Preliminaries

Anyone familiar with ℓ_p spaces can follow a healthy 50 percent of this book; if familiar with L_p spaces, the percentage rises to 75 percent. All the rest can be found in the text. Anyway, since once a man indulges himself in murder…, a reasonable list of prerequisites that could help smooth reading would be some acquaintance with classical Banach space theory; lack of fear when local convexity disappears; a certain bias towards abstraction; calm when non-linear objects show off and some fondness for exotic spaces. Better yet, perhaps, would be if, when in doubt, the reader is reminded about:

Sets and Functions

If S is a set, $\mathbf{1}_S$ denotes the identity on S, while 1_S stands for the characteristic function of S. If $S = \{s\}$ is a singleton, we write 1_s instead of $1_{\{s\}}$. We write $|S|$ for the cardinality of S. Sometimes 2 denotes the set $\{0, 1\}$. The power set of S is denoted $\mathcal{P}(S)$ or 2^S, and if $|S| = \aleph$ then $|2^S| = 2^\aleph$. We will write $\mathcal{P}_n(S) = \{a \in 2^S : |a| = n\}$ and $\operatorname{fin}(S) = \bigcup_n \mathcal{P}_n(S)$ for the family of all finite subsets of S, while $\mathcal{P}_\infty(S)$ denotes the family of all infinite subsets of S. The axiomatic system in which we work is ZFC, the usual Zermelo–Fraenkel axioms for set theory, including the axiom of choice. CH is the continuum hypothesis ($2^{\aleph_0} = \aleph_1$), and GCH is the generalised continuum hypothesis ($2^\aleph = \aleph^+$ for all infinite cardinals $\aleph$). Wherever additional axioms are assumed for some statement, the axioms appear in square brackets before the corresponding statement. Given a function $f : A \longrightarrow B$, its domain is $\operatorname{dom} f = A$, and its codomain is $\operatorname{codom} f = B$. Given functions $f : A \longrightarrow B$ and $g : C \longrightarrow D$, we write $f \times g : A \times C \longrightarrow B \times D$ for the function $(f \times g)(a, c) = (fa, gc)$. If $f : A \longrightarrow B$ and $g : A \longrightarrow D$, then $(f, g): A \longrightarrow B \times D$ is the function $(f, g)(a) = (fa, ga)$. If $f : A \longrightarrow B$ and $g : C \longrightarrow B$ then $f \oplus g : A \times C \longrightarrow B$ is the function $(f \oplus g)(a, c) = fa + gc$

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when this makes sense. Two functions $f, g\colon S \longrightarrow \mathbb{R}^+$ will be called equivalent if there are constants $C, c > 0$ such that $cg(s) \le f(s) \le Cg(s)$ for all $s \in S$.

Boolean Algebras

An algebra of sets is a non-empty subfamily of the family $\mathcal{P}(X)$ of all subsets of a set X which contains the sets $\varnothing$ and X and is closed under finite unions, finite intersections and taking complements. A Boolean algebra is a set $\mathfrak{B}$ endowed with two abstract binary operations of 'union' $\vee$ and 'intersection' $\wedge$, an operator of 'complementation' $A \mapsto A^{\mathrm{c}}$ and two distinguished elements 0 and 1 which satisfy the same laws of the union, intersection and complements as algebras of sets, with 0 in place of the empty set $\varnothing$ and 1 in place of the ambient set X. More precisely, it is required that union and intersection be commutative and associative, that each is distributive with respect to the other and that the absorption laws $A \vee (A \wedge B) = A$ and $A \wedge (A \vee B) = A$ and the complementation laws $A \wedge A^{\mathrm{c}} = 0$ and $A \vee A^{\mathrm{c}} = 1$ are satisfied for every $A, B \in \mathfrak{B}$. The simplest Boolean algebras are algebras of sets, and in fact, every Boolean algebra is isomorphic to some algebra of sets. A Boolean homomorphism $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ is a function that preserves the Boolean operations and the distinguished elements.

