# 8

# Extension of $C(K)$-Valued Operators

Homological and operator extension methods live in symbiosis, and the theory of extension of $\mathscr{C}$-valued operators shares their habitat. However, there are a number of techniques for obtaining extensions of $\mathscr{C}$-valued operators that are so firmly anchored to the existence of an underlying compact space that they cannot be translated to more general extension problems. Those techniques depend, one way or another, on the following facts:

- The norm of $C(K)$ is related to the order structure: $\|f\| \leq \lambda$ if and only if $-\lambda \leq f(s) \leq \lambda$ for every $s \in K$.
- An operator $\tau : X \longrightarrow C(K)$ is a family of functionals on $X$ parametrised by $K$. More precisely, if $\varphi\colon K \longrightarrow X^*$ is a weak*-continuous map, the formula $(\tau x)(s) = \langle \varphi(s), x\rangle$ defines an operator, and all $C(K)$-valued operators on $X$ arise in this way.
- If $g$ and $h$ are functions on $K$ such that $g$ is upper semicontinuous, $h$ is lower semicontinuous and $g \leq h$, then there is $f \in C(K)$ such that $g \leq f \leq h$.

Each extension problem involves a class of operators to be extended and, more or less implicitly, an often unnamed embedding. The following definitions emphasise the role of the different elements.

**$\mathscr{C}$-trivial embedding.** An embedding $\jmath$ is said to be $\mathscr{C}$-trivial if every $\mathscr{C}$-valued operator $\tau$ admits an extension through $\jmath$. It will be called $(\lambda, \mathscr{C})$-trivial if every $\mathscr{C}$-valued operator admits a $\lambda$-extension.

**$\mathscr{C}$-extension property.** A space $X$ has the $\mathscr{C}$-extension property if all embeddings into $X$ are $\mathscr{C}$-trivial. If all such embeddings are $(\lambda, \mathscr{C})$-trivial then $X$ is said to have the $(\lambda, \mathscr{C})$-extension property. A space with the $(1^+, \mathscr{C})$-extension property is said to have the almost isometric $\mathscr{C}$-extension property.



https://doi.org/10.1017/9781108778312.010 Published online by Cambridge University Press

$\mathscr{C}$**-extensible.** A separable space $X$ is said to be $\mathscr{C}$-extensible if all embeddings $\jmath: X \longrightarrow C(\Delta)$ are $\mathscr{C}$-trivial.

Throughout this chapter, we will only consider real separable Banach spaces: the general non-separable case is, at the time of writing these lines, just too difficult, even if we do make progress in the non-separable world, as recounted in the story of the CCKY problem in Section 8.7. This chapter, devoted to the single topic of extending $\mathscr{C}$-valued operators has, notwithstanding that and for very good reasons, a whirlpool organisation. Let us explain why. Section 8.1 presents the *global* approach to the extension of operators: Zippin's characterisation of $\mathscr{C}$-trivial embeddings by means of weak*-continuous selectors and a few remarkable applications. Section 8.2 is devoted to the Lindenstrauss–Pełczyński theorem, one of the cornerstones of the theory. Two very different proofs for this important result are presented: the first one combines homological techniques with the global approach, while the second is Lindenstrauss–Pełczyński's original proof. The analysis of their proof is indispensable for understanding Kalton's imaginative, not once but twice, inventions that lead to the so-called $(L^*)$ and $m_1$-type properties and to a decent list of $\mathscr{C}$-extensible spaces that subsumes all previously known cases. Kalton did not stop there: he further produced a no less impressive list of non-$\mathscr{C}$-extensible spaces. Kalton's approach to the $\mathscr{C}$-extension property was primarily designed to deal with Lipschitz maps. Accordingly, in Section 8.3 we present those points of the non-linear theory that are necessary to develop the linear theory. Kalton's subtle analysis crystallises into an asymptotic property of the norm, property $(L^*)$, which implies the almost isometric $\mathscr{C}$-extension property and is enjoyed by most classical sequence spaces … after suitable renormings. The list can be found in Section 8.4. The techniques in Section 8.5 are for the most part independent of those in the rest of the chapter, although the results are not. Section 8.6 contemplates different aspects of Zippin's problem about the extension of $\mathscr{C}$-valued operators from subspaces of $\ell_1$, a seemingly offline question that is, however, central for this book: Zippin's problem is to determine which separable Banach spaces $X$ satisfy $\operatorname{Ext}(X, C(K)) = 0$, where Ext is taken, here and for the rest of the chapter, in the category of Banach spaces. The question admits an interesting gradation in terms of the topological complexity of $K$, and so the chapter continues with a detailed analysis of the class of Banach spaces $X$ for which $\operatorname{Ext}(X, C(\Delta)) = 0$ and the larger class of those for which $\operatorname{Ext}(X, C(\omega^\omega)) = 0$. In Section 8.7, we report the complete solution of the problem of whether $\operatorname{Ext}(C(K), c_0) \neq 0$ for all non-metrisable compacta $K$. The preparations for this travel conclude with a simple and usually overlooked result:

**Lemma 8.0.1** *Finite-rank $c_0$-valued operators admit compact 1-extensions.*

*Proof* We first consider the extension from finite-dimensional subspaces. Let $F$ be a finite-dimensional subspace of a Banach space $X$, and let $\tau\colon F \longrightarrow c_0$ be an operator. Represent $\tau$ by a pointwise null sequence $(\tau_n)$ in $F^*$. For each $n$, let $T_n$ be a Hahn–Banach extension of $\tau_n$ to $X$. The sequence $(T_n)$ defines a 1-extension $T\colon X \longrightarrow \ell_\infty$ of $\tau$. But since $F$ is finite-dimensional, the pointwise null sequence $(\tau_n)$ is actually norm null, as is $(T_n)$, which makes $T$ compact and $c_0$-valued. The general case follows from this, but the reader is left to discover why: given a finite-rank operator $Y \longrightarrow c_0$ with range $F$ defined on a subspace $Y \subset X$, consider the pushout diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ F & \longrightarrow & \mathrm{PO} \\ \big\downarrow & & \\ c_0 & & \end{array}$$

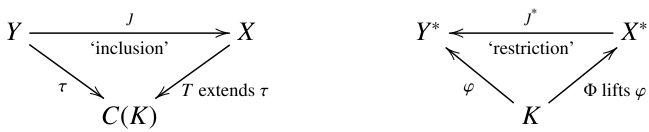
By the argumentation above, the inclusion $F \longrightarrow c_0$ admits a compact 1-extension to PO. And that is all. □

Our previous exposure to local injectivity in Chapter 5 now lets us indulge ourselves in saying:

**Lemma 8.0.2** *Every finite-rank operator taking values in a Lindenstrauss space admits* $1^+$*-extensions to any superspace.*

## 8.1 Zippin Selectors

We focus here on determining when a given operator $\tau\colon Y \longrightarrow C(K)$ has an extension through an embedding $\jmath\colon Y \longrightarrow X$. We look at the situation from the perspective provided by the adjunction in Note 4.6.1(d): the operator $\tau\colon Y \longrightarrow C(K)$ is associated with a weak*-continuous map $\varphi\colon K \longrightarrow Y^*$ by the formula $\langle\varphi(k), y\rangle = (\tau(y))(k)$ and, if the operator $T\colon X \longrightarrow C(K)$ is associated with $\Phi\colon K \longrightarrow X^*$, then $T$ extends $\tau$ through $\jmath$ if and only if $\Phi$ lifts $\varphi$ through $\jmath^*$:




If we assume that $\|\tau\| \le 1$ so that $\varphi[K] \subset B_Y^*$, it is clear that an overwhelmingly sufficient condition for $\tau$ to admit an extension is the existence of a weak*-continuous mapping $\omega\colon B_Y^* \longrightarrow X^*$ such that $\jmath^* \circ \omega$ is the identity on $B_Y^*$. It was Zippin who introduced this 'global approach' idea into the business of extending $\mathscr{C}$-valued operators.

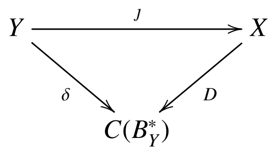
**Definition 8.1.1** Let $\jmath\colon Y \longrightarrow X$ be an embedding between Banach spaces. A Zippin selector for $\jmath$ is a weak*-continuous map $\omega\colon B_Y^* \longrightarrow X^*$ such that $\jmath^* \circ \omega$ is the identity on $B_Y^*$. If $\|\omega(y^*)\| \leq \lambda$ for every $y^* \in B_Y^*$ as well then we call it a $\lambda$-Zippin selector.

Since weak*-compact sets are bounded, we see that each Zippin selector can be labelled with some $\lambda$. The following proof shows that the canonical isometry $\delta\colon Y \longrightarrow C(B_Y^*)$, which was shown to be the best embedding into a $\mathscr{C}$-space regarding complementation in 2.12.2, is the most difficult operator to extend.

**Proposition 8.1.2** *Let $\jmath\colon Y \longrightarrow X$ be an embedding and $\lambda \geq 1$. The following are equivalent:*

(i) *$\jmath$ is $(\lambda, \mathscr{C})$-trivial.*
(ii) *$\delta\colon Y \longrightarrow C(B_Y^*)$ admits a $\lambda$-extension to $X$.*
(iii) *$\jmath$ admits a $\lambda$-Zippin selector.*

*Proof* The implication (i) $\Longrightarrow$ (ii) is trivial. To prove (ii) $\Longrightarrow$ (iii), let $D$ be an operator making commutative the diagram




Since $D\jmath = \delta$ then $\delta^* = \jmath^* D^*$, and thus the restriction of $D^*$ to $B_Y^*$ is a $\|D\|$-Zippin selector for $\jmath$. Note that $D^*|_{B_Y^*}\colon B_Y^* \longrightarrow X^*$ is the weak*-continuous mate of $D$ in the adjunction, while the mate of the inclusion $B_Y^* \longrightarrow Y^*$ is $\delta\colon Y \longrightarrow C(B_Y^*)$ .

The implication (iii) $\Longrightarrow$ (i) has been basically proved above, but we do it again anyway. Assume $\omega\colon B_Y^* \longrightarrow X^*$ is a $\lambda$-Zippin selector for $\jmath$, and let $\tau\colon Y \longrightarrow C(K)$ be an operator with $\|\tau\| = 1$. Define $T\colon X \longrightarrow C(K)$ by $Tx(k) = \langle \omega(\tau^*\delta_k), x\rangle$ for $x \in X$ and $k \in K$. Then $\|T\| \leq \lambda$ and $T\jmath y = \tau y$ since $T\jmath y(k) = \langle \omega(\tau^*\delta_k), \jmath y\rangle = \langle \jmath^*(\omega(\tau^*\delta_k)), y\rangle = \langle \tau^*\delta_k), y\rangle = \langle \delta_k), \tau y\rangle = (\tau y)(k)$. □

Having built an understanding in which we are reasonably confident of what it means to admit a Zippin selector, let us present some natural non-trivial examples of $\mathscr{C}$-trivial embeddings. A warning for the reader about these results: some of them will become obsolete by the end of the chapter.

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**Proposition 8.1.3** *The following embeddings admit Zippin selectors:*

(a) *The inclusion of any subspace of* $\ell_p$ *into* $\ell_p$ *for* $p \in (1, \infty)$.
(b) *The natural embedding of* $\ell_p$ *into the Kalton–Peck space* $\ell_p(\varphi)$ *when* $\varphi \in \mathrm{Lip}_0(\mathbb{R})$ *and* $p \in (1, \infty)$.
(c) *The inclusion of any finite-dimensional subspace whose unit ball is a polyhedron into any Banach space.*
(d) *The inclusion of any finite-dimensional subspace into a uniformly smooth Banach space.*

*Proof* We shall obtain homogeneous 1-Zippin selectors in all cases except (b). Note that in cases (a) and (b) the weak* and weak topologies coincide and that the required selectors in (c) and (d) have to be continuous in the norm topology. The proof of (a) depends on the unbeatable behaviour of the duality map on $\ell_p$. For a Banach space $X$ with strictly convex dual, the duality map (also called the support map) $J\colon X \longrightarrow X^*$ takes each $x \in X$ to the only $x^* \in X^*$ such that $\|x\| = \|x^*\|$ and $\langle x^*, x\rangle = \|x\|\,\|x^*\|$. When $X = \ell_p$ for $1 < p < \infty$, we can identify the dual of $\ell_p$ with $\ell_q$, where $q$ is the conjugate exponent of $p$ and the duality map $J_p\colon \ell_p \longrightarrow \ell_q$ is given by the clean formula $J_p(x) = \|x\|^{2-p}\,\mathrm{sgn}(x)\,|x|^{p-1}$ and turns out to be weak (= weak*) sequentially continuous. Let $Y$ be a subspace of $\ell_p$. Since $\ell_q$ is strictly convex, for each $y^* \in Y^*$, there is a unique $x^* \in \ell_q$ extending $y^*$ with $\|x^*\| = \|y^*\|$. Let $\omega\colon Y^* \longrightarrow \ell_q$ be the resulting map. We show that the restriction $\omega\colon B^*_Y \longrightarrow B^*_{\ell_p}$ is continuous in the weak topology. Indeed, the map $\omega$ is the composition of three maps,

$$\begin{array}{ccc} Y^* & \xrightarrow{\ \omega\ } & \ell_q \\ {\scriptstyle J^{-1}}\downarrow & & \uparrow{\scriptstyle J_p} \\ Y & \xrightarrow{\ \imath\ } & \ell_p \end{array}$$

where $J\colon Y \longrightarrow Y^*$ is the duality map of $Y$ and $\imath$ is plain inclusion. We know that $J_p$ is weakly sequentially continuous, as is the inclusion $\imath$. To see that $J^{-1}\colon B_{Y^*} \longrightarrow B_Y$ is weak* to weak continuous, just observe that $J\colon B_Y \longrightarrow B_{Y^*}$ is one to one and weak to weak* continuous since $J(y) = J_p(y)|_Y$ and $B_Y$ is weakly compact, while $B_{Y^*}$ is weak*-compact.

To prove (b), we revisit the duality of Kalton–Peck spaces $\ell_p(\varphi)$ in Section 3.8 and Kalton–Peck maps (3.27) now garbed in their full-grown centralizer clothes (Section 3.12): consider the mappings $\ell_p \longrightarrow \ell_\infty$ given by

$$\mathsf{KP}_{p,\varphi}(x) = x\,\varphi\left(p\log\frac{\|x\|_p}{|x|}\right), \qquad \mathsf{kp}_{p,\varphi}(x) = x\,\varphi(-p\log|x|),$$

and form the space $\ell_p(\varphi) = \{(y, x) \in \ell_\infty \times \ell_p \colon \|y - \mathsf{KP}_{p,\varphi}x\|_p + \|x\|_p < \infty\}$ endowed with the obvious quasinorm. Consider the canonical embedding

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$\jmath\colon \ell_p \longrightarrow \ell_p(\varphi)$ given by $\jmath(y) = (y, 0)$ and let us try to find a Zippin selector $\omega\colon \ell_p^* \longrightarrow \ell_p(\varphi)^*$ for $\jmath$. Identify $\ell_p(\varphi)^*$ with $\ell_q(-\varphi)$ as in Proposition 3.8.5 through the duality pairing $\langle (x^*, y^*), (y, x)\rangle = \langle x^*, x\rangle + \langle y^*, y\rangle$. Since $\jmath^*(x^*, y^*) = y^*$, we set the Zippin selector $\omega : \ell_q \longrightarrow \ell_q(\varphi)$ to be the map

$$\omega(y^*) = (-\mathsf{kp}_{q,\varphi}(y^*), y^*),$$

which takes values in the right space and is bounded, since, by estimate (3.7),

$$\begin{aligned}\|\omega(y^*)\| &= \left\|(-\mathsf{kp}_{q,\varphi}(y^*), y^*)\right\|_{-\mathsf{KP}_{q,\varphi}}\\ &= \|\mathsf{KP}_{q,\varphi}(y^*) - \mathsf{kp}_{q,\varphi}(y^*)\| + \|y^*\|\\ &\le q\,\mathrm{Lip}(\varphi)\|y^*\|_q \log \|y^*\|_q.\end{aligned}$$

Finally, $\omega$ is weak*-continuous on bounded sets because a sequence converges in the weak (= weak*) topology of $\ell_q(-\varphi)$ if and only if it is bounded and pointwise convergent, and $\mathsf{kp}_{q,\varphi}$ preserves pointwise convergence (the value of each coordinate of $\mathsf{kp}_{q,\varphi}(x)$ depends on the corresponding coordinate of $x$ in a continuous way).

To prove (c), let $\imath\colon F \subset X$ be a finite-dimensional subspace of the Banach space $X$, and let $\imath^*\colon X^* \longrightarrow F^*$ be the restriction map. If $B_F$ is a polyhedron (i.e., the convex hull of a finite set) then so is $B_F^*$. Let $(S_i)_{1\le i\le k}$ be a *triangulation* of $B_F^*$, which means that

- each $S_i$ is a simplex of the same dimension as $F^*$ and $B_F^* = \bigcup_{i=1}^k S_i$,
- the intersection of any pair of them is a (possibly empty) common face.

Let $V$ be the set of all vertices of the triangulation, and observe that not every vertex has to be in the boundary of the ball. For each $v \in V$, let $x_v^* \in X^*$ be a norm-preserving extension of $v$. Now, each $y^* \in B_F^*$ has a unique convex representation as $y^* = \sum_{v\in V} c_v(y^*)v$, and the coordinate functions $y^* \longmapsto c_v(y^*)$ are all continuous. The map $\omega\colon B_F^* \longrightarrow X^*$ given by

$$\omega(y^*) = \sum_{v\in V} c_v(y^*)x_v^*$$

is a 1-Zippin selector. Note that $\omega$ is piecewise linear but not necessarily homogeneous. One can obtain a homogeneous version of $\omega$ by just taking $\tilde{\omega}(0) = 0$ and

$$\tilde{\omega}(y^*) = \frac{\|y^*\|}{2}\left(\omega\left(\frac{y^*}{\|y^*\|}\right) - \omega\left(\frac{-y^*}{\|y^*\|}\right)\right).$$

This map is again weak* continuous since $F^*$ is finite-dimensional.

