\left|B_{R_1}\right| = |\Omega_+|$ and $\left|B_{R_2}\right| = |\Omega_-|$. Hence

$$\lambda_2(s,p,\Omega) > \max\left\{\lambda_1(s,p,B_{R_1}), \lambda_1(s,p,B_{R_2})\right\}. \tag{4.3.3}$$

The scaling properties of λ_1 imply that

$$\lambda_1(s,p,B_R) = R^{-s/p}\lambda_1(s,p,B_1);$$

also, $\left|B_{R_1}\right| + \left|B_{R_2}\right| \le |\Omega|$. As the right-hand side of (4.3.3) is minimised when $\left|B_{R_1}\right| = \left|B_{R_2}\right| = |\Omega|/2$, (4.3.2) follows.

To complete the proof, define Ω_k as in the statement of the Theorem; we may suppose that $B(x_k,R)$ and $B(y_k,R)$ are disjoint for all large enough k. Let u, v be the positive normalised first eigenvalues on $B(x_k,R)$, $B(y_k,R)$ respectively: their form does not depend on the centre of the ball. Put

$$a(x,y) = u(x) - u(y), b(x,y) = v(x) - v(y).$$

By Lemma 1.10,

$$\begin{aligned}
\lambda_2(s,p,\Omega_k) &\le \max_{|w_1|^p+|w_2|^p=1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w_1a - w_2b|^p}{|x-y|^{n+sp}}\,dx\,dy \\
&\le \max_{|w_1|^p+|w_2|^p=1} \left\{ \begin{array}{c} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w_1|^p|a|^p}{|x-y|^{n+sp}}\,dx\,dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w_2|^p|b|^p}{|x-y|^{n+sp}}\,dx\,dy \\ +c_p\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left(|w_1a|^2+|w_2a|^2\right)^{(p-2)/2}|w_1w_2ab|}{|x-y|^{n+sp}}\,dx\,dy \end{array} \right\} \\
&= \lambda_1(s,p,B_R) + c_p\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left(|w_1a|^2+|w_2a|^2\right)^{(p-2)/2}|w_1w_2ab|}{|x-y|^{n+sp}}\,dx\,dy.
\end{aligned}$$

Since $ab = -u(x)v(y) - u(y)v(x)$, the numerator in the last integral is nonzero only if $(x,y) \in B(x_k,R) \times B(y_k,R)$ or $(x,y) \in B(y_k,R) \times B(x_k,R)$. With

$$C := 2 \max_{|w_1|^p+|w_1|^p=1} \int_{B(x_k,R)}\int_{B(y_k,R)} \left(|w_1a|^2 + |w_2a|^2\right)^{(p-2)/2} |w_1w_2ab|\,dx\,dy,$$

we thus have

$$\lim_{k\to\infty} \lambda_2(s,p,\Omega_k) \le \lambda_1(s,p,B_R) + \lim_{k\to\infty} \frac{c_pC}{(|x_k-y_k| - 2C)^{n+sp}} = \lambda_1(s,p,B_R),$$

which completes the proof. □

Remark 4.17 This is the nonlocal version of the Hong–Krahn–Szegö inequality (see [26], Theorem 3.2) which claims that, among sets of prescribed measure, the second eigenvalue of the Dirichlet Laplacian is minimised when the underlying set is the disjoint union of two equal balls. Note that Theorem 4.16 implies that (in scalar invariant form) for any ball $B \subset \mathbb{R}^n$,

$$\lambda_2(s, p, \Omega) > (2\,|B|\,/\,|\Omega|)^{sp/n}\,\lambda_1(s, p, B).$$

A survey of eigenvalue bounds for fractional Laplacians and fractional Schrödinger operators is given in [79]. See also [103] and [137].