The space of ultrafilters on $\mathfrak{A}$, denoted by $\mathrm{ult}(\mathfrak{A})$, is the set of Boolean homomorphisms $\mathfrak{A} \longrightarrow \mathbf{2}$, where $\mathbf{2} = \{0, 1\}$ has the obvious Boolean structure. Clearly, $\mathrm{ult}(\mathfrak{A})$ is a closed subset of $2^{\mathfrak{A}}$, hence it is a compact space, which is called the Stone space of $\mathfrak{A}$.

Ordinals and Cardinals

Ordinals and cardinals are generalisations of natural numbers. Cardinals represent equivalence classes of sets under the equivalence relation: there is a bijective map between them. Ordinals represent equivalence classes of well-ordered sets (a well order on a set is an order such that any subset has a first element) under the equivalence relation that there exists a bijective order-preserving map. The axiom of choice in the equivalent form of the well ordering principle (every set admits a well order) means that every cardinal is an ordinal. One can identify cardinals with some ordinals as follows: the cardinal κ corresponds to the ordinal β of a set of cardinal κ that cannot be bijected with any set having ordinal $\alpha < \beta$. Every set S can be bijected with a unique cardinal $|S|$, called the cardinality of S: the cardinal (and ordinal) of any set with n elements is just n, the cardinal of $\mathbb{N}$ is called $\aleph_0$ and its ordinal (in its standard order) is ω. The set $\mathbb{N} \cup \{\bullet\}$ endowed with the well order in which $n < \bullet$ for all n has cardinal $\aleph_0$, but its ordinal is different from ω and is usually called $\omega + 1$. The cardinal of $\mathbb{R}$ is called $\mathfrak{c}$. Even while ignoring what they actually are, ordinals can be constructed *inductively* with three rules:

- 0 is an ordinal.
- Given an ordinal α, there is an ordinal $\alpha + 1$ or α^+ called its *successor*.
- Given any *set* of ordinals, there is an ordinal that is the supremum of the set.

Ordinals that are not successors are called *limit ordinals*. If one replaces the third rule by 'any *countable set* of ordinals has a supremum', one will never abandon the realm of countable sets. Thus 0, $1 = 0^+$, $2 = 1^+$ and $3 = 2^+$ are ordinals, as well as $\omega = \sup\{n\colon n \in \mathbb{N}\}$, $\omega + 1 = \omega^+$, etc. Sets of ordinals are well ordered, and thus ordinals can be used to perform inductive arguments even on uncountable sets. Precisely, to accept that a statement $P(\alpha)$ is valid for all ordinals α, it is enough to prove that $P(0)$ is true and then that if $P(\alpha)$ is true for every ordinal $\alpha < \beta$ then $P(\beta)$ is also true. Analogously, to define a function F on all ordinals, it is enough to describe how to determine $F(\beta)$ once $F(\alpha)$ is known for all $\alpha < \beta$. Given a cardinal κ, there exists a minimum cardinal κ^+ greater than κ. The first infinite cardinal is clearly $\aleph_0$. The cardinal $\aleph_0^+$ is denoted $\aleph_1$, and this notation continues by declaring $\aleph_{n+1} = \aleph_n^+$. The cardinal $\aleph_\omega$ is the supremum of the $\aleph_n$ for $n < \omega$, and so on. Given a cardinal κ, 2^κ is the cardinality of the product $\{0, 1\}^\kappa$. It is always the case that $\kappa < 2^\kappa$. The cardinality of the continuum $\mathbb{R}$ is $\mathfrak{c} = 2^{\aleph_0}$. The cofinality of a limit ordinal α is the least cardinal λ for which there is a subset of α of cardinality λ whose supremum is α. Thus, $\aleph_\omega$ has cofinality $\aleph_0$. The cofinality of $\mathfrak{c}$ is strictly greater than $\aleph_0$.