(d) Let $F$ be a finite-dimensional subspace of $X$. Then, for every $n \in \mathbb{N}$, there is a Zippin selector $\omega_n\colon B_F^* \longrightarrow (1 + \frac{1}{n})B_X^*$. A weak*-accumulation point of $\omega_n(y^*)$ must be an extension of $y^*$, which in a uniformly smooth space is unique. So it makes sense to define $\omega(y^*)$ as the weak*-accumulation point of $\omega_n(x^*)$, and this yields a selector $\omega\colon B_F^* \longrightarrow B_X^*$. Again by uniform smoothness, for every normalised $y^* \in F^*$, we have $\omega(y^*) = \lim_n \omega_n(x^*)$ in the norm topology of $X^*$. Hence $\omega$ is weak*-continuous on the unit sphere of $F^*$. □

## 8.2 The Lindenstrauss–Pełczyński Theorem

We now present the Lindenstrauss–Pełczyński theorem, which is perhaps the first significant result about the extension of $\mathscr{C}$-valued operators. We shall provide two (actually three) proofs for this important result. The first has a homological flavour but ultimately depends on the simple structure of the subspaces and quotients of $c_0$ and comes without any explicit bound on the norm of the extension. Here it is.

**Theorem 8.2.1** *Every $\mathscr{C}$-valued operator defined on a subspace of $c_0$ can be extended to $c_0$.*

*Proof* The proof proceeds in several steps.

*Step 1* We test our abilities when the subspace and the embedding have the simplest conceivable form. For each integer $k$, let $\jmath_k\colon A_k \longrightarrow \ell_\infty^{n(k)}$ be an isometry and let $\jmath\colon c_0(\mathbb{N}, A_k) \longrightarrow c_0(\mathbb{N}, \ell_\infty^{n(k)})$ be the isometry given by $\jmath((a_k)_k) = (\jmath_k(a_k))_k$. For each $k \in \mathbb{N}$, let $\omega_k\colon B_{A_k}^* \longrightarrow \ell_1^{n(k)}$ be the homogeneous 1-Zippin selector constructed in the proof of Proposition 8.1.3(c). Paste together all the maps $\omega_k$ into one map $\omega\colon B_{\ell_1(\mathbb{N}, A_k^*)} \longrightarrow B_{\ell_1(\mathbb{N}, \ell_1^{n(k)})}$ by setting $\omega((x_k^*)_k) = (\omega_k(x_k^*))_k$. This is clearly a selector for $\jmath$, and it turns out to be weak*-continuous: observe that its domain is metrisable and that a bounded sequence in the space $\ell_1(\mathbb{N}, A_k)$ is weak*-null if and only if the norms of its projections into each $A_k$ are convergent to 0. This shows that the embedding $\jmath$ is $\mathscr{C}$-trivial.

*Step 2* We keep the subspace isomorphic to some $c_0(\mathbb{N}, A_k)$, with $A_k$ finite-dimensional, but now consider any possible embedding $\imath\colon c_0(\mathbb{N}, A_k) \longrightarrow c_0$. Let us pick an almost isometric embedding $\jmath\colon c_0(\mathbb{N}, A_k) \longrightarrow c_0(\mathbb{N}, \ell_\infty^{n(k)}) = c_0$ once again. Replacing $c_0$ by its square if necessary, we can assume that both $\jmath$ and $\imath$ have infinite codimensional ranges, and so the automorphic character of $c_0$ comes to the rescue by providing an automorphism $u$ that intertwines $\jmath$ and $\imath$ as in the commutative diagram

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$$\begin{array}{ccc} c_0(\mathbb{N},A_k) & \xrightarrow{\ \jmath\ } & c_0 \\ & \searrow^{\imath} & \big\downarrow u \\ & & c_0(\mathbb{N},\ell_\infty^{n(k)}) = c_0 \end{array}$$

Needless to say, an operator defined on $c_0(\mathbb{N}, A_k)$ can be extended through $\imath$ if and only if it can be extended through $\jmath$.

*Step 3* Finally, if $Y$ is any arbitrary closed subspace of $c_0$, we use the Johnson–Rosenthal–Zippin result in Proposition 5.3.1 to decompose $Y$ as a twisted sum $0 \longrightarrow c_0(\mathbb{N}, A_k) \longrightarrow Y \longrightarrow c_0(\mathbb{N}, B_k) \longrightarrow 0$ in which $A_k$ and $B_k$ are finite-dimensional. Now we draw the pushout diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & c_0(\mathbb{N},A_n) & = & c_0(\mathbb{N},A_n) & & & & \\ & & \downarrow \imath & & \downarrow & & & & \\ 0 & \longrightarrow & Y & \xrightarrow{\ \jmath\ } & c_0 & \longrightarrow & c_0/Y & \longrightarrow & 0 \\ & & \downarrow \rho & & \downarrow \overline{\rho} & & \| & & \\ 0 & \longrightarrow & c_0(\mathbb{N},B_n) & \xrightarrow[\ \overline{\jmath}\ ]{} & \mathrm{PO} & \longrightarrow & c_0/Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Let $\tau\colon Y \longrightarrow E$ be a $\mathscr{C}$-valued operator. Its restriction $\tau\,\imath$ can be extended to $c_0$. Let $T_1\colon c_0 \longrightarrow E$ be any extension of $\tau\,\imath$. The difference $\tau - T_1\,\jmath$ vanishes on the image of $\imath$, and thus it factors through the quotient $\rho$ in the form $\tau - T_1\,\jmath = \tau_2\,\rho$ for some operator $\tau_2\colon c_0(\mathbb{N}, B_k) \longrightarrow E$. Since every quotient of $c_0$ is isomorphic to a subspace of $c_0$, $\tau_2$ admits an extension, say $T_2$, to PO. The operator $T = T_1 + T_2\,\overline{\rho}$ is an extension of $\tau$ since, according to our records, for every $y \in Y$, we have $T_2\,(\overline{\rho}(y)) = T_2\,(\overline{\jmath}(\rho(y)) = \tau_2(\rho(y)) = \tau(y) - T_1(y)$. □

The statement just proved implies the existence of a constant $C$ such that every $\mathscr{C}$-valued operator defined on any subspace of $c_0$ has a $C$-extension to $c_0$: this follows from an obvious amalgamation argument, taking into account that if $(H_n)$ is a sequence of subspaces of $c_0$ then so is $c_0(\mathbb{N}, H_n)$, and that if $(E_n)$ is a sequence of $\mathscr{C}$-spaces then so is $\ell_\infty(\mathbb{N}, E_n)$. However, there is no easy way to follow the track of $C$ throughout the proof since one has no control over the parameters of the left vertical sequence in the preceding diagram nor over the norm of the automorphism in the second step of the proof. In [465, Proposition 3], Zippin provides a complete proof for the following: *every $\mathscr{C}$-valued operator defined on a subspace of $c_0$ admits a $4^+$-extension*. The proof uses his global approach, very much in the spirit of the previous proof, but without the good (homological) parts.

We will stop burdening the reader with these worries and next recover the original proof supplied by Lindenstrauss and Pełczyński [330], which provides sharp bounds for the norm of the extension. The ripples from this stone thrown in the $\mathscr{C}$-extension pond will spread out through the entire chapter.

Let $K$ be a compact space, and suppose we are given two bounded (but not necessarily continuous) functions $f, g \colon K \longrightarrow \mathbb{R}$ such that $f \leq g$. Under what circumstances can we separate them by a continuous $h \colon K \longrightarrow \mathbb{R}$ in the sense that $f(s) \leq h(s) \leq g(s)$ for all $s \in K$? The Hahn–Tong theorem (see the proof of Lemma 2.2.2) states that this is the case if $f$ is upper-semicontinuous and $g$ is lower-semicontinuous. It quickly follows that such an $h \in C(K)$ exists if and only if the upper semicontinuous envelope $f^{\text{usc}}(s) = \min\left(f(s), \limsup_{t \to s} f(t)\right)$ of $f$ and the lower semicontinuous envelope $g_{\text{lsc}}(s) = \max\left(f(s), \liminf_{t \to s} g(t)\right)$ of $g$ satisfy $f^{\text{usc}} \leq g_{\text{lsc}}$. Enough said:

**Theorem 8.2.2** *Every $\mathscr{C}$-valued operator defined on a subspace of $c_0$ admits a $1^+$-extension to the whole $c_0$.*

*Proof* Let $\tau \colon H \longrightarrow C(K)$ be an operator, where $H$ is a subspace of $c_0$ and $K$ is a compact space. Since every separable subset of $C(K)$ is contained in a separable subalgebra and these are $\mathscr{C}$-spaces, we may assume that $K$ is metrisable and also that $\|\tau\| = 1$. It is enough to prove that $\tau$ can be extended to an operator $T$ on $H + [x]$ having norm at most $\lambda$ for each $x \in c_0 \setminus H$ and $\lambda > 1$. Searching for an admissible value $h = T(x)$ means showing that there exists $h \in C(K)$ such that $\|h - \tau(y)\| \leq \lambda \|x - y\|$ for all $y \in H$. Using the order structure of $C(K)$, this is equivalent to

$$\tau(y) - \lambda \|y - x\| 1_K \leq h \leq \tau(y) + \lambda \|y - x\| 1_K. \tag{8.1}$$

Letting

$$F = \bigvee_{y \in H} \left(\tau(y) - \lambda \|x - y\| 1_K\right) \quad \text{and} \quad G = \bigwedge_{y \in H} \left(\tau(y) + \lambda \|x - y\| 1_K\right),$$

we see that $F \leq G$ and that $h$ fits in (8.1) if and only if $F \leq h \leq G$. The proof will be complete if we show that $F^{\text{usc}} \leq G_{\text{lsc}}$. Here is where $c_0$ comes into play. If for some $s \in K$ we have $F^{\text{usc}}(s) > G_{\text{lsc}}(s)$ then there are sequences $(s_n), (t_n)$ in $K$ converging to $s$ for which

$$\lim_n F(s_n) > \lim_n G(t_n).$$

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From the definition of $F$ and $G$, it can easily be deduced that there exist $y_n, z_n \in H$ such that

$$\lim_n \Big(\tau(y_n)(s_n) - \lambda\|x - y_n\|\Big) > \lim_n \Big(\tau(z_n)(t_n) + \lambda\|x - z_n\|\Big). \tag{8.2}$$

Now we consider the adjoint $\tau^*\colon C(K)^* \longrightarrow H^*$. For each $n$, $\tau^*(\delta_{s_n})$ is a linear functional on $H$ that admits an equal norm extension to $c_0$, which we will call $u_n \in \ell_1$. Similarly, for each $n$, let $v_n \in \ell_1$ be a norm-preserving extension of $\tau^*(\delta_{t_n})$ to $c_0$. Note that $\|u_n\|, \|v_n\| \leq 1$. Passing to subsequences if necessary, we can assume that both $(u_n)_n$ and $(v_n)_n$ are weak*-convergent in $\ell_1$. If we denote the corresponding limits by $u$ and $v$, then $u|_H = v|_H = \tau^*(\delta_s)$ since $\tau^*$ is continuous in the weak* topologies and $\delta_{t_n}, \delta_{s_n}$ converge to $\delta_s$ in the weak* topology of $C(K)^*$. On the other hand, if $(w_n)_n$ is weak*-null in $\ell_1$, it is almost obvious that for every $w \in \ell_1$ we have $\lim_n (\|w_n + w\| - \|w_n\| - \|w\|) = 0$. In particular,

$$\lim_{n\to\infty} \Big(\|u_n\| - \|u_n - u\| - \|u\|\Big) = \lim_{n\to\infty} \Big(\|v_n\| - \|v_n - v\| - \|v\|\Big) = 0. \tag{8.3}$$

If we restrict these functionals to $H$ then the norms of $u_n$ and $v_n$ do not vary (they are norm-preserving extensions of $\tau^*\delta_{s_n}$ and $\tau^*\delta_{t_n}$), while the norms of the other functionals in (8.3) only decrease. Letting $r = \big\| u|_H \big\| = \big\| v|_H \big\| = \big\| \tau^*(\delta_s) \big\|$, we have $\limsup_n \|u_n - u\| \leq 1 - r$ and also $\limsup_n \|v_n - v\| \leq 1 - r$, both with respect to the $\ell_1$ norm. Since $\tau(y_n)(s_n) = \langle \tau^* s_n, y_n\rangle = \langle u_n, y_n\rangle$ and $\tau(z_n)(t_n) = \langle \tau^* t_n, z_n\rangle = \langle v_n, z_n\rangle$, we can rewrite (8.3) as

$$L = \lim_n \Big(\langle v_n, z_n\rangle - \langle u_n, y_n\rangle + \lambda(\|x - z_n\| + \|x - y_n\|)\Big) < 0. \tag{8.4}$$

But $\lim_n\langle u_n - u, z\rangle = \lim_n\langle v_n - v, z\rangle = 0$ for every $z \in c_0$, so

$$\begin{aligned}
L &= \lim_n \Big(\langle v_n, z_n\rangle - \langle u_n, y_n\rangle + \lambda(\|x - z_n\| + \|x - y_n\|)\Big)\\
&= \lim_n \Big(\langle v_n - v, z_n\rangle - \langle u_n - u, y_n\rangle + \langle \tau^*(\delta_s), z_n - y_n\rangle + \lambda(\|x - z_n\| + \|x - y_n\|)\Big)\\
&= \lim_n \Big(\langle v_n - v, z_n - x\rangle - \langle u_n - u, y_n - x\rangle + \langle \tau^*\delta_s, z_n - y_n\rangle + \lambda(\|x - z_n\| + \|x - y_n\|)\Big)\\
&= \lim_n (\alpha_n + \beta_n + \gamma_n),
\end{aligned}$$

with the obvious choices

$$\begin{aligned}
\alpha_n &= \langle v_n - v, z_n - x\rangle + (\lambda - r)\|z_n - x\|,\\
\beta_n &= -\langle u_n - u, y_n - x\rangle + (\lambda - r)\|y_n - x\|,\\
\gamma_n &= \langle \tau^*(\delta_s), z_n - y_n\rangle + r(\|x - z_n\| + \|x - y_n\|).
\end{aligned}$$

Now, $\alpha_n$ and $\beta_n$ are non-negative for large $n$, while $\gamma_n \geq 0$ for all $n$. This contradicts (8.4), which concludes the proof. □

We cannot usually achieve 1-extensions, as the following example shows.

**8.2.3** *There is a compact operator from an hyperplane of $c_0$ into $c$ which has no norm-preserving extension to $c_0$.*

*Proof* Fix $0 < \lambda < \frac{1}{2}$ and define $f = \sum_{n=1}^{\infty} \lambda^{n-1}(e_{2n-1} + e_{2n})$. Clearly, $f \in \ell_1$. Treating $\ell_1$ as the dual of $c_0$ in the obvious way, we set $H = \ker f$. Let

$$g = \sum_{n=1}^{\infty} \lambda^{n-1} e_{2n} - \sum_{n=2}^{\infty} \lambda^{n-1} e_{2n-1},$$

and for $n = 2, 3, ...$, we set

$$g_n = \sum_{j=1}^{n} \lambda^{j-1} e_{2j} - \sum_{j=2}^{n} \lambda^{j-1} e_{2j-1},$$

$$h_n = \sum_{j=1}^{n} \lambda^{j-1} e_{2j} - \sum_{j=2}^{n} \lambda^{j-1} e_{2j-1} + \lambda^{n-1}(e_{2n-1} + e_{2n+2}).$$

It is clear that $g = \lim g_n = \lim h_n$ in the norm of $\ell_1$. In particular, the convergence is maintained in $H^*$. We put

$$G_n = \frac{g_n|_H}{\|g_n|_H\|}, \quad H_n = \frac{h_n|_H}{\|h_n|_H\|}, \quad G = \frac{g|_H}{\|g|_H\|}.$$

The set $K = \{G\} \bigcup \{G_n, H_n : n \in \mathbb{N}\}$ is a norm-closed subset of $H^*$ homeomorphic to $\alpha\mathbb{N}$, the one point compactification of $\mathbb{N}$, and the inclusion of $K$ into the unit ball of $H^*$ is *norm* continuous so it induces a *compact* operator $\tau : H \longrightarrow C(K)$. By the considerations made at the beginning of the section, everything we have to show is that there is no weak*-continuous map $\phi : K \longrightarrow B_{\ell_1}$ that satisfies $\phi(k)|_H = k$ for all $k \in K$. To show that, it is enough to show that all $G_n$ and $H_n$ have unique Hahn–Banach extensions $g_n$ and $h_n - f$, respectively. Since $\lim g_n = g$ and $\lim h_n - f = g - f$, the proof is essentially done. To fill in the details, observe that any extension of $G_n$ must have the form $g_n + tf$. Since

$$\|g_n + tf\| = |t|\left(1 + \frac{2\lambda^n}{1-\lambda}\right) + |1-t|\frac{1-\lambda^n}{1-\lambda} + \lambda|1+t|\frac{1-\lambda^{n-1}}{1-\lambda}$$

and the minimum of this quantity occurs only at $t = 0$, it follows that the unique norm-preserving extension of $G_n$ is $g_n$. Also, any extension of $H_n$ must have the form $h_n + tf$. Since

$$\|h_n + tf\| = |t|\left(1 + \frac{2\lambda^{n+1}}{1-\lambda}\right) + |1-t|\frac{1-\lambda^n}{1-\lambda} + \lambda|1+t|\frac{1-\lambda^{n-1}}{1-\lambda} + 2\lambda^{n-1}|1-\lambda t|$$

and the minimum of this quantity occurs only at $t = 1$, it follows that the unique Hahn–Banach extension of $H_n$ is $h_n - f$. □

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## 8.3 Kalton's Approach to the $\mathscr{C}$-Extension Property

Kalton's approach to the extension of $\mathscr{C}$-valued operators is rooted in his studies of Lipschitz maps in Banach spaces. Lipschitz maps have appeared in Banach space theory since its inception, sporadically at first but occupying an increasingly central role. The authoritative book by Benyamini and Lindenstrauss [41] is responsible to a great extent for this turnaround. In the series of papers [271; 272; 273; 274], Kalton revisits the topic and, as he always did, sets new standards. Of course, the linear and non-linear theories are different, and thus we will present here only the facts of the non-linear theory that are indispensable for understanding the linear part. The interested reader can behold a more complete picture in Section 8.8.3. The following observation helps avoid possible misunderstandings concerning the role of the space $c$ in the ensuing exposition.