Compact Spaces

Each compact space has an associated important cardinal, its *weight*, and an important ordinal, its *height*. The *weight* is the smallest cardinal of a base of open sets. To define the height, we explain the derivation process: the derived set S' of a topological space S is the subset of its accumulation points. Given a compact K, its αth derived space K^α is defined by transfinite induction $K^0 = K$, $K^{\alpha+1} = (K^\alpha)'$ and $K^\beta = \bigcap_{\alpha<\beta} K^\alpha$ for a limit ordinal β. The *height* is the smallest ordinal (if it exists) α such that the αth derived set K^α is empty. A compact space satisfying $K' = K$ is called *perfect*. A compact is said to be scattered if each subset admits a relatively isolated point. When K is scattered, its height exists and is always a successor ordinal. Ordinals can be viewed as compact spaces when endowed with the order topology, defined as follows: a fundamental system of neighbourhoods of α is formed by the sets $(\gamma, \alpha]$ for $\gamma < \alpha$. Limit ordinals are accumulation points, while successor ordinals are isolated points. The ordinal β has weight $|\beta|$, while the ordinals ω^N have height $N+1$ and ω^ω has height $\omega+1$. A compact space is said to be zero-dimensional or totally disconnected if its clopen (closed and open) subsets form a base of the topology, or, equivalently, if, for every two different points $x, y \in K$, there exists a clopen set A such that $x \in A$, $y \notin A$. Scattered compacta are totally disconnected, while the Cantor set is totally disconnected and perfect. The topological structure of zero-dimensional compacta is completely described by the algebraic structure of their family of clopen sets through Stone duality: if K is a compact space, then its clopen sets $\mathfrak{cl}(K)$ form a Boolean algebra whose Stone space is naturally homeomorphic to K itself when K is zero-dimensional. See Note 4.6.1 for more on this duality.

Quasinormed Spaces and Operators

We will work with quasi and p-normed spaces (see Section 1.1), which can be either real or complex. If there is no need to specify whether the ground field is $\mathbb{R}$ or $\mathbb{C}$ then it will simply be $\mathbb{K}$. A map $f\colon X \longrightarrow Y$ acting between linear spaces is called *homogeneous* if $f(\lambda x) = \lambda f(x)$. It is called *positively homogeneous* if $f(\lambda x) = |\lambda| f(x)$ for every $\lambda \in \mathbb{K}$ and every $x \in X$. A map $f\colon X \longrightarrow Y$ acting between quasinormed spaces is bounded if there is a constant K such that $\|f(x)\| \leq K\|x\|$ for all $x \in X$. The least possible K for which this holds is denoted by $\|f\|$ and provides a quasinorm on the space $\mathsf{B}(X, Y)$ of bounded homogeneous maps $X \longrightarrow Y$. Linear continuous maps are called operators, and the space of all operators $X \longrightarrow Y$ is denoted by $\mathfrak{L}(X, Y)$, or just $\mathfrak{L}(X)$ when $Y = X$. A (linear) isomorphism is an operator admitting an inverse operator. We write $X \simeq Y$ when the spaces X, Y are isomorphic, i.e., there exists an isomorphism between them. We say that they are C-isomorphic if there is an isomorphism $u\colon X \longrightarrow Y$ such that $\|u\|\|u^{-1}\| \leq C$. The Banach–Mazur distance between X and Y is defined as $d(X, Y) = \inf\{\|u\|\|u^{-1}\|\colon u \text{ is an isomorphism between } X \text{ and } Y\}$. An isometry is an operator $u\colon X \longrightarrow Y$ such that $\|u(x)\| = \|x\|$ for all $x \in X$. Isometries are not assumed to be surjective! However, we will say that X and Y are isometric, $X \approx Y$, if there is a surjective isometry between X and Y. An ε-isometry, $\varepsilon \in [0, 1)$, is an operator u satisfying $(1+\varepsilon)^{-1}\|x\| \leq \|u(x)\| \leq (1+\varepsilon)\|x\|$ for all x in the domain of u. The range or image of an operator $u\colon X \longrightarrow Y$ is denoted by $u[X]$, and we say that u is an embedding when it is an isomorphism between X and $u[X]$, in which case we also say that $u[X]$ is a copy of X in Y. We say that Y contains a copy of X if there is an embedding $u\colon X \longrightarrow Y$. If $u\colon X \longrightarrow Y$ is an ε-isometry then u^{-1} is an ε-isometry from $u[X]$ to X. A quotient map is a surjective operator that is open. A quotient map $u\colon X \longrightarrow Y$ is called isometric if $\|y\| = \inf_{y=u(x)} \|x\|$ for all $y \in Y$. If Y is a closed subspace of X then the quotient quasinorm on the quotient space X/Y is $\|x + Y\| = \inf_{y\in Y} \|x + y\|$. Let $\jmath\colon Y \longrightarrow X$ be an embedding, and let $\tau\colon Y \longrightarrow E$ be an operator. An extension of τ is an operator $T\colon X \longrightarrow E$ such that $T\jmath = \tau$. A λ-extension of τ is an extension T such that $\|T\| \leq \lambda\|\tau\|$. Given a p-normed space X, we will denote the set of all its finite-dimensional subspaces by $\mathscr{F}(X)$. Sometimes we will use $\mathscr{F}^{(p)}$ to represent the class of all finite-dimensional p-Banach spaces, with $\mathscr{F}^{(1)}$ shortened to $\mathscr{F}$. We will denote the set of all its separable subspaces by $\mathscr{S}(X)$. Sometimes we will use $\mathscr{S}^{(p)}$ to represent the class of all separable p-Banach spaces, with $\mathscr{S}^{(1)}$ shortened to $\mathscr{S}$.

In this book we will use an unusual notation: by the dimension $\dim(X)$ of a quasi-Banach space X, we shall not mean the dimension of its underlying vector space but rather the cardinal of a smallest subset spanning a dense subspace.

Classical Spaces

Most of the time we work with quite honest spaces. Given a set I and $p \in (0, \infty)$, we write $\ell_p(I)$ for the space of all functions $f\colon I \longrightarrow \mathbb{K}$ such that $\|f\|_p = \left(\sum_{i\in I} |f(i)|^p\right)^{1/p} < \infty$. This is a quasi-Banach space with the obvious quasinorm and is a Banach space when $p \geq 1$. The space of all bounded functions $f\colon I \to \mathbb{K}$ endowed with the supremum norm is denoted by $\ell_\infty(I)$, and $c_0(I)$ is the closed subspace spanned by the characteristic functions of the singletons of I. The isometry type of $\ell_p(I)$ depends only on $\aleph = |I|$, and sometimes we write $\ell_p(\aleph)$ with the obvious meaning. Similar conventions apply to $c_0(\aleph)$. When $I = \mathbb{N}$ or $\aleph = \aleph_0$, we just write ℓ_p and c_0, while c denotes the subspace of ℓ_∞ formed by all convergent sequences. Clearly, $c_0 \simeq c$. Given a compact Hausdorff space K, we write $C(K)$ for the Banach space of all continous functions $f\colon K \longrightarrow \mathbb{K}$, with the sup norm. A $\mathscr{C}$-space is a Banach space isometric to $C(K)$ for some (often unspecified) compact K. Given a sequence (F_n) of finite-dimensional p-Banach spaces that are dense in $\mathscr{F}^{(p)}$ in the Banach–Mazur distance one can form the by now classical spaces $C_\infty^{(p)} = \ell_\infty(\mathbb{N}, F_n)$, $C_r^{(p)} = \ell_r(\mathbb{N}, F_n)$ and $C_0^{(p)} = c_0(\mathbb{N}, F_n)$. Let T be a compact operator on a fixed, separable Hilbert space H. The singular numbers of T are the eigenvalues of $|T| = (T^*T)^{1/2}$, arranged in decreasing order and counting multiplicity. The Schatten class S_p consists of those operators on H whose sequence of singular numbers $(s_n(T))$ belongs to ℓ_p. It is a quasi-Banach space endowed with the quasinorm $\|T\|_p = |(s_n(T))|_p$. Each $T \in S^p$ can be represented as $T = \sum_n s_n(T)\, x_n \otimes y_n$ for some orthonormal

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sequences $(x_n), (y_n)$ in H. The Schatten class S_p can be considered as the non-commutative version of ℓ_p. The terms *function space* and *sequence space* have very specific meanings in this book. The definition of function space is in 1.1.4, while that of sequence space is in 3.2.4. This otherwise coherent notation unfortunately introduces some points that may perplex the casual reader: $C[0,1]$ is not a function space, and ℓ_∞ is not a sequence space.