**Proposition 8.3.1** *There is a* 2*-Lipschitz retraction of* $\ell_\infty$ *onto* $c$*.*

*Proof* Given $x \in \ell_\infty$, set

$$x^- = \liminf_n x(n), \quad x^- = \liminf_n x(n), \quad m(x) = \frac{x^- + x^+}{2}, \quad r(x) = \frac{x^+ - x^-}{2},$$

and define $\tilde{x} \colon \mathbb{N} \longrightarrow \mathbb{R}$ by

$$\tilde{x}(n) = \begin{cases} m(x) & \text{if } x(n) \in [x^-, x^+], \\ x(n) - r(x) & \text{if } x(n) > x^+, \\ x(n) + r(x) & \text{if } x(n) < x^-. \end{cases}$$

It is clear that $\tilde{x} \in c$ for all $x \in \ell_\infty$ and also that $\tilde{x} = x$ when $x \in c$, so the mapping $x \longrightarrow \tilde{x}$ extends the identity of $c$. To prove that this map is 2-Lipschitz, one has to check that $|\tilde{x}(n) - \tilde{y}(n)| \leq 2\|x - y\|$ for all $n$. This is very easy, taking into account that $|m(x) - m(y)|, |r(x) - r(y) \leq \|x - y\|$ and reasoning on a case-by-case basis. □

However, no Lipschitz retraction of $\ell_\infty$ onto $c$ can be linear at all because $c$ is not injective in the category of Banach spaces (even if it is injective in 'the Lipschitz category of Banach spaces'). Is separability the problem? Not on its own. In fact, $c$ is injective in both the Lipschitz and the linear categories of separable Banach spaces, but the constants are different: $c$ is 3-separably injective as a Banach space, while it is 2-injective in the Lipschitz category. So, even in this setting, the previous proposition highlights a subtle difference between the extension of Lipchitz maps and the extension of linear operators.

**Corollary 8.3.2** *Let* $Y$ *be a subset of a metric space* $X$*. Every Lipschitz map* $\tau \colon Y \longrightarrow c$ *admits a* 2*-extension to* $X$*.*

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*Proof* Extend a contraction $\tau$ to a contraction $X \longrightarrow \ell_\infty$ and then compose with the map provided by Proposition 8.3.1. □

## Extension of $\mathscr{C}$-Valued Lipchitz Maps for Dummies

For a first contact with the problems we face, let us cheat a bit about the simplest conceivable extension problem for non-linear Lipschitz $\mathscr{C}$-valued maps by picking arguably the simplest infinite compactum there is: the one-point compactification $\alpha\mathbb{N}$ of $\mathbb{N}$ and its associated space of convergent sequences $c = C(\alpha\mathbb{N})$. Let $Y$ be a subset of a metric space $X$, and let $\tau\colon Y \longrightarrow c$ be a fixed contraction, which is not assumed to be linear for obvious reasons. Given $x \in X\backslash Y$ and $\lambda \geq 1$, under what conditions can $\tau$ be extended to a $\lambda$-Lipschitz map $Y \cup \{x\} \longrightarrow c$? As we already know by Corollary 8.3.2, this can always be achieved when $\lambda \geq 2$. Of course, what has to be done is to assign an admissible value $\xi \in c$ to $x$ such that $\|\xi - \tau(y)\| \leq \lambda d(x, y)$ for every $y \in Y$. This means, taking advantage of the order structure of $c$, that $\xi$ must satisfy

$$\tau(y) - \lambda d(x, y)1_{\mathbb{N}} \leq \xi \leq \tau(y) + \lambda d(x, y)1_{\mathbb{N}}. \tag{8.5}$$

We define two bounded functions $\tau^-, \tau^+$ on $\mathbb{N}$ by

$$\tau^- = \bigvee_{y\in Y} \tau(y) - \lambda d(x, y)1_{\mathbb{N}}, \qquad \tau^+ = \bigwedge_{z\in Y} \tau(z) + \lambda d(x, z)1_{\mathbb{N}},$$

where the supremum and the infimum are defined pointwise on $\mathbb{N}$. It is clear that $\tau^- \leq \tau^+$ and also that $\xi \in \ell_\infty$ satisfy (8.5) if and only if $\tau^- \leq \xi \leq \tau^+$. Now, the point is that some $\xi \in c$ exists separating $\tau^-$ from $\tau^+$ if and only if

$$\limsup_{n\to\infty} \tau^-(n) \leq \liminf_{n\to\infty} \tau^+(n),$$

and this condition is clearly equivalent to:

- For every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for every $n, k > N$ and every $y, z \in Y$, one has $|\tau(y)(n) - \tau(z)(m)| \leq \lambda(d(y, x) + d(z, x)) + \varepsilon$.

Corollary 8.3.2 says that this condition is satisfied by all Lipschitz maps on all metric spaces when $\lambda = 2$. It is up to the reader whether to decide whether or not this was unforeseen.

## Extension of $c$-Valued Lipschitz Maps

Let's keep up our steady trotting pace with a slight modification to the previous problem: we now want to fix a $\lambda$ and extend all contractions $Y \longrightarrow c$ to one

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more point while keeping the Lipschitz constant of the extension bounded by $\lambda$. This will be the key step in the extension of $\mathscr{C}$-valued Lipschitz maps.

**Lemma 8.3.3** *Let $Y$ be a separable subset of a metric space $X$, $x \in X \backslash Y$ and $\lambda \geq 1$. The following are equivalent:*

(i) *Every Lipschitz map $\tau \colon Y \longrightarrow c$ admits a $\lambda$-extension to $Y \cup \{x\}$.*
(ii) *Given sequences $(y_n), (z_n)$ in $Y$ and $\varepsilon > 0$, there is $u \in Y$ such that*
$$d(u, y_n) + d(u, z_n) \leq \lambda\big(d(x, y_n) + d(x, z_n)\big) + \varepsilon \tag{8.6}$$
*for infinitely many $n$.*
(iii) *Every $\mathscr{C}$-valued Lipschitz map on $Y$ admits a $\lambda$-extension to $Y \cup \{x\}$.*

*Proof* We will work with contractions. To prove the implication (i) $\Longrightarrow$ (ii), let us fix a countable dense subset $(e_n)_{n\geq 1}$ of $Y$. The following condition is clearly stronger than (ii):

(†) Given $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that for any $y, z \in Y$ we have
$$\min_{1\leq j\leq n} \Big(d(e_j, y) + d(e_j, z)\Big) \leq \lambda\,(d(x, y) + d(x, z)) + \varepsilon.$$

Now we establish the implication (i) $\Longrightarrow$ (†). If (†) fails, we can construct sequences $(y_n)$ and $(z_n)$ in $Y$ such that, for $j < n$, we have
$$\begin{aligned}
\lambda\,(d(x, y_n) + d(x, z_n)) + \varepsilon &< d(y_j, y_n) + d(y_j, z_n),\\
\lambda\,(d(x, y_n) + d(x, z_n)) + \varepsilon &< d(z_j, y_n) + d(y_j, z_n),\\
\lambda\,(d(x, y_n) + d(x, z_n)) + \varepsilon &< d(e_j, y_n) + d(e_j, z_n).
\end{aligned} \tag{8.7}$$

Given $e \in Y$, the sequence $(d(e, y_n) - d(x, y_n))$ is bounded by $d(e, x)$, and thus, since $Y$ is separable, by merciless diagonalisation, we can assume that $\lim_n d(e, y_n) - d(x, y_n)$ exists for all $e \in Y$. We define now a sequence of Lipschitz maps $f_n \colon Y \longrightarrow \mathbb{R}$. First the odd terms:
$$f_{2n-1}(y) = d(y, y_n) - d(x, y_n),$$
which have $\mathrm{Lip}(f_{2n-1}) = 1$. To define the even terms, we first define new Lipschitz maps $\varphi_n \colon \{y_k\}_{k=1}^n \cup \{z_k\}_{k=1}^n \cup \{e_k\}_{k=1}^n \longrightarrow \mathbb{R}$ by
$$\begin{cases} \varphi_n(y_j) = f_{2n-1}(y_j), & j < n,\\ \varphi_n(z_j) = f_{2n-1}(z_j), & j < n,\\ \varphi_n(e_j) = f_{2n-1}(e_j), & j \leq n,\end{cases}$$
and
$$\begin{cases} \varphi_n(y_n) = f_{2n-1}(y_n) + \lambda\big(d(y_n, x) + d(z_n, x)\big) - d(y_n, z_n) + \varepsilon,\\ \varphi_n(z_n) = f_{2n-1}(z_n) + \lambda\,(d(y_n, x) + d(z_n, x)) - d(y_n, z_n) + \varepsilon.\end{cases}$$

**Claim** $\mathrm{Lip}(\varphi_n) \leq 1$.

*Proof of the claim* It suffices to estimate $\varphi_n(w)-\varphi_n(w')$ when $w \in \{y_n, z_n\}$ and $w' \in \{y_1, \ldots, y_{n-1}, z_1, \ldots, z_{n-1}, e_1, \ldots, e_n\}$. Because if so,

$$\varphi_n(w) - \varphi_n(w') \geq f_{2n-1}(w) - f_{2n-1}(w') \geq -d(w, w'),$$

and the claim is proved using the inequalities (8.7) to get

$$\varphi_n(w) - \varphi_n(w') \leq d(w, y_n) + d(z_n, w') - d(y_n, z_n) \leq d(w, w'). \qquad \square$$

The even term $f_{2n}$ can be any contractive extension of $\varphi_n$ to $Y$. Now let $\tau\colon Y \longrightarrow \ell_\infty$ be the contractive map given by $\tau(y) = (f_n(y))_n$. Note that $\tau(e_j) \in c$ for every $j$, and since the $e_j$ generate a dense subspace in $Y$, it turns out that $\tau$ takes values in $c$. We prove that $\tau$ cannot be extended to a $\lambda$-Lipschitz mapping $\tilde{\tau}\colon Y \cup \{x\} \longrightarrow c$. Suppose that $\tilde{\tau} = (g_n)_{n\leq 1}$ is such an extension. Then it follows from

$$\begin{cases} g_{2n}(x) \geq f_{2n}(z_n) - \lambda d(x, z_n) \\ g_{2n-1}(x) \leq f_{2n-1}(y_n) + \lambda d(x, y_n) \end{cases}$$

that $g_{2n}(x) - g_{2n-1}(x) \geq \varepsilon$, which flagrantly contradicts our assumption that $\tilde{\tau}$ takes values in $c$.

Let us prove the much easier implication (ii) $\Longrightarrow$ (iii). Let $\tau : Y \longrightarrow C(K)$ be a contractive map that we want to extend to a $\lambda$-Lipschitz map on $Y \cup \{x\}$. This amounts to finding $f \in C(K)$ such that $\|f - \tau(y)\|_\infty \leq \lambda d(x, y)$ for all $y \in Y$; that is,

$$\tau(y) - \lambda d(x, y)1_K \leq f \leq \tau(y) - \lambda d(x, y)1_K. \qquad (8.8)$$

Define $F, G \in \ell_\infty(K)$ by

$$F = \bigvee_{y \in Y} \tau(y) - \lambda d(x, y)1_K, \qquad G = \bigwedge_{z \in Y} \tau(z) + \lambda d(x, z)1_K.$$

Clearly, $F \leq G$, and $f$ satisfies (8.8) if and only if $F \leq f \leq G$. Thus the proof will be complete if we show for each $s \in K$ that

$$\limsup_{t \to s} F(t) \leq \liminf_{t \to s} G(t).$$

Assume on the contrary that $\limsup_{t\to s} F(t) > \liminf_{t\to s} G(t)$ for some $s \in K$. Then there is $\varepsilon > 0$ and a pair of sequences $(s_n), (t_n)$ converging to $s$ such that $F(s_n) > G(t_n) + 2\varepsilon$, and then we may choose sequences $(y_n), (z_n)$ in $Y$ such that

$$(\tau(y_n))(s_n) - \lambda d(x, y_n) > \tau(z_n)(t_n) + \lambda d(x, z_n) + 2\varepsilon.$$

Now we apply (ii) with these $(y_n), (z_n)$ and $\varepsilon$ to find $u \in Y$ such that

$$d(u, y_n) + d(u, z_n) \leq \lambda\big(d(x, y_n) + d(x, z_n)\big) + \varepsilon$$

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for infinitely many $n$. Let us take a look at the function $g = \tau(u)$. We have $\tau(y_n) \leq g + d(u, y_n)$ and $\tau(z_n) \geq g - d(u, z_n)$. For those $n$ we thus obtain

$$\begin{aligned} g(s_n) - g(s_n) &\geq \tau(y_n)(s_n) - d(u, y_n) - \tau(z_n)(t_n) - d(u, z_n) \\ &> \lambda\big(d(x, y_n) + d(x, z_n)\big) + 2\varepsilon - d(u, y_n) - d(u, z_n) \geq \varepsilon, \end{aligned}$$

which contradicts the continuity of $g$ at $s$. □

## Back to Linear Operators

Let us see if the connections we have sown between linear and non-linear extensions bring forth a harvest by returning to the main topic of the chapter. First of all, we isolate the metric configuration supporting Lemma 8.3.3:

**Definition 8.3.4** Let $Y$ be a subset of a metric space $X$ and $\lambda \geq 1$. We say the pair $(Y, X)$ satisfies condition $\Sigma_1(\lambda)$ if, given sequences $(y_n), (z_n)$ in $Y$, $x \in X$ and $\varepsilon > 0$, there exists $u \in Y$ such that

$$d(u, y_n) + d(u, z_n) \leq \lambda\big(d(x, y_n) + d(x, z_n)\big) + \varepsilon$$

for infinitely many $n$.

Thus, the real content of Lemma 8.3.3 is that if $Y$ is a subset of a separable Banach space $X$ then the pair $(Y, X)$ satisfies $\Sigma_1(\lambda)$ if and only if $\mathscr{C}$-valued (or $c$-valued) Lipschitz maps defined on $Y$ admit $\lambda$-extensions to one further point in $X$. A simple remark provides the necessary irrigation for our seeds.

**Lemma 8.3.5** *Let $H$ be a closed hyperplane of a Banach space $X$, and let $\tau\colon H \longrightarrow E$ be a contractive operator. Let $x \in X \setminus H$ and $\lambda \geq 1$. If $\tau$ has a $\lambda$-Lipschitz extension to $H \cup \{x\}$ then it has a $\lambda$-linear extension $T\colon X \longrightarrow E$.*

*Proof* If $\tilde{\tau}\colon H \cup \{x\} \longrightarrow E$ is $\lambda$-Lipschitz extension, pick $\xi = \tilde{\tau}(x)$ and set $T(y + tx) = \tau(y) + t\xi$. This $T$ is a linear extension of $\tau$ and $\|T\| \leq \lambda$ since

$$\|T(y + tx)\| = |t|\,\|\tau(y/t) + \xi\| = |t|\,\|\xi - \tau(-y/t)\| \leq |t|\,\lambda\,\|x + y/t\| = \lambda\|y + tx\|. \quad □$$

Time to provide some restorative shadow confirming that linear extension properties are much more demanding than their Lipschitz counterparts.

**Theorem 8.3.6** *Let $E$ be a closed subspace of a separable Banach space $X$ and let $\lambda > 1$. If every operator $\tau\colon E \longrightarrow c$ admits a $\lambda$-extension then $(E, X)$ has property $\Sigma_1(\lambda)$.*

*Proof* If the pair $(Y, X)$ does not satisfy $\Sigma_1(\lambda)$ then there are two sequences $(y_n), (z_n)$ in $Y$, a point $x \in X \setminus Y$ and an $\varepsilon > 0$ such that for every $u \in Y$ the set of $n$ for which

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$$\|u - y_n\| + \|u - z_n\| < \lambda(\|x - y_n\| + \|x - z_n\|) + 2\varepsilon$$

is finite. This implies that for each compact subset $K \subset Y$ there is an $n(K)$ such that for all $u \in K$ and $n \geq n(K)$ we have

$$\|u - y_n\| + \|u - z_n\| > \lambda(\|x - y_n\| + \|x - z_n\|) + \varepsilon.$$

Since $Y$ is separable, there is an increasing sequence of compact subsets of $Y$ containing the origin, say $(K_m)$, whose union is dense in $Y$. It then follows that we can choose a subsequence $\mathbb{M} = \{n(1), n(2), \dots\}$ such that when $u \in K_m$,

$$\|u - y_{n(m)}\| + \|u - z_{n(m)}\| > \lambda(\|x - y_{n(m)}\| + \|x - z_{n(m)}\|) + \varepsilon. \tag{8.9}$$

Next, observe that if $A$ is a Banach space and $V$ is a compact, convex subset of $A$ such that $\|v - a\| > c$ for some $a \in A$ and $c \geq 0$ and all $v \in V$, then there exists a functional $a^* \in A^*$ with $\|a^*\| \leq 1$ such that $\langle a^*, v - a\rangle \geq c$ for all $v \in V$. To see this, just apply the Hahn–Banach theorem to the effect of separating the compact convex set $V - a$ from the closed ball of radius $c$ centered at the origin. If we now interpret $\|u - y_{n(m)}\| + \|u - z_{n(m)}\|$ as the norm of the difference $(u, u) - (y_{n(m)}, z_{n(m)})$ in the space $X \oplus_1 X$, since $(X \oplus_1 X)^* = X \oplus_\infty X$, from (8.9), we get the existence of $y_m^*, z_m^* \in X^*$ with $\|y_m^*\|, \|z_m^*\| \leq 1$ such that when $u \in K_m$,

$$\langle y_m^*, u - y_{n(m)}\rangle + \langle z_m^*, u - z_{n(m)}\rangle \geq \lambda(\|x - y_{n(m)}\| + \|x - z_{n(m)}\|) + \varepsilon. \tag{8.10}$$

Hence, for $u \in K_m$,

$$\begin{aligned}\langle y_m^* + z_m^*, u\rangle &\geq \langle y_m^*, y_{n(m)}\rangle + \langle z_m^*, z_{n(m)}\rangle + \lambda(\|x - y_{n(m)}\| + \|x - z_{n(m)}\|) + \varepsilon\\ &\geq \langle y_m^*, y_{n(m)}\rangle + \langle z_m^*, z_{n(m)}\rangle + \lambda(\langle y_m^*, x - y_{n(m)}\rangle + \langle z_m^*, x - z_{n(m)}\rangle) + \varepsilon\\ &\geq \langle y_m^*, x\rangle + \langle z_m^*, x\rangle + \varepsilon. \end{aligned} \tag{8.11}$$

At this point, we pass to a further subsequence $\mathbb{L}$ of $\mathbb{M}$ such that $(y_m^*)_{m\in\mathbb{L}}$ and $(z_m^*)_{m\in\mathbb{L}}$ are weak*-convergent. Take limits in the weak* topology to set

$$y^* = \lim_{m\in\mathbb{L}} y_m^* \qquad \text{and} \qquad z^* = \lim_{m\in\mathbb{L}} z_m^*.$$