Approximation Properties

Let X be a quasi-Banach space. A sequence $(x_n)_{n\geq 1}$ is a (Schauder) basis of X if for every $x \in X$ there is a unique sequence of scalars $(c_n)_{n\geq 1}$ such that $x = \sum_{n=1}^{\infty} c_n x_n$. If all these series are unconditionally convergent, then the basis is called *unconditional*, and there is a constant C, called the *unconditionality constant* of (x_n), such that $\left\|\sum_{1\leq i\leq k} t_i x_i\right\| \leq C \left\|\sum_{1\leq i\leq k} c_i x_i\right\|$ provided $|t_i| \leq c_i$ for every i. In this case an equivalent quasinorm with unconditionality constant 1 can be defined as $\left|\!\left|\!\left| \sum_{i=1}^{\infty} c_i x_i \right|\!\right|\!\right| = \sup_{|t_i|\leq|c_i|} \left\| \sum_{i=1}^{\infty} c_i x_i \right\|$. If (x_n) is a normalised 1-unconditional basis of X, then X can be seen as a sequence space in the obvious way. A sequence (x_n) of elements of a quasi-Banach space is called a basic sequence if it is a basis of its closed linear span. The basic constant of a basic sequence (x_n) is the smallest $K > 0$ such that $\|\sum_{n=1}^{N} \lambda_n x_n\| \leq K \|\sum_{n=1}^{M} \lambda_n x_n\|$ for all $N \leq M$.

Basic sequence criterion *If (x_n) is a normalised basic sequence with constant K in a p-Banach space and $\sum \|x_n - y_n\|^p < (2K)^{-p}$ then (y_n) is a basic sequence.*

The meaning of *unconditional basic sequence* should be obvious. A Banach space with unconditional basis contains c_0, or ℓ_1, or a reflexive subspace, and thus the same is true for a Banach space containing an unconditional basic sequence. Quasi-Banach spaces with unconditional bases have (many) infinite-dimensional complemented subspaces: if (x_n) is an unconditional basis of X and $I \subset \mathbb{N}$, then $X(I) = \left\{x \in X : x = \sum_{i\in I} c_i x_i\right\}$ is a complemented subpace of X. The next best thing to a basis is a finite-dimensional decomposition (FDD): a sequence of finite-dimensional subspaces $(X_n)_{n\geq 1}$ is a FDD if each $x \in X$ has a unique expansion of the form $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in X_n$ for all n. If these series converge unconditionally then the FDD is said to be an UFDD.

Definition 0.0.1 A quasi-Banach space X is said to have the λ-approximation property (λ-AP) if for each finite-dimensional subspace $F \subset X$ and each $\varepsilon > 0$ there is a finite-rank operator $T \in \mathfrak{L}(X)$ such that $\|T\| \leq \lambda$ and $\|f - Tf\| \leq \varepsilon \|f\|$ for each $f \in F$. The space is said to have the bounded approximation property (BAP) if it enjoys the λ-AP for some λ.

One could alternatively require that the operator fix F, that is, $Tf = f$ for $f \in F$, transferring the error to the norm of the operator so that we can just ask for $\|T\| \leq \lambda + \varepsilon$. When X is separable, the BAP (resp. the λ-AP) is equivalent to the existence of a sequence of finite-rank operators $T_n \in \mathfrak{L}(X)$ converging pointwise to the identity (resp. with $\sup_n \|T_n\| \leq \lambda$). A Banach space X is said to have the uniform approximation property (UAP) when it has the λ-AP and there exists a 'control function' $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that, given F and $\lambda' > \lambda$, we can choose T such that $\operatorname{rk}(T) \leq f(\dim F)$ and $Tf = f$ for all $f \in F$, with $\|T\| \leq \lambda'$. This means exactly that every ultrapower of X has the BAP. A sequence (E_n) of spaces is said to have the joint-UAP if all the spaces E_n have the λ-UAP with the same control function. This occurs if and only if $\ell_\infty(\mathbb{N}, E_n)$ has the λ-UAP. Since X^{**} is complemented in some ultrapower of X, when X has the UAP, then all even duals have the UAP. And since the BAP passes from the dual to the space, when X has the UAP, all its duals have the UAP. See either [227, p. 60] or [83, Section 7] for details.