Using (8.11), we get that for $u \in \bigcup_m K_m$,

$$\langle y^*, u\rangle + \langle z^*, u\rangle \geq \langle y^*, x\rangle + \langle z^*, x\rangle + \varepsilon,$$

and since $(K_m)$ has dense union in $Y$, we conclude that $\langle y^* + z^*, u\rangle \geq \langle y^* + z^*, x\rangle$ for every $u \in Y$. Since $Y$ is a linear subspace of $X$, this implies that $(y^* + z^*)|_Y = 0$. We define a contractive operator $\tau\colon X \longrightarrow \ell_\infty(\{1, -1\} \times \mathbb{L})$ by

$$\tau(z) = \big(\langle y_m^*, z\rangle - \langle z_m^*, z\rangle\big)_{m\in\mathbb{L}}.$$

Then $\tau$ maps $Y$ into $c(\{1,-1\}\times\mathbb{L})$, and the hypothesis provides an extension $T\colon X\longrightarrow c(\{1,-1\}\times\mathbb{L})$, with $\|T\|\le\lambda$. Let us write $T$ in the form

$$T(z) = \left(\langle\tilde{y}_m^*, z\rangle - \langle\tilde{z}_m^*, z\rangle\right)_{m\in\mathbb{L}}.$$

For each $n$, we have $\|T(x-y_n)\|+\|T(x-z_n)\|\le\lambda(\|x-y_n\|+\|x-z_n\|)$. Hence, for any $n\in\mathbb{N}, m\in\mathbb{L}$ and any choice of signs, we also have

$$\pm\langle\tilde{y}_m^*, x-y_n\rangle \pm \langle\tilde{z}_m^*, x-z_n\rangle \le \lambda(\|x-y_n\|+\|x-z_n\|). \tag{8.12}$$

But since every value of $T$ is a convergent sequence, we have that $(\tilde{y}_m^*)_m$ and $(\tilde{z}_m^*)_m$ converge in the weak* topology of $X^*$. If we let

$$\tilde{y}^* = \lim_{m\in\mathbb{L}}\tilde{y}_m^* \qquad\text{and}\qquad \tilde{z}^* = \lim_{m\in\mathbb{L}}\tilde{z}_m^*,$$

then $\tilde{y}^*+\tilde{z}^*=0, \tilde{y}^*|_Y = y^*|_Y, \tilde{z}^*|_Y = z^*|_Y$. In particular,

$$\lim_{m\in\mathbb{L}}\langle\tilde{y}_m^*+\tilde{z}_m^*, x\rangle = \langle\tilde{y}^*+\tilde{z}^*, x\rangle = 0.$$

Thus, if we take both signs in (8.12) to be negative and set $n=n(m)$, we obtain

$$\begin{aligned}\limsup_{m\in\mathbb{L}}\left(\langle\tilde{y}_m^*, y_{n(m)}\rangle + \langle\tilde{z}_m^*, z_{n(m)}\rangle\right) &= \limsup_{m\in\mathbb{L}}\left(\langle y_m^*, y_{n(m)}\rangle + \langle z_m^*, z_{n(m)}\rangle\right)\\ &\le \lambda(\|x-y_{n(m)}\|+\|x-z_{n(m)}\|),\end{aligned}$$

which contradicts (8.10) when $u=0$. □

## Types and the Almost Isometric $\mathscr{C}$-Extension Property

While the $(\lambda,\mathscr{C})$-extension property seems elusive, the almost isometric $\mathscr{C}$-extension property for separable spaces can be readily characterised. It is all a matter of showing that when Lipschitz maps extend to Lipschitz maps, operators extend to operators. This, which was very simple for extensions to one more dimension when done in Lemma 8.3.5, is considerably harder in the general situation. The passage can be smoothed using the language of types. Usually, a *type* on a Banach space $X$ is defined to be a function $\sigma\colon X\longrightarrow[0,\infty)$ having the form $\sigma(x)=\|x-a\|$ for some $a$ belonging to some ultrapower of $X$. However, in separable spaces, we can adopt the following equivalent formulation:

**Definition 8.3.7** A type on a separable Banach space $X$ is a function $\sigma$ : $X\longrightarrow[0,\infty)$ having the form

$$\sigma(x) = \lim_{n\to\infty}\|x+x_n\|$$

for some bounded sequence $(x_n)_n$ of $X$. We will say in that case that $\sigma$ is defined by $(x_n)$ and when $(x_n)\subset Y\subset X$, $\sigma$ is said to be supported on $Y$.

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It is a matter of an elementary Ramsey-like argument that every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in a separable space $X$ contains a subsequence $(x_n)_{n\in\mathbb{M}}$ such that $\lim_{n\in\mathbb{M}} \|x + x_n\|$ exists for all $x \in X$, thus defining a type on $X$. Condition $\Sigma_1(\lambda)$ can be reformulated as:

**Lemma 8.3.8** *Let $X$ be a separable Banach space, $Y$ a subset of $X$ and $\lambda \geq 1$. If $(Y, X)$ satisfies condition $\Sigma_1(\lambda)$ then, for every pair of types $\sigma, \beta$ supported on $Y$, we have*

$$\inf_{u\in Y} (\sigma(u) + \beta(u)) \leq \lambda \inf_{x\in X} (\sigma(x) + \beta(x)) . \tag{8.13}$$

*The converse is true if $\lambda > 1$.*

*Proof* We first remark that the reason for not having an equivalence in the case $\lambda = 1$ is that, while sequences defining types have to be bounded, the same is not the case for the sequences appearing in condition $\Sigma_1(\lambda)$. For $\lambda > 1$, the inequality required in condition $\Sigma_1(\lambda)$ is automatic when $(y_n)$ or $(z_n)$ is unbounded, and we can thus work with bounded sequences only. Now assume that $(Y, X)$ satisfies $\Sigma_1(\lambda)$ and let $\sigma$ and $\beta$ be types on $X$ supported on $Y$. Then there exist sequences $(y_n), (z_n)$ of $Y$ such that

$$\sigma(x) = \lim_{n\to\infty} \|x + y_n\| \qquad \text{and} \qquad \beta(x) = \lim_{n\to\infty} \|x + z_n\|$$

for all $x \in X$. Fix $\varepsilon > 0$ and pick $x_0 \in X$ such that

$$\sigma(x_0) + \beta(x_0) < \inf_{x\in X} (\sigma(x) + \beta(x)) + \varepsilon.$$

Applying condition $\Sigma_1(\lambda)$ to $(y_n), (z_n), x_0$ and $\varepsilon$, we obtain a point $u \in Y$ such that $\sigma(u) + \beta(u) \leq \lambda(\sigma(x_0) + \beta(x_0)) + \varepsilon$, which is enough. As for the other part, suppose $\lambda > 1$, and let $(y_n), (z_n)$ be bounded sequences of $Y$. Then there is an increasing function $n : \mathbb{N} \longrightarrow \mathbb{N}$ such that the functions

$$\sigma(x) = \lim_{k\to\infty} \|x + y_{n(k)}\| \qquad \text{and} \qquad \beta(x) = \lim_{k\to\infty} \|x + z_{n(k)}\|$$

are correctly defined types on $X$. Now, if (8.13) holds then for every $x \in X$ and every $\varepsilon > 0$, there is $u \in Y$ and $N \in \mathbb{N}$ such that for all $k \geq N$ we have

$$\|u - y_{n(k)}\| + \|u - z_{n(k)}\| \leq \lambda(\|x - y_{n(k)}\| + \|x - z_{n(k)}\|) + \varepsilon. \qquad \square$$

**Lemma 8.3.9** *Let $X$ be a separable Banach space and $\lambda > 1$. If the pair $(H, X)$ satisfies $\Sigma_1(\lambda)$ for every hyperplane $H$ of $X$ then so does $(Y, X)$ for every closed subspace $Y \subset X$.*

*Proof* If $(Y, X)$ fails to have property $\Sigma_1(\lambda)$ then there exist types $\sigma, \tau$ supported on $Y$, $\varepsilon > 0$ and $x_\varepsilon \in X$ such that for all $y \in Y$ we have

$$\sigma(y) + \tau(y) > \lambda(\sigma(x_\varepsilon) + \tau(x_\varepsilon)) + 2\varepsilon.$$

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Let $D = \{x \in X : \sigma(x) + \tau(x) < \lambda(\sigma(x_\varepsilon) + \tau(x_\varepsilon)) + \varepsilon\}$. Note that $x_\varepsilon \in D$. Since $D + \varepsilon B_X$ does not meet $Y$ we can arrange the geometric form of the Hahn–Banach theorem to obtain $x^* \in X^*$ such that $Y \subset \ker x^*$ and $\langle x^*, x\rangle > 0$ for all $x \in D$. Thus, $(\ker x^*, X)$ also fails to have $\Sigma_1(\lambda)$. □

The following result should be compared with the theorem in Section 8.8.3.

**Theorem 8.3.10** *Let $X$ be a separable Banach space. The following are equivalent:*

(i) *For every hyperplane $H \subset X$ and every $\lambda > 1$, every operator $\tau : H \longrightarrow c$ admits a $\lambda$-extension.*
(ii) *For every hyperplane $H \subset X$ and every $\lambda > 1$, the pair $(H, X)$ has property $\Sigma_1(\lambda)$.*
(iii) *For every subspace $Y \subset X$ and every $\lambda > 1$, the pair $(Y, X)$ has property $\Sigma_1(\lambda)$.*
(iv) *$X$ has the almost isometric $\mathscr{C}$-extension property.*

*Proof* The proof is an assembly of previous results. The implications (iv) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (ii) are trivial. The implication (ii) $\Longrightarrow$ (iii) is the content of Lemma 8.3.9, and Theorem 8.3.6 provides both (iv) $\Longrightarrow$ (iii) and (i) $\Longrightarrow$ (ii). Thus, it suffices to show that (iii) $\Longrightarrow$ (iv), which is an easy consequence of Lemma 8.3.3 inlaid with Lemma 8.3.5. To see why, let $\tau : Y \longrightarrow E$ be a $\mathscr{C}$-valued operator, $\lambda > 1$ and let $(\lambda_n)$ be a sequence with $\lambda_n > 1$ for every $n$ and such that $\prod_{n\geq 1} \lambda_n \leq \lambda$. Let $(Y_n)_{n\geq 0}$ be an increasing sequence of subspaces of $X$ such that $Y_0 = Y$, each $Y_n$ has codimension 1 (or 0) in $Y_{n+1}$ and $\bigcup_n Y_n$ is dense in $X$. Assume that $\tau$ has been extended to a linear operator $\tau_n : Y \longrightarrow E$, with $\|\tau_n\| \leq \prod_{1\leq k\leq n} \lambda_k$. Since $(Y_n, X)$ has property $\Sigma(\lambda_{n+1})$, Lemma 8.3.3 provides an extension $\tilde{\tau}_n : Y_{n+1} \longrightarrow E$ with Lipschitz constant at most $\prod_{1\leq k\leq n+1} \lambda_k$. Applying Lemma 8.3.5, we get a linear extension, say $\tau_{n+1} : Y_{n+1} \longrightarrow E$, with $\|\tau_{n+1}\| \leq \prod_{1\leq k\leq n+1} \lambda_k$. Continuing in this way, we arrive at a linear extension on $\bigcup_n Y_n$ bounded by $\lambda$, and we are done. □

## Kalton's First Reading and Property $(L^*)$

No matter if one is aware or not, part of $c_0$'s dowry in the proof of the Lindenstrauss–Pełczyński theorem is that weak*-null sequences $(x_n)_{n\geq 1}$ in $\ell_1$ behave this way: for every $x \in \ell_1$, we have

$$\lim_{n\to\infty} (\|x - x_n\| - \|x_n\| - \|x\|) = 0. \tag{8.14}$$

Kalton reads this condition two times. The first reading [273] is in the language of types: if $\sigma$ is a weak*-null type on $\ell_1$ (that is, one realised by a weak*-null

https://doi.org/10.1017/9781108778312.010 Published online by Cambridge University Press

sequence) then one has $\sigma(x) = \sigma(0) + \|x\|$, and therefore $c_0$ (not $\ell_1$!) has the following property:

**Definition 8.3.11** A Banach space $X$ has property $(L^*)$ if any two weak*-null types on $X^*$ that agree at the origin are equal.

Thus, $f \in X^* \longmapsto \|f\|$ is to be the only type on $X^*$ vanishing at 0, and so the idea could cross one's mind that $f \longmapsto c+\|f\|$ is the only type taking the value $c$ at 0. This is false, but only because we cannot guarantee that $c + \|\cdot\|$ is a type (types do not form a vector space). Property $(L^*)$ can be stated equivalently with inequalities, namely, if $\sigma$ and $\tau$ are weak*-null types on $X^*$ and $\sigma(0) \leq \tau(0)$ then $\sigma \leq \tau$. The proof is trivial. This property is not as innocent as it seems: if $X$ has property $(L^*)$ then all types $\sigma$ on $X^*$ are symmetric (i.e. $\sigma(f) = \sigma(-f)$ since $\sigma(f) = \sigma(0) + \|f\|$) and therefore all bidual types (i.e. all types having the form $\sigma(f) = \|f + g\|$ for some $g$ in the bidual of $X^*$) are symmetric. Maurey shows [357] that this implies that $X^*$ contains $\ell_1$.

**Proposition 8.3.12** *Every $\mathscr{C}$-valued operator defined on a subspace of a separable space with property $(L^*)$ admits a $1^+$-extension to the whole space.*

*Proof* Let $X$ be a separable space with property $(L^*)$. Given two weak*-null types $\sigma, \tau$ on $X^*$ and $u^*, v^* \in X^*$, there exists $0 \leq \theta \leq 1$ such that

$$\max\{\sigma(w^*), \tau(w^*)\} \leq \max\{\sigma(u^*), \tau(v^*)\} \tag{8.15}$$

whenever $w^* = (1-\theta)u^* + \theta v^*$. Indeed, assume without loss of generality $\|u^*\| \leq \|v^*\|$. Property $(L^*)$ and $\theta = 1$ yield $\sigma(v^*) \leq \tau(v^*)$. To conclude, use Theorem 8.3.10 after the next lemma. □

**Lemma 8.3.13** *If inequality* (8.15) *holds, every c-valued operator defined on a hyperplane of $X$ admits a $1^+$-extension to $X$.*

*Proof* With the same notation as in Lemma 1.8.4, if $K_M = \limsup_M K_n$ then all we need to show is that $\bigcap_M K_M \neq \emptyset$. Since $H$ is an hyperplane of $X$, it suffices to prove that $K_M \cap K_N \neq \emptyset$ for any two infinite subsets $M, N \subset \mathbb{N}$. Indeed, let $C = \{M\colon K_M \text{ is convex}\}$; by Helly's theorem, $\bigcap_{M\in C} K_M \neq \emptyset$. But the second assertion of Lemma 1.8.4 implies that also $\bigcap_M K_M \neq \emptyset$. For every $n \in \mathbb{N}$, pick $f_n \in K_n$ with $\|f_n\| = 1$. The sequence $(f_n)_{n\in M}$ contains a subsequence $(f_n)_{n\in M_1}$ that is weak*-convergent to some point, say, $x^*$. Analogously, the sequence $(f_n)_{n\in N}$ contains a subsequence $(f_n)_{n\in N_1}$ that is weak*-convergent to some point, say, $y^*$. By an argument 'à la Brunel-Sucheston', there is no loss of generality in assuming that for every $z^* \in [x^*, y^*]$, the limits $\lim_{M_1} \|z^* + f_n - x^*\|$ and $\lim_{N_1} \|z^* + f_n - x^*\|$ exist. By condition (8.15), there exists some $u^* = (1-\theta)x^* + \theta y^*$ such that

$$\max\left\{\lim_{M_1}\|u^* + f_n - x^*\|, \lim_{N_1}\|u^* + f_n - x^*\|\right\} \le 1.$$

Since $y^* - x^* \in H^\perp$, it follows that $u^* - x^* \in H^\perp$, and therefore $u^* + f_n - x^* \in K_n$ for large $n \in M_1$. Similarly, $u^* + f_n - y^* \in K_n$ for large $n \in N_1$. Taking weak*-limits, we have $u^* \in K_{M_1} \cap K_{N_1}$. □

Since $c_0$ has property $(L^*)$, we recover the Lindenstrauss–Pełczyński theorem 8.2.1 from Proposition 8.3.12 with sharp bounds. Besides, it is nearly obvious that the $\ell_p$-spaces have property $(L^*)$: if $\sigma\colon \ell_q \longrightarrow \mathbb{R}$ is a weak* (= weak) null type and $f \in \ell_q$ then

$$\sigma(f) = (\sigma(0)^q + \|f\|^q)^{1/q}. \tag{8.16}$$

In this way, Proposition 8.3.12 implies that isometries $Y \longrightarrow \ell_p$ are $(1^+, \mathscr{C})$-trivial (compare with Proposition 8.1.3 (a)).

**Proposition 8.3.14** *Let $X$ be a Banach space with property $(L^*)$ whose dual is separable. Every $\mathscr{C}$-valued operator defined on a weak*-closed subspace of $X^*$ admits a $1^+$-extension.*

*Proof* The goal is to show that property $(L^*)$ implies that for every weak*-closed subspace $E$ of $X^*$, the pair $(E, X^*)$ has all properties $\Sigma_1(\lambda)$ for $\lambda > 1$. In view of Lemma 8.3.8, it suffices to see that if $\sigma, \tau$ are types on $X^*$, supported on $E$, then

$$\inf_{u^*\in E}\left(\sigma(u^*) + \tau(u^*)\right) = \inf_{x^*\in X^*}\left(\sigma(x^*) + \tau(x^*)\right). \tag{8.17}$$

Observe that since all weak*-null types $\sigma\colon X^* \longrightarrow \mathbb{R}$ are symmetric (i.e. they are even maps), given $u^*, v^* \in X^*$, we have

$$\sigma\left(\frac{u^* - v^*}{2}\right) + \sigma\left(\frac{v^* - u^*}{2}\right) \le \sigma(u^*) + \sigma(v^*).$$

Let us show that 'the same' happens with any two weak*-null types $\sigma, \tau$, namely that given points $u^*, v^*$, there exists $\theta \in [0, 1]$ such that

$$\sigma(\theta(u^* - v^*)) + \tau((1-\theta)(v^* - u^*)) \le \sigma(u^*) + \tau(v^*). \tag{8.18}$$

Indeed, taking $\theta \in [0, 1]$ such that $(1-\theta)\sigma(0) = \theta\tau(0)$, we have

$$\begin{aligned}
\sigma\left(\theta(u^* - v^*)\right) &= \lim \|\theta u^* - \theta v^* + x_n^*\| \\
&\le \lim \|\theta u^* + \theta x_n^*\| + \lim \| - \theta v^* + (1-\theta)x_n^*\| \\
&= \lim \|\theta u^* + \theta x_n^*\| + \lim \|\theta v^* + \theta y_n^*\| \\
&= \theta\left(\sigma(u^*) + \tau(v^*)\right).
\end{aligned}$$

Analogously, $\tau\left((1-\theta)(v^*-u^*)\right) \leq (1-\theta)\left(\sigma(u^*)+\tau(v^*)\right)$. We finally show that (8.18) implies (8.17). Let $E$ be a weak*-closed subspace of $X^*$ and $\sigma, \tau \colon X^* \longrightarrow \mathbb{R}$ arbitrary types supported on $E$. Since the unit ball of $E$ is weak*-compact, for some $e^*, f^* \in E$, we have that $\sigma_0(x^*) = \sigma(x^*+e^*)$ and $\tau_0(x^*) = \tau(x^*+f^*)$ are weak*-null types on $X^*$. For arbitrary $x^* \in X^*$, taking $u^* = x^* - e^*$ and $v^* = x^* - f^*$, it is possible by (8.18) to find some $\theta \in [0,1]$ such that

$$\sigma_0\left(\theta(f^*-e^*)\right) + \tau_0\left((1-\theta)(e^*-f^*)\right) \leq \sigma_0(x^*-e^*) + \tau_0(x^*-f^*).$$

Put $w^* = (1-\theta)e^* + \theta f^*)$ to conclude that $\sigma(w^*) + \tau(w^*) \leq \sigma(x^*) + \tau(x^*)$. □

Even if our next assertion does not mean much at this moment, we want to remark that the Johnson–Zippin theorem 8.6.2 can be derived from here. The interest of this remark is to observe that Proposition 8.3.12 unifies most classical results on the extension of $\mathscr{C}$-valued operators. But since we did not arrive thus far merely to reprove oldies, the next section will contain different, and quite spectacular, applications.

## 8.4 Sequence Spaces with the $\mathscr{C}$-Extension Property

Many familiar sequence spaces, including most Orlicz sequence spaces and modular spaces, can be renormed to have property $(L^*)$. This furnishes us with a significant number of spaces having the $\mathscr{C}$-extension property, something quite remarkable when one considers how difficult it is to establish the $\mathscr{C}$-extension property for those spaces in their native norms, and more remarkable yet when we take into account that we will actually use not property $(L^*)$ but instead a close relative:

**Definition 8.4.1** A Banach space has property $(L)$ if any two weakly null types that agree at the origin are equal.

As the reader can imagine, some work is necessary to relate properties $(L)$ and $(L^*)$ since, at first glance, the latter depends on the behaviour of weak*-null sequences of $X^*$, while the former depends on the behaviour of weakly null sequences in $X$. All the connection we need is provided by:

**Proposition 8.4.2** *A Banach space with a* 1*-unconditional shrinking basis and property* $(L)$ *also has property* $(L^*)$.

The shrinking property of the basis is necessary: the space $\ell_1$ obviously has property $(L)$ and clearly fails $(L^*)$. Before beginning the proof, let us isolate the key fact linking weak*-null types on $X^*$ and weakly null types on $X$:

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**Lemma 8.4.3** *Assume $X$ has a 1-unconditional, shrinking basis. Suppose that $(x_n)_{n\geq 1}$ is weak*-null in $X^*$ and that $u^* \in X^*$ is such that $\lim_n \|u^* + x_n^*\|$ exists. Then there is an infinite $\mathbb{M} \subset \mathbb{N}$, a weakly null sequence $(x_m)_{m\in\mathbb{M}}$ in the unit ball of $X$ and a point $u \in X$ such that $\|u + x_m\| = 1$ for all $m \in \mathbb{M}$ and*

$$\langle u^*, u\rangle + \lim_{m\in\mathbb{M}} \langle x_m^*, x_m\rangle = \lim_{n\in\mathbb{N}} \|u^* + x_n^*\|. \tag{8.19}$$

*Proof of the lemma* Let $(e_n)_{n\geq 1}$ be the basis of $X$ which we consider normalised. The hypotheses guarantee that the coordinate functionals $(e_n^*)$ constitute a basis of $X^*$. Thus, we may assume that $(x_n^*)_n$ is a block sequence and also that $u^*$ is finitely supported, say $u^* = \sum_{1\leq i\leq k} u_i e_i^*$. In particular, $x_n^*$ and $u^*$ are 'disjoint' for $n > k$. Now, using the 1-unconditional property of the basis, for each $n$, we can pick a normalised $v_n \in X$ such that

$$\langle u^* + x_n^*, v_n\rangle = \|u^* + x_n^*\|, \qquad \operatorname{supp}(v_n) \subset \operatorname{supp}(u^* + x_n^*).$$

Let $P = \sum_{1\leq i\leq k} e_i^* \otimes e_i$ be the projection of $X$ onto the first $k$ coordinates, and let $\mathbb{M} \subset \mathbb{N}$ be an infinite subset for which the limit $u = \lim_{n\in\mathbb{M}} P(v_n)$ exists in the norm topology of $X$. Now, set $x_n = v_n - u$. Clearly, $\|u + x_n\| = 1$ for all $n$, while $(x_n)_{n\in\mathbb{M}}$ is weakly null, with $\|x_n\| \leq 1$ for all $n$. The proof concludes with

$$\|u^* + x_n^*\| = \langle u^* + x_n^*, v_n\rangle = \langle u^* + x_n^*, u + x_n\rangle = \langle u^*, u\rangle + \langle u^*, x_n\rangle + \langle x_n^*, u\rangle + \langle x_n^*, x_n\rangle.$$

□

*Proof of Proposition 8.4.2* It suffices to check that if $(x_n^*)$ and $(y_n^*)$ are normalised weak*-null sequences of $X^*$ then

$$\lim_{n\to\infty} \|u^* + x_n^*\| \leq \lim_{n\to\infty} \|u^* + y_n^*\|,$$

as long as both limits exist. Let us apply the just proved lemma to $(x_n^*)$ and $u^*$ to get an infinite subset $\mathbb{M} \subset \mathbb{N}$, a weak*-null sequence $(x_n)_{n\in\mathbb{M}}$ and a $u \in X$, such that $\|u + x_n\| = 1$ for all $n \in \mathbb{M}$ and

$$\langle u^*, u\rangle + \lim_{n\in\mathbb{M}} \langle x_n^*, x_n\rangle = \lim_{n\in\mathbb{N}} \|u^* + x_n^*\|.$$

Applying the lemma a second time to $(y_n^*)_{n\in\mathbb{M}}$ and $0 \in Y^*$, we obtain an infinite $\mathbb{L} \subset \mathbb{M}$ and a normalised weakly null sequence $(y_n)_{n\in\mathbb{L}}$ such that $\lim_{n\in\mathbb{L}} \langle y_n^*, y_n\rangle = 1$. Since $(\|x_n\| y_n)_{n\in\mathbb{L}}$ is weakly null, the property $(L)$ of $X$ yields $\lim_{n\in\mathbb{L}} \big\| u + \|x_n\| y_n \big\| = 1$. Thus,

$$\begin{aligned}
\lim_{n\in\mathbb{L}} \|u^* + y_n^*\| &\geq \limsup_{n\in\mathbb{L}} \langle u^* + y_n^*, u + \|x_n\| y_n\rangle \geq \langle u^*, u\rangle + \limsup_{n\in\mathbb{L}} \|x_n\| \\
&\geq \langle u^*, u\rangle + \lim_{n\in\mathbb{L}} \|x_n\| \geq \langle u^*, u\rangle + \lim_{n\in\mathbb{L}} \langle x_n^*, x_n\rangle \\
&= \lim_{n\in\mathbb{L}} \|u^* + x_n^*\|.
\end{aligned}$$

□

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We are ready to display Banach spaces with shrinking basis and property $(L)$. Start with a sequence of finite-dimensional spaces $(V_k)_{k\geq 1}$. For each $k$, let $N_k$ be a norm on $\mathbb{R} \times V_k$ (not $V_k$!) having the following properties:

★ $N_k(t, x) \geq \max(N_k(t, 0), N_k(0, x))$ for every $t \in \mathbb{R}$ and every $x \in V_k$,
★ $N_k(1, 0) = 1$ and $N_k(-t, x) = N_k(t, x)$ for every $(t, x) \in \mathbb{R} \times V_k$.

Next, we define an upper triangular infinite matrix of seminorms $(N_{m,n})_{m\leq n}$ on the space $c_{00}(V_k)$ as follows:

- For $(n, n)$ on the diagonal, we set $N_{n,n}(v) = N_n(0, v_n)$, where $v = (v_k)_{k\geq 1}$.
- Then, if $(m, n)$ is above the diagonal, that is, if $m < n$, we inductively define $N_{m,n}(v) = N_m(N_{m+1,n}(v), v_m)$ until reaching the diagonal from above.

We define a genuine norm on $c_{00}(V_k)$ by taking $\|v\|_L = \sup_{m\leq n} N_{m,n}(v)$. Let $\Lambda_L$ be the completion of $c_{00}(V_k)$ with respect to $\|\cdot\|_L$. It is clear that each $\Lambda_L$ can be regarded as the space of all infinite sequences $v = (v_k)_{k\geq 1}$ such that

- $\|v\|_L = \sup_k \|(v_1, \ldots, v_k, 0, \ldots)\|_L < \infty$,
- $\lim_{k\to\infty} \sup_{n>k} \|(0, \ldots, 0, v_k, \ldots, v_n, 0, \ldots)\|_L = 0$.

The time is now ripe for

**Proposition 8.4.4** *The space $\Lambda_L$ has property $(L)$.*

*Proof* The proof is almost trivial after realising how the norm of $\Lambda_L$ works. We must see that if $(y^n)_{n\geq 1}, (z^n)_{n\geq 1}$ are weakly null sequences in $\Lambda$ such that the limits

$$\sigma(x) = \lim_n \|x + y^n\|_L \qquad \text{and} \qquad \rho(x) = \lim_n \|x + z^n\|_L$$

exist for every $x \in \Lambda_L$ and $\sigma(0) = \rho(0)$, then $\sigma(x) = \rho(x)$ for all $x \in \Lambda_L$. After a moment's reflection, we see that it suffices to prove our statements for finitely supported $x$ assuming that $y^n$ and $z^n$ are finitely supported and that the sequences $(y^n)_{n\geq 1}, (z^n)_{n\geq 1}$ have the property that for each $k$ there is $m$ such that $y_i^n = z_i^n = 0$ for $i \leq k$ and $n \geq m$. Besides, we may and do assume that $\|y^n\|_L = \|z^n\|_L = \sigma(0) = \rho(0)$ for all $n$. In this way, property $(L)$ falls victim to the following property of the norm of $\Lambda_L$: if $x, y, z$ are finitely supported vectors such that

- there is an $m$ such that $x_i = 0$ for $i > m$, while $y_i = z_i = 0$ for $i \leq m$, and
- $\|y\|_L = \|z\|_L$,

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then $\|x + y\|_L = \|x + z\|_L$. Note that the crucial hypothesis is that when the supports of $y$ and $z$ 'start', $x$ is already null. To prove this, note that one has $N_{m,n}(v) \geq N_{m+1,n}(v)$ by the very definition of the norms $N_{m,n}$ when $m < n$, so $\|v\|_L = \sup_n N_{1,n}(v)$. On the other hand, $N_{1,n+1}(v) \geq N_{1,n}(v)$, so actually,

$$\|v\|_L = \lim_{n\to\infty} N_{1,n}(v).$$

But if $v_{n+1} = 0$ then $N_{1,n+1}(v) = N_{1,n}(v)$, and so, for $v = (v_1, \ldots, v_k, 0, \ldots)$,

$$\|v\|_L = N_{1,k}(v) = N_1(N_2(\ldots N_{k-1}(N_k(0, v_k), v_{k-1})\ldots, v_2), v_1). \tag{8.20}$$

Now, if $k$ is so large that we can compute the $L$-norm of the points $x, y, z, x + y, x + z$ using $N_{1,k}$, then

$$\|x + y\|_L = N_1(N_2(\ldots N_m\,\overbrace{(N_{m+1}(\ldots N_k(0, y_k), \ldots y_{m+1})}^{\|y\|_L}, x_m)\ldots, x_2), x_1),$$
$$\|x + z\|_L = N_1(N_2(\ldots N_m\,\underbrace{(N_{m+1}(\ldots N_k(0, z_k), \ldots z_{m+1})}_{\|z\|_L}, x_m)\ldots, x_2), x_1),$$

hence if $\|y\|_L = \|z\|_L$ then the proof is complete by showing

$$\begin{aligned}\|x + y\|_L &= N_1(N_2(\ldots N_m(\|y\|_L, x_m)\ldots, x_2), x_1)\\ &= N_1(N_2(\ldots N_m(\|z\|_L, x_m)\ldots, x_2), x_1)\\ &= \|x + z\|_L.\end{aligned}$$

□

To identify the spaces $\Lambda_L$ as modular sequence spaces, one can consider the family of functions $\Phi_k : V_k \longrightarrow \mathbb{R}_+$ given by $\Phi_k(x) = N_k(1, x) - 1$. It is clear that each $\Phi_k$ is a Young function and so it makes sense to consider the corresponding modular space

$$h((\Phi_k)_k) = \left\{ v \in \prod_{k\geq 1} V_k : \sum_{k=1}^{\infty} \Phi_k(tv_k) < \infty \text{ for all } t > 0 \right\}$$

(see Section 1.8.2) with the Luxemburg norm

$$\|v\|_{(\Phi_k)_k} = \inf\left\{ t > 0 : \sum_{k=1}^{\infty} \Phi_k(v_k/t) \leq 1 \right\}.$$

**Proposition 8.4.5** *One has $\Lambda_L = h((\Phi_k)_k)$, with equivalence of norms.*

*Proof* We first prove that $h((\Phi_k)_k)$ contains $\Lambda_L$, and that the inclusion is bounded. Assume $v \in \Lambda_L$, with $\|v\|_L \leq 1$. Then, for $1 \leq k < n$, using that

$N_{k+1,n}(v) \leq 1$ and the convexity of $\Phi_k$, we have

$$\begin{aligned} N_{k,n}(v) &= N_k(N_{k+1,n}(v), v_k)) \\ &= N_{k+1,n}(v) N_k\left(1, \frac{v_k}{N_{k+1,n}(v)}\right) \\ &= N_{k+1,n}(v)\left(1 + \Phi_k\left(\frac{v_k}{N_{k+1,n}(v)}\right)\right) \\ &= N_{k+1,n}(v) + N_{k+1,n}(v)\,\Phi_k\left(\frac{v_k}{N_{k+1,n}(v)}\right) \\ &\geq N_{k+1,n}(v) + \Phi_k(v_k). \end{aligned} \tag{8.21}$$

Thus, for each $n \geq 1$,

$$1 \geq N_{1,n}(v) = \sum_{k=1}^{n-1} (N_{k,n}(v) - N_{k+1,n}(v)) \geq \sum_{k=1}^{n-1} \Phi_k(v_k).$$

It quickly follows that $v \in h((\Phi_k))$, with norm at most 1. To prove the other inclusion, first suppose that $v = (v_k)$ is finitely non-zero and that $\sum_k \Phi_k(v_k) < 1$, and let us see that $\|v\|_L \leq 2e$. If $\|v\|_L \leq 2$, there is nothing to prove. Note that $N_{n,n}(v) = N_n(0, v_n) \leq N_n(1, v_n) = 1 + \Phi_n(v_n) \leq 2$, thus if $\|v\|_L > 2$ then $N_{m,n}(v) > 2$ for some $m < n$. Let $r$ be the smallest index such that $N_{r,n}(v) \leq 2$. Clearly, $m < r \leq n$. We have

$$N_{r-1,n}(v) = N_{r-1}(N_{r,n}(v), v_{r-1}) \leq N_{r-1}(2, v_{r-1}) \leq 2(1 + \Phi_{r-1}(v_{r-1}/2)),$$

and then, for any $m < j < r$,

$$N_{j-1,n}(v) = N_{j-1}(N_{j,n}(v), v_{j-1}) \leq N_{j,n}(v)(1 + \Phi_{j-1}(v_{j-1}/2)).$$

Hence,

$$N_{m,n}(v) \leq 2 \prod_{k=m}^{r-1} (1 + \Phi_k(v_k)) \leq 2e,$$

as required. Now assume $v \in h((\Phi_k))$, that is, $\sum_{k\geq 1} \Phi_k(tv_k) < \infty$ for all $t > 0$. Then, for every $\varepsilon > 0$, there is an $r$ such that $\sum_{r<k} \Phi_k(2ev_k/\varepsilon) < \infty$. Thus, for $r < k < n$ and $v \in \Lambda_L$, we have $\|(0, \ldots, 0, v_k, \ldots, v_n, 0, \ldots)\|_L < \varepsilon$. □

What have we obtained? That all modular sequence spaces and all Fenchel–Orlicz spaces fall within the range of application of Proposition 8.4.5! We first give the proof for modular spaces to then provide a description of the argument for Fenchel–Orlicz spaces, which is long and winding, although not terribly difficult.

**Corollary 8.4.6** *Let $\Phi_k : V_k \longrightarrow \mathbb{R}_+$ be a sequence of Young functions, where each $V_k$ is a finite-dimensional space. Assume that for each k, there is a norm $N_k$ on $\mathbb{R} \times V_k$ such that, for every $x \in V_k$,*

- $N_k(s, x) \leq N_k(t, x)$ *for* $|s| \leq |t|$,
- $N_k(1, x) = 1 + \Phi_k(x)$.