Selected Operator Ideals

Many interesting classes of operators between Banach spaces are *operator ideals*, classes $\mathfrak{A}$ such that (a) finite-rank operators are in $\mathfrak{A}$, (b) $\mathfrak{A} + \mathfrak{A} \subset \mathfrak{A}$ and (c) The composition of any operator with an operator in $\mathfrak{A}$ is in $\mathfrak{A}$. Following Pietsch's traditional notation, we use "fraktur" types for operator ideals. The fundamental operator ideals that appear often in these pages are:

- The ideal $\mathfrak{L}$ of all operators.
- The ideal $\mathfrak{F}$ of finite-rank operators: an operator $T\colon X \longrightarrow Y$ is in $\mathfrak{F}$ if $T[X]$ is finite-dimensional; finite-rank operators can be represented as $T = \sum_{n\leq N} x_n^* \otimes y_n$, where $x_i \in X^*$, $y_i \in Y$ and $N \in \mathbb{N}$.
- The ideal $\mathfrak{N}_p$ of p-nuclear operators, $0 < p \leq 1$: those admitting a representation $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$, with $\sum_n \|x_n^*\|^p \|y_n\|^p < \infty$.
- The ideal $\mathfrak{K}$ of compact operators: those transforming the unit ball of X into a relatively compact subset of Y.
- The ideal $\mathfrak{V}$ of completely continuous operators: those transforming the unit ball of X into a relatively compact subset of Y.
- The ideal $\mathfrak{W}$ of weakly compact operators between Banach spaces: those transforming the unit ball of X into a weakly compact subset of Y.
- The ideal $\mathfrak{S}$ of strictly singular operators: those whose restriction to an infinite-dimensional closed subspace is never an isomorphism.

0.0.2 *An operator $T\colon X \longrightarrow Y$ acting between Banach spaces is strictly singular if and only if, for every $\varepsilon > 0$, every infinite-dimensional subspace $A \subset X$ contains an infinite-dimensional subspace $B \subset A$ such that $\|T|_B\| \leq \varepsilon$.*

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The result fails for quasi-Banach spaces, as explained in Note 9.4.4.

- The ideal $\mathfrak{P}_p$ of p-summing operators: $T\colon X \longrightarrow Y$ is p-summing if it transforms weakly p-summable sequences (sequences (x_n) such that $(\langle x^*, x_n\rangle)_n \in \ell_p$ for all $x^* \in X^*$) into absolutely p-summable sequences (sequences (y_n) such that $(\|y_n\|)_n \in \ell_p$).

The Grothendieck–Pietsch domination / factorisation theorem establishes that T is p-summing if and only if there is a factorisation

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \longrightarrow & Y^{**} \\ \downarrow & & & \nearrow & \\ L_\infty(\mu) & \longrightarrow & L_p(\mu) & & \end{array}$$

for some measure μ on B_{X^*} so that $L_\infty(\mu) \longrightarrow L_p(\mu)$ is the canonical inclusion. When $p = 2$, the right upwards arrow goes $L_2(\mu) \longrightarrow Y$. This means that 2-summing operators factorise through Hilbert spaces (operators that factorise through a Hilbert space are sometimes called 2-factorable) and that they extend anywhere [153, 4.15]. Grothendieck's theorem [153, Theorem 3.7] establishes that every operator between an $\mathscr{L}_1$ and an $\mathscr{L}_2$ space is 2-summing.

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