*Then $\Lambda_L((N_k)) = h((\Phi_k))$, with equivalence of norms.*

*Proof* This is a direct consequence of Proposition 8.4.5, since the norms $N_k$ clearly have the properties marked with ★ at the begining of this section. □

**Theorem 8.4.7** *Every modular sequence space $h((\phi_k)_k)$ has an equivalent norm with property $(L)$. If $h((\phi_k)_k)$ has separable dual then this norm has property $(L^*)$ as well.*

*Proof* We can assume that $\phi_k(1) = 1$ for every $k$. This implies that the right derivative of each $\phi_k$ at $\frac{1}{2}$ is at most 2. Then we only need to replace the sequence $(\phi_k)$ with an equivalent sequence $(\varphi_k)$ where each $\varphi_k$ has the form $\varphi_k(t) = N_k(1, t) - 1$, where each $N_k$ a is norm on $\mathbb{R}^2$ satisfying the hypotheses of Corollary 8.4.6. We will choose $\varphi_k$ to be convex, with $(\varphi_k(t)+1)/t$ decreasing for $t > 0$, say

$$\varphi_k(t) = \begin{cases} \phi_k(t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \phi_k(\frac{1}{2}) + 2(t - \frac{1}{2}) & \text{for } t \geq \frac{1}{2}. \end{cases}$$

Now we define the required norm on $\mathbb{R}^2$ by

$$N_k(s, t) = \begin{cases} |s|(1 + \phi_k(|t|/|s|)) & \text{for } s \neq 0 \\ 2|t|, & \text{for } s = 0. \end{cases}$$

This concludes the proof of the first part. The second statement is implied by Proposition 8.4.2 and James' classical result [216] asserting that a Banach space with unconditional basis has separable dual if and only if it does not contain $\ell_1$ and if and only if the given (or any other) basis is shrinking. □

**Proposition 8.4.8** *Every modular sequence space not containing $\ell_1$ has the $\mathscr{C}$-extension property.*

Note that *degenerate* Young functions have not been excluded so that $c_0$ can be considered as the modular space generated by the function $\varphi(t) = 2(t - \frac{1}{2})$ for $t \geq \frac{1}{2}$ and $\varphi(t) = 0$ for $0 \leq t \leq \frac{1}{2}$. Thus, finite products $X_1 \times \cdots \times X_k$, where each $X_i$ is either $c_0$ or $\ell_p$, with $1 < p < \infty$, can be renormed to enjoy the almost isometric $\mathscr{C}$-extension property. Observe that there is no direct proof for this fact because, in general, neither do the spaces $\ell_p \oplus_s \ell_r$ have property

$(L^*)$, nor is the duality map weakly continuous (even when $p = s = 2, r = 4$). We conclude with the case of Fenchel–Orlicz spaces, thus providing a shallow description of the fourth section of [12]:

**Proposition 8.4.9** *Every Fenchel–Orlicz space not containing $\ell_1$ and built on a non-degenerate Young function has the $\mathscr{C}$-extension property.*

These include the spaces $\ell_p(\varphi)$ for $p > 1$, by Theorem 10.8.1. The key point is that each non-degenerate Young function $\Phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^+$ is equivalent near zero to another Young function $\Psi$ that arises as $\Psi(x) = N(1, x) - 1$, where $N$ is a norm on $\mathbb{R} \times \mathbb{R}^n$ satisfying the first condition of Corollary 8.4.6. The proof, which requires considerable skill in convex geometry, goes as follows. Let $B$ be the Euclidean ball of $\mathbb{R}^n$. Starting with a Young function $\Phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^+$ vanishing only at zero, one constructs an even, convex function $\Gamma\colon B \longrightarrow \mathbb{R}^+$ vanishing only at zero, which is $C^1$ away from zero and equivalent to $\Phi$ on $B$. The lion's share of the proof consists of showing that $\Gamma$ can be extended from a neighbourhood of zero to a Young function $\Psi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^+$ such that

- for each $x \in \mathbb{R}^n$, the map $t \longmapsto (1 + \Psi(tx))/t$ is decreasing on $\mathbb{R}^+$,
- the function $x \longmapsto \lim_{t\to\infty}(1 + \Psi(tx))/t$ is a norm on $\mathbb{R}^n$.

The function $\Psi$, which is again equivalent to $\Phi$ near zero, is finally used to define a norm $N$ on $\mathbb{R} \times \mathbb{R}^n$ through the formula

$$N(s, x) = \begin{cases} |s|\,(1 + \Psi\,(x/|s|))\,, & \text{if } s \neq 0,\\ \lim_{t\to 0} |t|\,(1 + \Psi\,(x/|t|))\,, & \text{otherwise.} \end{cases}$$

From this, it is clear that $N(s, x) \leq N(t, x)$ for $|s| \leq |t|$ and also that $N(1, x) = 1+\Psi(x)$ for all $x \in \mathbb{R}^n$. Corollary 8.4.6 then implies that $h(\Phi) = h(\Psi) = \Lambda_L(N)$. Whether the result remains true for families of Young functions is unclear because the size of the neighbourhood of zero in the second step seems to depend on the given Young function.

## 8.5 $\mathscr{C}$-Extensible Spaces

Since, because of Lemma 1.6.2, $C(\Delta)$ contains 1-complemented copies of all separable $\mathscr{C}$-spaces as well as isometric copies of all separable Banach spaces, $\mathscr{C}$-extendibility is a game played in $C(\Delta)$: $X$ is $\mathscr{C}$-extensible if and only if every operator $\tau\colon X \longrightarrow C(\Delta)$ extends to $C(\Delta)$ through whatever embedding $X \longrightarrow C(\Delta)$. As we know, every separable Banach space $X$ admits some $\mathscr{C}$-trivial embedding into $C(\Delta)$: the composition of the canonical embedding

$\delta\colon X \longrightarrow C(B_X^*)$ with an isomorphism between $C(B_X^*)$ and $C(\Delta)$. To be $\mathscr{C}$-extensible means that all embeddings into $C(\Delta)$ are $\mathscr{C}$-trivial. Sobczyk's theorem records $c_0$ as the first $\mathscr{C}$-extensible space, while the Lindentrauss–Pełczyński theorem files all its subspaces in the list of $\mathscr{C}$-extensible spaces. The following result is, formally at least, a generalisation of both. A crucial step in its proof is the use of a *homogeneous* Zippin selector. Upgrading Zippin selectors in this manner is possible for separable Banach spaces (see Note 8.8.1) but far from trivial.

**Lemma 8.5.1** *If $E$ is $\mathscr{C}$-extensible then so is $c_0(E)$.*

*Proof* Assume $c_0(E)$ has been embedded as a subspace of a separable $Y$, and let $\pi_n\colon c_0(E) \longrightarrow E$ be the projection onto the $n$th-coordinate; when necessary, we will write $E_n = \pi_n[c_0(E)]$. Let $(F_k)$ be an increasing sequence of finite-dimensional subspaces of $Y$ spanning a dense subspace and such that $F_1 = 0$, $F_k \cap c_0(E) = 0$. Let $W_k = F_k + c_0(E)$.

**Claim** For each $k$, there is $n_0(k)$ such that for all $n \geq n_0(k)$, the operator $T_n\colon W_k \longrightarrow E$ defined as $f + z \longmapsto \pi_n(z)$ has norm at most $2 + \varepsilon$.

*Proof of the claim* If not, for some $k$, there are sequences $(f_n) \subset F_k$, $(z_n) \subset c_0(E)$ and $(m(n)) \subset \mathbb{N}$ such that $\|f_n + z_n\| < 1$ but $\pi_{m(n)}(z_n) > 2 + \varepsilon$. Since $F_k \cap c_0(E) = 0$, the sequence $(f_n)$ is bounded and it must contain a convergent subsequence, which, after relabelling, is itself. But this is in contradiction with the fact that $\|z_m - z_n\| > 2 + \varepsilon$ for large $m$, which means $\|f_m - f_n\| > \varepsilon$. □

Let $T_n\colon F_{k(n)} + c_0(E) \longrightarrow E$ be the operators $T_n(f + z) = \pi_n(z)$ with $\|T_n\| \leq 2 + \varepsilon$. Consider the pushout diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_{k(n)} & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T_n} & & \downarrow{\scriptstyle \tau_n} & & \| & & \\ 0 & \longrightarrow & E & \xrightarrow{\ \iota_n\ } & \mathrm{PO}_n & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

The open mapping theorem implies that every embedding $\jmath\colon E \longrightarrow C(\Delta)$ comes with a constant

$$\lambda_\jmath = \inf\left\{\lambda\colon \forall \tau \in \mathfrak{L}(E, C(\Delta))\ \exists\ T \in \mathfrak{L}(C(\Delta)) \text{ such that } \tau = T\jmath, \|T\| \leq \lambda\|\tau\|\right\}.$$

Showing that $\sup_\jmath \lambda_\jmath < \infty$ is a bit trickier. Let $\jmath_n$ be a sequence of 'uniform' embeddings, each with constant $\lambda_n$. Form their (multiple) pushout PO, and let $\iota\colon E \longrightarrow \mathrm{PO}$ be the resulting embedding. Since PO is separable, pick 'the canonical' embedding $\delta\colon \mathrm{PO} \longrightarrow C(\Delta)$ and then the embedding $\jmath = \delta\iota$. It is clear that $\lambda_\jmath \geq \sup_n \lambda_n$, which shows the assertion. Returning to the proof, we have obtained a constant $\lambda$ such that the lower sequences in the preceding diagram $(\lambda, \mathscr{C})$-split and therefore all of them admit $\lambda$-Zippin selectors. For the time being, we will take for granted the existence of a homogeneous $\lambda$-Zippin selector $\psi_n$ for $\imath_n$. With its aid, we can define a homogeneous map $\phi_n\colon c_0(E)^* \longrightarrow Y^*$ in the form $\phi_n(z^*) = \tau_n^*\psi_n(z^*|_{E_n})$. This map is weak*-continuous on bounded sets and satisfies the estimate $\|\phi_n(z^*)\| \leq (2+\varepsilon)\lambda\|\pi_n^*(z^*)\|$ since $\|\tau_n\| \leq 1$ (by the usual properties of pushouts). Moreover, if $z \in c_0(E)$ and $z^* \in c_0(E)^*$ then

$$\langle \phi_n(z^*), z\rangle = \langle \tau_n^*\psi_n(z^*|_{E_n}), z\rangle = \langle \psi_n(z^*|_{E_n}), \tau_n z\rangle = \langle \psi_n(z^*|_{E_n}), \imath_n\pi_n z\rangle = \langle z^*, \pi_n(z)\rangle.$$

Now define $\omega\colon c_0(E)^* \longrightarrow Y^*$ by $\omega(z^*) = \sum \phi_n(z^*)$; this has $\|\omega\| \leq (2+\varepsilon)\lambda$ and is a selector since $\omega(z^*)(z) = z^*$. Moreover, it is weak*-continuous on bounded sets: if $f \in \bigcup F_k$ then eventually, $T_n f = 0$. Thus, $z^* \longmapsto \omega(z^*)(f)$ is weak*-continuous on bounded sets, as well as $\omega$. □

Now, the typical 3-space result:

**Lemma 8.5.2** *$\mathscr{C}$-extensibility is a 3-space property.*

*Proof* Let $X$ be a Banach space with a $\mathscr{C}$-extensible subspace $Y$ such that $X/Y$ is $\mathscr{C}$-extensible. Let $\tau\colon X \longrightarrow E$ be a $\mathscr{C}$-valued operator and $\jmath\colon X \longrightarrow U$ an embedding in which $U$ is separable. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y & = & Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X & \xrightarrow{\jmath} & U & \longrightarrow & U/\jmath[X] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} & & \| & & \\
0 & \longrightarrow & X/Y & \xrightarrow{\overline{\jmath}} & U/\jmath[Y] & \longrightarrow & U/\jmath[X] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We sympathise with any reader believing this is groundhog day, since this diagram is formally identical to those that already appeared in the proofs of Theorem 8.2.1 and Lemma 2.14.3 (central diagram; the hypotheses now are about sequences 2 and 4 and the thesis about sequence 1); moreover, we will proceed exactly as we did then: we first extend $\tau|_Y$ to an operator $T_1 : U \longrightarrow E$ such that $T_1(\jmath(y)) = \tau(y)$ for $y \in Y$. As the difference $\tau - T_1\jmath$ vanishes on $Y$,

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there is a $\tau_2\colon X/Y \longrightarrow E$ such that $\tau - T_1 \jmath = \tau_2 \pi$. If $T_2\colon U/\jmath[Y] \longrightarrow E$ is an extension of $\tau_2$ through $\bar{\jmath}$ then $T = T_1 + T_2 \pi$ is the required extension of $\tau$. □

## Kalton's Second Reading and the Secret Life of $\ell_1$

We now show that $\ell_1$ is also $\mathscr{C}$-extensible. To understand why, let us recall that the key fact (8.14) about weak*-null sequences of $\ell_1$ that appears during the proof of the Lindenstrauss–Pełczyński theorem was transformed in Kalton's hands into property ($L^*$). In that same paper, Kalton returned to the crime scene and read it again in a new way: let $\sigma\colon \ell_1 \longrightarrow \mathbb{R}_+$ be *any* type defined by a bounded sequence $(x_n)$. The sequence can be assumed to be weak*-convergent to some point $u \in \ell_1$, hence $(x_n - u)$ is weak*-null, and we have

$$\lim_{n\to\infty} (\|x + u - x_n\| - \|u - x_n\| - \|x\|) = 0$$

for every $x \in \ell_1$. Replacing $x$ by $x - u$ yields

$$\lim_{n\to\infty} (\|x - x_n\| - \|u - x_n\| - \|x - u\|) = 0,$$

equivalently, $\sigma(x) - \sigma(u) = \|x - u\|$. This peculiarity of $\ell_1$ deserves a name:

**Definition 8.5.3** A Banach space $X$ has the $m_1$-type property if, for every type $\sigma$ on $X$, there exists $u \in X$ such that for all $x \in X$, we have

$$\sigma(x) = \|x - u\| + \sigma(u).$$

Of course, $\ell_1$ has the $m_1$-type property. It is not the first time we have encountered something similar: recall from (8.16) that $\ell_p$ spaces, $1 < p < \infty$, have the analogous property that *any* type $\sigma\colon \ell_p \longrightarrow \mathbb{R}_+$ has the form $\sigma(x) = (\|x-u\|^p + \sigma(u)^p)^{1/p}$. The exact value of $p$ is essential because (compare with Proposition 8.5.8):

**Theorem 8.5.4** *A separable Banach space with the $m_1$-type property is $\mathscr{C}$-extensible. In particular, $\ell_1$ is $\mathscr{C}$-extensible.*

*Proof* We shall proceed in two steps. The first step is a construction that allows one to replace a given nasty enlargement by a more pleasant one. The trick for doing that is as follows. Let $Z$ be a Banach space with a subspace $X$. An $X$-seminorm on $Z$ is a convex, symmetric function $\varphi\colon Z \longrightarrow [0,\infty)$ such that

- $\varphi(z) \le \|z\|$ for $z \in Z$,
- $\varphi(x) = \|x\|$ for $x \in X$.

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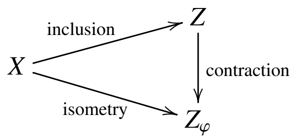
Observe that a minimal $X$-seminorm (with respect to the pointwise order) is actually a seminorm on $Z$, because if $0 < t < 1$ then $\phi(y) = t^{-1}\varphi(ty)$ defines another $X$-seminorm with $\phi \leq \varphi$. Hence $\phi = \varphi$ and since $t\phi(tx) = \varphi(ttx) = \phi(ttx)$, it follows that $t\phi(x) = \phi(tx)$ for $|t| < 1$. For $t > 1$, just set $\phi(tx) = \phi(t^{-1}ttx) = t^{-1}\phi(ttx)$ to get $t\phi(tx) = \phi(ttx)$ and we are done again. So, $\varphi(tx) = |t|\varphi(x)$ for $t$ real, and it is in fact a seminorm. Let $\varphi$ be any minimal $X$-seminorm on $Z$ and let $Z_\varphi$ be the completion of $Z/\ker\varphi$ with respect to the induced norm. It is clear that there is a contractive operator $Z \longrightarrow Z_\varphi$ whose restriction to $X$ is an isometry:




Thus, it suffices to show that $\mathscr{C}$-valued operators on $X$ admit $1^+$-extensions to $Z_\varphi$. Before going any further, let us remark that $Z_\varphi$ has the property that the only $X$-seminorm on $Z_\varphi$ is the norm itself. Clearly, $Z_\varphi$ is separable. Let $(Z_n)_{n\geq 0}$ be an increasing sequence of subspaces of $Z_\varphi$ whose union is dense, with $Z_0 = X$ and $\dim Z_{n+1}/Z_n = 1$ for $n \geq 1$. Now, the plan is to prove that each $\mathscr{C}$-valued operator on $Z_n$ has a $1^+$-extension to $Z_{n+1}$. To do this, it suffices to show that for every $n \geq 0$, the pair $(Z_n, Z_\varphi)$ has property $\Sigma_1(\lambda)$ for all $\lambda > 1$, and then to apply Lemma 8.3.3 followed by Lemma 8.3.5. We get the idea, right? So the proof will be complete after showing that for each $n \geq 0$, the pair $(Z_n, Z_\varphi)$ satisfies the inequality (8.13) of Lemma 8.3.8 for $\lambda = 1$. Fix $n \geq 0$ and let $\sigma$ be any type on $Z_\varphi$ that is supported on $Z_n$. Since $Z_n/X$ is finite-dimensional, there is a $u_0 \in Z_n$ such that $\sigma(z) = \sigma'(z - u_0)$ for some type $\sigma'$ supported on $X$. Since $X$ has the $m_1$-type property, there exists $u_1 \in X$ such that $\sigma'(x) - \sigma(u_1) = \|x - u_1\|$ for all $x \in X$. Letting $u = u_0 + u_1$, define $\sigma_0 \colon Z_\varphi \longrightarrow [0, \infty)$ by $\sigma_0(z) = \sigma(z+u)$. Then, for $x \in X$, we have $\sigma_0(x) = \|x\| + \sigma(u)$ and, in general,

$$\sigma_0(z) \leq \sigma_0(0) + \|z\| = \sigma(u) + \|z\|. \tag{8.22}$$

Consider the function

$$\psi(z) = \frac{\sigma_0(z) + \sigma_0(-z) - 2\sigma(u)}{2}.$$

This is an $X$-seminorm on $Z_\varphi$: that it is convex and symmetric is obvious; similarly that $\psi(0) = 0$, and since $\mathrm{Lip}(\psi) = 1$, we have $\psi(z) \leq \|z\|$ and, clearly, $\psi(x) = \|x\|$ for all $x \in X$. But every $X$-seminorm on $Z_\varphi$ is actually the norm, and so $\psi(z) = \|z\|$ for all $z \in Z_\varphi$. Taking (8.22) into account, it quickly follows that $\sigma_0(z) - \sigma(u) = \|z\|$ for all $z \in Z_\varphi$, which can be written as $\sigma(z) = \|z - u\| + \sigma(u)$.

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Finally, if $\beta$ is another type on $Z_\varphi$ that is supported on $Z_n$, then one can find $v \in Z_n$ such that $\beta(z) = \|z - v\| + \beta(v)$. Hence,

$$\begin{aligned}\sigma(u) + \beta(u) &= \sigma(u) + \beta(v) + \|u - v\| \\ &\leq \sigma(u) + \beta(v) + \|z - u\| + \|z - v\| \\ &= \sigma(u) + \beta(v) + \sigma(z) - \sigma(u) + \beta(z) - \beta(v) \\ &= \sigma(z) + \beta(z).\end{aligned}$$

□

Speechless? What has actually been proved is

**Corollary 8.5.5** *Every $\mathscr{C}$-valued operator defined on a separable space with the $m_1$-type property admits a $1^+$-extension to every separable superspace.*

The 3-space argument given in Lemma 8.5.2 immediately yields that weak*-closed subspaces of $\ell_1$ are $\mathscr{C}$-extensible because spaces of the form $\ell_1(\mathbb{N}, F_n)$ with each $F_n$ finite-dimensional have the $m_1$-type property and thus they must be $\mathscr{C}$-extensible, while each weak*-closed subspace of $\ell_1$ is a twisted sum of two subspaces of that form (Proposition 5.3.1). This argument does not, however, provide an estimate for the norm of the extending operator; on the other hand, that is unnecessary since Theorem 8.5.4 and Proposition 8.3.14 combine to yield $1^+$ in all cases! It is perhaps about time to present the carnival parade of all $\mathscr{C}$-extensible spaces currently known:

**Theorem 8.5.6** *The following Banach spaces are $\mathscr{C}$-extensible:*

- *$c_0$ and all its subspaces,*
- *$\ell_1$ and all its weak*-closed subspaces,*
- *all spaces with property (L*) as well as their duals,*
- *every space with the $m_1$-type property,*
- *$c_0(E)$ for every $E$ in the list,*
- *twisted sums of two spaces in the list.*

## What about $\ell_p$?

It is difficult not to believe that $\ell_p$ spaces are $\mathscr{C}$-extensible too; after all, it is true for the extreme values $p = 1, \infty$, no matter how we interpret the case $\infty$: either as $c_0$ or, bending the rules, as $\ell_\infty$. So, what could go wrong in the middle? Well, what goes wrong is that $\ell_p$ is not $\mathscr{C}$-extensible for $1 < p < \infty$.

**Proposition 8.5.7** *Let $X$ be a separable Banach space containing $\ell_p$, where $p \in (1, \infty)$. If every $\mathscr{C}$-valued operator on $\ell_p$ can be extended to $X$ then $X$*

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*can be given an equivalent norm $|\cdot|$ such that for every weakly null type $\sigma$ supported on $\ell_p$ and all $x \in X$, we have*

$$\left(|x|^p + \sigma(0)^p\right)^{1/p} \le \sigma(x). \tag{8.23}$$

*Proof* If the embedding $\ell_p \longrightarrow X$ is $\mathscr{C}$-trivial then there is a homogeneous Zippin selector $\omega\colon \ell_q \longrightarrow X^*$, where $q$ is the conjugate exponent of $p$ and $\ell_q$ is treated as the dual space of $\ell_p$. We consider the following seminorm on $X$:

$$|x|_0 = \sup\left\{\langle x, \omega(u^*)\rangle : \ \|u^*\|_q \le 1\right\}.$$

It is clear that $|u|_0 = \|u\|$ for $u \in \ell_p$ and that $|x|_0 \le \|\omega\|\,\|x\|$, where, as usual, $\|\omega\| = \sup\{\|\omega(u^*)\| \colon \|u^*\|_q \le 1\}$. Assume $(u_n)$ is a weakly null sequence in $\ell_p$. We want to see that

$$\liminf_{n\to\infty} \left(|x|_0^p + \|u_n\|^p\right)^{1/p} \le \limsup_{n\to\infty} |x + u_n|_0 \tag{8.24}$$

for each $x \in X$. Pick normalised $u_n^* \in \ell_q$ such that $\langle u_n^*, u_n\rangle = \|u_n\|$. Fix $x \in X$ and $\varepsilon > 0$. Take a normalised $u^* \in \ell_q$ such that $\langle x, \omega(u^*)\rangle = |x|_0$. We have

$$\left\langle x + u_n, \omega(u^* + \varepsilon u_n^*)\right\rangle \le \|u^* + \varepsilon u_n^*\|\,|x + u_n|_0,$$

hence

$$\liminf_{n\to\infty} \left\langle x + u_n, \omega(u^* + \varepsilon u_n^*)\right\rangle \le \limsup_{n\to\infty} \|u^* + \varepsilon u_n^*\|\,|x + u_n|_0. \tag{8.25}$$

For the terms on the right, we have $\lim \|u^* + \varepsilon u_n^*\| = (1 + \varepsilon^q)^{1/q}$ as we are working in $\ell_q$. The terms on the left can be decomposed as $\langle x, \omega(u^* + \varepsilon u_n^*)\rangle + \langle u_n, \omega(u^* + \varepsilon u_n^*)\rangle$ and thus $\lim_n \langle x, \omega(u^* + \varepsilon u_n^*)\rangle = \langle x, \omega(u^*)\rangle = |x|_0$ since $\omega$ is weak*-continuous and $u_n^*$ is weak*-null, while $\langle u_n, \omega(u^* + \varepsilon u_n^*)\rangle = u^*(u_n) + \varepsilon\|u_n\|$ because $\omega$ is a selector. Thus (8.25) implies

$$|x|_0 + \varepsilon \liminf_{n\to\infty} \|u_n\| \le (1 + \varepsilon^q)^{1/q} \limsup_{n\to\infty} |x + u_n|_0,$$

and Hölder's inequality yields (8.24). Now, we introduce a true renorming on $X$ as follows:

$$|x| = \inf\{\|x - v\| + |x - v|_0 + \|v\| : v \in \ell_p\}.$$

Note that $\|x\| \le |x| \le (1 + \|\omega\|)\|x\|$ for all $x \in X$ and that $|x| = \|x\| = \|x\|_p$ for $x \in \ell_p$. This norm has 'the same' property as $|\cdot|_0$, namely that if $(u_n)$ is weakly null in $\ell_p$ then

$$\liminf_{n\to\infty} \left(|x|^p + \|u_n\|^p\right)^{1/p} \le \limsup_{n\to\infty} |x + u_n|_0, \tag{8.26}$$

which is just a restatement of (8.23). Fix $(u_n)$ and $x$. Passing to a subsequence if necessary, we may assume that both $|x + u_n|$ and $\|u_n\|$ converge. Choose a sequence $(v_n)$ of points in $\ell_p$ such that

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$$\lim_n |x+u_n| = \lim_n \Big(\|x+u_n-v_n\| + |x+u_n-v_n|_0 + \|v_n\|\Big).$$

The sequence $(v_n)$ is bounded in $\ell_p$. Hence, passing to further subsequences, we may assume it is weakly convergent to, say $v \in \ell_p$ so that

$$\lim_n \|v_n\| = \lim_n \Big(\|v-v_n\|^p + \|v\|^p\Big)^{1/p}.$$

The relevant property of $|\cdot|_0$ now intervenes: assuming that all limits exist, we have

$$\lim_n \Big(|x-v|_0^p + \|u_n+v-v_n\|^p\Big)^{1/p} \le \lim_n |y-v+v-v_n+u_n|_0. \tag{8.27}$$

In conclusion, assuming again that all limits exist, we have

$$\begin{aligned}
&\lim_n |x+u_n| \\
&= \lim_n \Big(\|x+u_n-v_n\| + |x+u_n-v_n|_0 + \|v_n\|\Big) \\
&\ge \lim_n \Big(\|x+u_n-v_n\| + \Big(|x-v|_0^p + \|u_n+v-v_n\|^p\Big)^{1/p} + \|v_n\|\Big) \\
&\ge \lim_n \Big(\|x+u_n-v_n\| + \Big(|x-v|_0^p + \|u_n+v-v_n\|^p\Big)^{1/p} + \Big(\|v-v_n\|^p + \|v\|^p\Big)^{1/p}\Big) \\
&\ge \lim_n \Big(\Big(\|x-v\| + |x-v|_0 + \|v\|\Big)^p + \|u_n\|^p\Big)^{1/p} \\
&\ge \lim_n \Big(|x|^p + \|u_n\|^p\Big)^{1/p}.
\end{aligned}$$

□

**Proposition 8.5.8** *The space $\ell_p$ is not $\mathscr{C}$-extensible for $1 < p < \infty$.*

*Proof* The idea is to construct a separable enlargement $X(p)$ of $\ell_p$ for which no renorming satisfies (8.23). Let $\mathcal{D}$ be the dyadic tree, whose elements are all finite sequences $a = (s_1, \dots, s_m)$ of zeros and ones, including the empty sequence, denoted by $\emptyset$. The number $m$ is called the *depth* of $a$. The depth of $\emptyset$ is zero. Given $a = (s_1, \dots, s_m)$ and $b = (t_1, \dots, t_n)$, we write $a \le b$ if $m \le n$ and $s_i = t_i$ for $1 \le i \le m$. If $a \le b$ and $a \ne b$, are write $a \prec b$. A *segment* of $\mathcal{D}$ is a finite subset of the form $\{a_1, \dots, a_k\}$ with $a_1 \prec a_2 \prec \cdots \prec a_k$. The collection of all segments of $\mathcal{D}$ will be denoted $S$. Let $c_{00}(\mathcal{D})$ be the space of all finitely supported functions $\xi\colon \mathcal{D} \longrightarrow \mathbb{K}$. As usual, we use $e_a$ to denote the function that takes the value 1 at $a$, and zero otherwise. We consider the natural bilinear pairing on $c_{00}(\mathcal{D})$ given by

$$\langle \xi, \eta \rangle = \sum_{a \in \mathcal{D}} \xi(a)\eta(a).$$

We now introduce two mutually dual norms on $c_{00}(\mathcal{D})$. The first is

$$\|\xi\|_* = \max\Big(\|\xi\|_{\ell_q(\mathcal{D})}, \sup_{\beta \in S} \sum_{a \in \beta} |\xi(a)|\Big), \tag{8.28}$$

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where $q^{-1} + p^{-1} = 1$. This norm will play a supporting role in the construction. The lead role is played by

$$\|\eta\| = \sup\{\langle \xi, \eta \rangle : \|\xi\|_* \le 1\}.$$

Let $X(p)$ be the completion of $c_{00}(\mathcal{D})$ with respect to $\|\cdot\|$. Although these norms may remind us of that of the James tree space, they are unconditional in the sense that if $\eta, \xi \in c_{00}(\mathcal{D})$ are such that $|\xi| \le \eta$, then $\|\xi\| \le \|\eta\|$ and $\|\xi\|_* \le \|\eta\|_*$. Clearly, if $\beta = \{a_1, \dots, a_k\}$ is a segment, then $\|1_\beta\|_* = k$ and $\|1_\beta\| = 1$. Our immediate task is to find a copy of $\ell_p$ inside $X(p)$. To this end, given $a \in \mathcal{D}$, let $a0, a1$ be the two successors of $a$ and set $u_a = e_{a0} + e_{a1}$; that is,

$$u_a(b) = \begin{cases} 1 & \text{if } b \text{ is a successor of } a \\ 0 & \text{otherwise.} \end{cases}$$

We now prove that the sequence $(u_a)_{a \in \mathcal{D}}$ spans a subspace isomorphic to $\ell_p$ in $X(p)$. More precisely, we shall show that, for $\eta \in c_{00}(\mathcal{D})$, we actually have

$$\frac{\|\eta\|_{\ell_p(\mathcal{D})}}{C_p} \le \Big\| \sum_{a \in \mathcal{D}} \eta(a) u_a \Big\|_X \le 2^{1/p} \|\eta\|_{\ell_p(\mathcal{D})}, \tag{8.29}$$

where $C_p > 0$ is a constant depending only on $p$. The right inequality is obvious from (8.28). The left inequality follows from the following:

**Claim** For every $\eta \in c_{00}(\mathcal{D})$, there is $\xi \in c_{00}(\mathcal{D})$ such that $\langle \xi, u_a \rangle = \eta(a)$ and $\|\xi\|_* \le C_p \|\eta\|_{\ell_q(\mathcal{D})}$, where $C_p = \max\left(1, (2^p - 2)^{-1/p}\right)$.

*Proof of the claim* It suffices to consider the case in which $\eta, \xi \ge 0$. For $n \ge 0$, let $\mathcal{D}_n$ be the subset of those points of $\mathcal{D}$ whose depth is at most $n$. For fixed $p$, let $C_n$ be the best constant such that, whenever $\eta \ge 0$ is supported on $\mathcal{D}_n$, there is $\xi \ge 0$ satisfying $\langle \xi, u_\alpha \rangle = \eta(a)$ and $\|\xi\|_* \le C_n \|\eta\|_{\ell_q(\mathcal{D})}$. Note that $C_0 = 1$. We will estimate $C_n$ in terms of $C_{n-1}$. Let us split $\mathcal{D}_n$ into three disjoint subsets $\mathcal{D}_n = \{\emptyset\} \cup 0\mathcal{D}_{n-1} \cup 1\mathcal{D}_{n-1}$, where, as should be obvious, $k\mathcal{D}_{n-1} = \{(a_1, \dots, a_n) \colon a_1 = k\}$, for $k = 0, 1$. If $\eta$ is supported on $\mathcal{D}_n$ and we call $\eta^{(k)} = \eta 1_{k\mathcal{D}_{n-1}}$ for $k = 0, 1$ then $\eta = \eta(\emptyset) e_\emptyset + \eta^{(0)} + \eta^{(1)}$. In view of the obvious symmetries of the norms $\|\cdot\|_*$ and $\|\cdot\|$, and the definition of $C_{n-1}$, for $k = 0, 1$, we can find $\xi^{(k)} \ge 0$, supported on $k\mathcal{D}_{n-1}$, such that

$$\|\xi^{(k)}\|_* \le C_{n-1} \|\eta^{(k)}\|_{\ell_{p^*}(\mathcal{D})} \quad \text{and} \quad \langle \xi^{(k)}, u_a \rangle = \eta(a) \quad \text{for} \quad a \in k\mathcal{D}_{n-1}.$$

Note that for every $a \in \mathcal{D}_n$ different from $\emptyset$, we do have $\langle \xi^{(0)} + \xi^{(1)}, u_a \rangle = \eta(a)$. Thus, we try to construct $\xi$ by just tuning $\xi^{(0)} + \xi^{(1)}$, keeping $\xi(\emptyset) = 0$. To do this, for $k = 0, 1$, let $\beta^{(k)}$ be the segment starting at $(k)$ such that

$$\sum_{a\in\beta^{(k)}} \xi^{(k)}(a) = \max_{\beta\in S} \sum_{a\in\beta} \xi^{(k)}(a).$$

Assume, without loss of generality, that $\sum_{a\in\beta^{(0)}} \xi^{(0)}(a) \le \sum_{a\in\beta^{(1)}} \xi^{(1)}(a)$. Thus,

- if $\eta(\emptyset) \le \sum_{a\in\beta^{(1)}} \xi^{(1)}(a) - \sum_{a\in\beta^{(0)}} \xi^{(0)}(a)$, set $\xi = \eta(\emptyset)e_{(0)} + \xi^{(0)} + \xi^{(1)}$;
- otherwise, set

$$\xi = \eta(\emptyset)\frac{e_{(0)} + e_{(1)}}{2} + \left(\sum_{a\in\beta^{(1)}} \xi^{(1)}(a) - \sum_{a\in\beta^{(0)}} \xi^{(0)}(a)\right)\frac{e_{(0)} - e_{(1)}}{2} + \xi^{(0)} + \xi^{(1)}.$$

In either case, we have

$$\begin{aligned}\|\xi\|_{\ell_q(\mathcal{D})} &\le \left(\eta(\emptyset)^q + C_{n-1}\|\eta^{(0)}\|^q_{\ell_q(\mathcal{D})} + C_{n-1}\|\eta^{(1)}\|^q_{\ell_q(\mathcal{D})}\right)^{1/q} \\ &\le \max(1, C_{n-1})\,\|\eta\|_{\ell_q(\mathcal{D})}\end{aligned}$$

since, in the first case, $\max_{\beta\in S}\sum_{a\in\beta}\xi(a) \le \sum_{a\in\beta^{(1)}}\xi^{(1)}(a) \le C_{n-1}\|\eta^{(0)}\|_{\ell_q(\mathcal{D})}$; and, in the second case,

$$\begin{aligned}\max_{\beta\in S}\sum_{a\in\beta}\xi(a) &= \frac{\eta(\emptyset) + \sum_{a\in\beta^{(0)}}\xi^{(0)}(a) + \sum_{a\in\beta^{(1)}}\xi^{(1)}(a)}{2} \\ &\le \frac{\eta(\emptyset) + C_{n-1}\|\eta^{(0)}\|_{\ell_q(\mathcal{D})} + C_{n-1}\|\eta^{(1)}\|_{\ell_q(\mathcal{D})}}{2} \\ &\le \frac{(1 + 2C^p_{n-1})^{1/p}}{2}\,\|\eta\|_{\ell_q(\mathcal{D})}\,.\end{aligned}$$

Hence $C_n \le \max\left(1, C_{n-1}, \frac{1}{2}(1 + 2C^p_{n-1})^{1/p}\right)$. It follows that for $2^p \ge 3$, we get $C_n \le 1$, while if $2^p < 3$ then $C_n \le (2^p - 2)^{-1/p}$. □

The left estimate in (8.29) is now easy. Pick a finitely supported $\eta$, which we assume positive without loss of generality. Let $\xi$ be the output of the claim when the input is $\eta^{p-1}$ so that

$$\begin{aligned}&\langle \xi, u_a\rangle = \eta(a)^{p-1}, \\ &\|\xi\|_* \le C_p\|\eta^{p-1}\|_{\ell_q(\mathcal{D})} = C_p\|\eta\|^{p/q}_{\ell_p(\mathcal{D})}.\end{aligned}$$

For the last equality, note that $q(p-1)/p = 1$. Thus,

$$\Big\|\sum_{a\in\mathcal{D}}\eta(a)u_a\Big\| \ge \frac{\langle\xi, \sum_a \eta(a)u_a\rangle}{\|\xi\|_*} \ge \frac{\sum_a \eta^p(a)}{C_p\|\eta\|^{p/q}_{\ell_p(\mathcal{D})}} \ge \frac{\|\eta\|^p_{\ell_p(\mathcal{D})}}{C_p\|\eta\|^{p/q}_{\ell_p(\mathcal{D})}} = \frac{\|\eta\|_{\ell_p(\mathcal{D})}}{C_p}.$$

Up to here, we have the proof that $X(p)$ is a separable enlargement of $\ell_p$. The remainder of the proof is to establish that the inclusion of $\ell_p$ into $X(p)$ is not

$\mathscr{C}$-trivial. This is done by exploiting the criterion in Proposition 8.5.7 against the sequence $(\frac{1}{2}u_a)_{a\in\mathcal{D}}$: if every $\mathscr{C}$-valued operator on $\ell_p$ can be extended to $X(p)$, some equivalent norm $|\cdot|$ on $X(p)$ should satisfy

$$(|\eta|^p + 2c^p)^{1/p} \le \liminf_a \left|\eta + \tfrac{1}{2}u_a\right| \tag{8.30}$$

for some constant $c > 0$ and every $\eta \in X(p)$. One can actually take $c = |u_\emptyset|/2^{1+1/p}$. In particular, for each finitely supported $\eta$ and every $a \in \mathcal{D}$, there is $a^* \succ a$ such that

$$(|\eta|^p + c^p)^{1/p} < \left|\eta + \tfrac{1}{2}u_{a^*}\right|. \tag{8.31}$$

No norm on $c_{00}(\mathcal{D})$ having this property can be equivalent to the norm of $X(p)$; let us see why. If (8.31) holds then we can find, for every $n$, a segment $\beta$ (of cardinality $n$, if we want) such that $|1_\beta| \ge cn^{1/p}$, where $1_\beta = \sum_{a\in\beta} e_a$. On the other hand, $\|1_\beta\| = 1$ for every segment $\beta$: it is trivially true for $n = 1$, so we can assume that $\beta = \{a_1, \dots, a_n\}$ is a segment such that $|1_\beta| \ge cn^{1/p}$. Letting $\eta = 1_\beta$ in (8.31), we can find $a^* \succ a_n$ such that

$$c(n+1)^{1/p} \le (|1_\beta|^p + c^p)^{1/p} < \left|1_\beta + \tfrac{1}{2}u_{a^*}\right|.$$

Since $u_{a^*} = e_{a^*0} + e_{a^*1}$, either

$$c(n+1)^{1/p} < \left|1_\beta + e_{a^*0}\right| \qquad \text{or} \qquad c(n+1)^{1/p} < \left|1_\beta + e_{a^*1}\right|.$$

In the first case, the new segment is $\{a_1, \dots, a_n, a^*0\}$; in the second case, it is $\{a_1, \dots, a_n, a^*1\}$. □

We now want to obtain a superreflexive version of $X(p)$. The idea is to obtain an intermediate space between $X(p)$ and $\ell_p(\mathcal{D})$. A comfortable way to do so would be by using the *complex* interpolation method, which requires a temporary lift of the ban on complex scalars. But the alleged simplification that this way of acting brings is mostly a delusion, resulting in the reader's disappointment. We prefer instead to avoid that tactical move and just explain the bare facts as they are:

**Corollary 8.5.9** *For each $1 < p < \infty$, there is an embedding of $\ell_p$ into a superreflexive space that is not $\mathscr{C}$-trivial.*

*Proof* Let $X_0$ and $X_1$ be two Banach lattices of functions defined on the same set $S$. For $0 < \theta < 1$, we define the intermediate space $X_\theta = X_0^{1-\theta}X_1^\theta$ of functions $f\colon S \longrightarrow \mathbb{R}$ that admit a decomposition $|f| = g^{1-\theta}h^\theta$ for non-negative $g \in X_0$ and $h \in X_1$, endowed with the norm

$$\|f\|_\theta = \inf \|g\|_0^{1-\theta}\|h\|_1^\theta,$$

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where the infimum is taken over all $g, h$ in the decomposition above. The space $X_\theta$ is again a Banach lattice, and it is uniformly convex if either $X_0$ or $X_1$ is: this we know because a nice computation performed by Cwikel and Reisner in [143, Theorem 1 (i)] shows that the modulus of convexity of $X_\theta$ obeys an estimate of the form $\delta_{X_\theta}(\varepsilon) \geq c_1 \delta_{X_1}(c_2 \varepsilon^{1/\theta})$ for some $c_1, c_2 > 0$ and all $0 < \varepsilon < 2$. That said, let us specialise to the case $X_0 = X(p)$ and $X_1 = \ell_p(\mathcal{D})$. We claim that any of the spaces $X_\theta = X(p)^{1-\theta} \ell_p(\mathcal{D})^\theta$ for $0 < \theta < 1$ provides the required example as a consequence of the following facts:

- $X_\theta$ is uniformly convex because $\ell_p(\mathcal{D})$ is.
- The sequence $(u_a)_{a \in \mathcal{D}}$ is equivalent to the unit basis of $\ell_p$ in $X_\theta$ since this happens both in $\ell_p(\mathcal{D})$ and in $X(p)$, by the very definition of the norm of $X_\theta$.
- If $\beta$ is a segment of length $n$, then the norm of $1_\beta$ in $\ell_p(\mathcal{D})$ is $n^{1/p}$, while its norm in $X(p)$ is 1. Therefore, the norm of $1_\beta$ in $X_\theta$ is at most $n^{\theta/p}$.

The embedding of $\ell_p$ into $X_\theta$ provided by the sequence $(u_a)$ cannot be $\mathscr{C}$-trivial since it was established during the proof of Proposition 8.5.8 that for any norm $|\cdot|$ on $c_{00}(\mathcal{D})$ satisfying the condition (8.30) there is a constant $c > 0$ such that, for every $n$, there is a segment $\beta$ of length $n$ such that $|1_\beta| \geq cn^{1/p}$. This contradicts $\|1_\beta\|_\theta$ being dominated by $n^{\theta/p}$. □

## 8.6 The Dark Side of the Johnson–Zippin Theorem

The Johnson–Zippin theorem has its origin in the papers [464; 465]. At the end of [464] Zippin poses three problems; one of them, Problem 2, reads: *Is it true that every $\mathscr{C}$-valued operator defined on any subspace of $\ell_1$ admits a $\lambda$-extension for some $\lambda \geq 2$?* Why has $\lambda < 2$ been ruled out? Because Zippin shows in [465] that if $E$ is the kernel of the sum functional on $\ell_1$, the inclusion $E \longrightarrow \ell_1$, which obviously admits a 2-Zippin selector, does not admit a $\lambda$-Zippin selector for any $\lambda < 2$. That is startling. Problem 2 is explicitly implicit in [466, pp. 1732–1735], has a negative solution as we all know (don't we?) and mutates along the way into

**8.6.1 Zippin's problem** Characterise the subspaces $E$ of $\ell_1$ such that every $C(K)$-valued operator defined on $E$ can be extended to $\ell_1$.

We already know that $\mathscr{C}$-valued operators defined on weak*-closed subspaces of $\ell_1$ admit extensions. The following theorem asserts the same for $\mathscr{L}_\infty$-valued operators. Just use Proposition 5.2.9 (b) to get

**8.6.2 The Johnson–Zippin theorem** *$\mathscr{L}_\infty$-valued operators defined on any weak*-closed subspace of $\ell_1$ can be extended to $\ell_1$.*

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*Proof* $\mathrm{Ext}(H^*, \mathscr{L}_\infty) = 0$ for any subspace $H \subset c_0$. □

The original Johnson–Zippin theorem [237] yields an estimate of $3^+$ for the norm of the extending operator. Under the additional assumption that the quotient space has the AP, they reduce the estimate to an optimal $1^+$. The question of whether the apparent duality between the Lindenstrauss–Pełczyński and the Johnson–Zippin theorems is a side effect of the structure of subspaces of $c_0$ or, as some of the authors think, a real duality that deserves careful consideration.

In doing so, one must take into account that the sequence $0 \longrightarrow \ell_2 \longrightarrow C(B^*_{\ell_2}) \longrightarrow \diamond \longrightarrow 0$ is $\mathscr{C}$-trivial (with the meaning that the embedding is $\mathscr{C}$-trivial) but has non-$\mathscr{C}$-trivial dual or bidual sequences: indeed, the dual sequence has the form $0 \longrightarrow \diamond^* \longrightarrow L_1(\mu) \longrightarrow \ell_2 \longrightarrow 0$ and is not $\mathscr{C}$-trivial because it is semi-equivalent to any projective presentation $0 \longrightarrow \kappa(l_2) \longrightarrow \ell_1 \longrightarrow \ell_2 \longrightarrow 0$ since $\kappa(\ell_2)$ is an ultrasummand by Lemma 10.4.1. Thus, either both sequences are $\mathscr{C}$-trivial or neither of them is. But a $\mathscr{C}$-trivial projective presentation of $\ell_2$ means $\mathrm{Ext}(\ell_2, C(\Delta)) = 0$, which is simply false for many reasons, among which is the next Proposition 8.6.4. The bidual sequence cannot be $\mathscr{C}$-trivial because the natural embedding $\delta\colon \ell_2 \longrightarrow C(B^*_{\ell_2})$ cannot extend to an operator $C(B^*_{\ell_2})^{**} \longrightarrow C(B^*_{\ell_2})$: the former space is injective, thus it enjoys the Dunford–Pettis and Grothendieck properties. The latter makes every operator from it into a separable space weakly compact; the former makes it completely continuous. Thus, its restriction to $\ell_2$ should be compact and cannot be $\delta$. Similar reasoning shows that while any sequence $0 \longrightarrow H \longrightarrow c_0 \longrightarrow c_0/H \longrightarrow 0$ is $\mathscr{C}$-trivial, as is its dual sequence, its bidual sequence cannot be $\mathscr{C}$-trivial. Since weak*-closed subspaces of $\ell_1$ are $\mathscr{C}$-extensible, the 'converse Johnson–Zippin theorem' would be to decide whether weak*-closed subspaces are *the only* $\mathscr{C}$-extensible subspaces of $\ell_1$. But even this formulation remains fishy: does one mean *subspaces isomorphic to weak*-closed subspaces*? The additional information that $\mathscr{C}$-valued operators on weak*-closed subspaces admit almost isometric extensions paves the way for a true converse:

**Proposition 8.6.3** *A subspace $E \subset \ell_1$ is weak*-closed if and only if every $\mathscr{C}$-valued operator defined on $E$ admits $1^+$-extensions to $\ell_1$.*

*Proof* The 'only if' part was proved in Proposition 8.3.14, so we prove the 'if' part. Let $(y_n)$ be a sequence in a closed subspace $E \subset \ell_1$. Assume that the sequence is weak*-convergent to some point $y \in \ell_1$. By passing to a subsequence, we may assume that $(y_n)$ defines a type $\sigma$ on $\ell_1$, which, by the $m_1$-type property, must have the form $\sigma(x) = \|x - y\| + \|y\|$. Since the pair $(E, \ell_1)$ has all properties $\Sigma_1(\lambda)$ for $\lambda > 1$, one has $\inf_{u\in E} \sigma(u) \le \lambda\sigma(y)$, hence

$\inf_{u\in E}\sigma(u) \le \sigma(y)$. These two things together yield $\inf_{u\in E}\|u-y\| + \|y\| \le \sigma(y)$, which necessarily means $y \in E$. □

Subspaces of $\ell_1$ that are weak*-closed with respect to other dual pairings are not necessarily $\mathscr{C}$-extensible. To show that, we anticipate from Proposition 8.6.4 that there are heavy restrictions on a subspace $E$ of $\ell_1$ in order for it to be $\mathscr{C}$-extensible: $\ell_1/E$ has to be at least a Schur space. Thus, given a non-Schur separable space $X$ and a quotient map $Q\colon \ell_1 \longrightarrow X$, there exists an operator $\ker Q \longrightarrow C(\Delta)$ that cannot be extended to $\ell_1$. Now pick an isometric embedding of Schreier's space $\mathsf{S}$ into $C(\omega^\omega)$. The kernel of the quotient map $\ell_1 \longrightarrow \mathsf{S}^*$ is obviously weak*-closed when $\ell_1$ is treated as the dual of $C(\omega^\omega)$. But $\mathsf{S}^*$ is not Schur since $\mathsf{S}$ does not have the Dunford–Pettis property [102].

## When Is $\mathrm{Ext}(X, C(\Delta)) = 0$?

A reformulation of Zippin's problem is: *Characterise the Banach spaces $X$ such that* $\mathrm{Ext}(X, C(\Delta)) = 0$. Actually, it is not known whether there exists a single 'non-trivial' example of a Banach space $X$ for which $\mathrm{Ext}(X, C(\Delta)) = 0$. What do we mean by a non-trivial example? Well, something like '$X$ does not have the form $\ell_1(F_n)$ for finite-dimensional $F_n$, or is not a twisted sum of such spaces … or it is not too close to $\ell_1$ … somehow'.

**Proposition 8.6.4** *If* $\mathrm{Ext}(X, C(\Delta)) = 0$ *then $X$ has the Schur property.*

*Proof* Let $X$ be a separable Banach space. If $X$ is not Schur then it contains a weakly null normalised basic sequence $(x_n)_n$. By considering the basis expansion, we obtain a map $\tau_0\colon \overline{[x_n : n \in \mathbb{N}]} \longrightarrow c_0$ such that $\tau_0(x_n) = e_n$. Since $c_0$ is separably injective, $\tau_0$ admits a 2-extension $\tau\colon X \longrightarrow c_0$. Pick the Foiaş–Singer sequence (2.5) and draw the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(\Delta) & \longrightarrow & D & \longrightarrow & c_0 & \longrightarrow & 0\\
 & & \| & & \uparrow{\scriptstyle \underline{\tau}} & & \uparrow{\scriptstyle \tau} & & \\
0 & \longrightarrow & C(\Delta) & \longrightarrow & \mathrm{PB} & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

The pullback sequence cannot split because a linear continuous section $s\colon X \longrightarrow \mathrm{PB}$ would produce a weakly null lifting $(\underline{\tau} s(x_n))_n$ of $(e_n)$, which is impossible by Lemma 2.2.4. □

Actually, [267, Theorem 5.1] shows that $X$ must have a stronger version of the Schur property, but we will not continue on that path. The paper [267] provides, in addition to the previous one, the best available answer to Zippin's problem: if $X$ is a separable space with a UFDD and $\mathrm{Ext}(X, C(\Delta)) = 0$ then

$X$ is isomorphic to the dual of a subspace of $c_0$ (see the next section). This, in turn, provides the best evidence we have for a positive answer to the following conjecture [271, Problem 1]: *let $X$ be a separable Banach space.* $\operatorname{Ext}(X, C(\Delta)) = 0$ *if and only if $X$ is isomorphic to the dual of a subspace of* $c_0$. Equivalently, a subspace of $\ell_1$ is $\mathscr{C}$-extensible if and only if it occupies a weak*-closed *position*, that is, if there is an automorphism $\tau\colon \ell_1 \longrightarrow \ell_1$ such that $\tau[E]$ is weak*-closed. A different approach to Zippin's problem was undertaken in [113] by characterising when $\operatorname{Ext}(X, C(K)) = 0$ via properties of a metric projection $m\colon Q^{(1)}(X, \mathbb{R}) \longrightarrow \mathsf{L}(X, \mathbb{R})$. The crucial point in all this is that while $\operatorname{Ext}(X, C(\Delta)) = 0$ imposes tight conditions on $X$, $\operatorname{Ext}(X, c) = 0$ imposes none when $X$ is separable. What about intermediate cases? The simplest $\mathscr{C}$-spaces intermediate between $c$ and $C(\Delta)$, or $C[0,1]$, are the spaces $C(\alpha)$ for countable ordinals $\alpha$. Since $C(\alpha)$ is a complemented subspace of $C(\Delta)$, it is clear that $\operatorname{Ext}(X, C(\Delta)) = 0 \implies \operatorname{Ext}(X, C(\alpha)) = 0$. It is then natural to look for properties of $X$, weaker than Schur although probably of the same type, that might characterise $\operatorname{Ext}(X, C(\alpha)) = 0$. Brunel and Sucheston [55] introduced the notion of a *spreading model*, which we will use only as a tool, using Ramsey-style arguments to observe that each normalised sequence in a Banach space has a subsequence $(x_n)_{n\geq 1}$ such that the limit $\lim_{\substack{n_1\longrightarrow\infty\\ n_1<\cdots<n_k}} \left\|\sum_{i=1}^k \lambda_i x_{n_i}\right\|$ exists for every finite sequence of scalars $\lambda_1, \dots, \lambda_k$. This limit defines a norm on the space of finitely supported sequences if and only if the subsequence $(x_n)_n$ is not convergent [34, I. 1. Proposition 2]. The spreading model generated by the sequence $(x_n)$ is the completion of that space.

**Proposition 8.6.5** *Let $X$ be a separable Banach space. If* $\operatorname{Ext}(X, C(\omega^\omega)) = 0$ *then, for some $\mu > 0$, every weakly null normalised sequence $(x_n)$ and every $k \in \mathbb{N}$, there are integers $n_1 < n_2 < \cdots < n_k$ and signs $\varepsilon_k = \pm 1$ such that $\|\sum_j^k \varepsilon_j \lambda_j x_{n_j}\| \geq \mu \sum_{j=1}^k \lambda_j$ for all choices of positive scalars $\lambda_j$. Therefore, every spreading model of $X$ generated by a normalised weakly null sequence is isomorphic to $\ell_1$.*

*Proof* Consider the embedding of $\omega^\omega$ into $[0,1]$ as in Lemma 2.2.6, form the sequence $0 \longrightarrow C(\omega^\omega) \longrightarrow D(\omega^\omega; (\omega^\omega)') \longrightarrow c_0((\omega^\omega)') \longrightarrow 0$ and observe that since $(\omega^\omega)'$ is countable, we can identify the quotient space with $c_0$.

Let $(x_n)$ be a basic sequence in $X$. Proceeding as in the proof of Proposition 8.6.4, we obtain an operator $\tau\colon X \longrightarrow c_0$ such that $\tau(x_n) = e_n$, where $(e_n)$ is the unit basis of $c_0$, and we can form the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(\omega^\omega) & \longrightarrow & D(\omega^\omega; (\omega^\omega)') & \xrightarrow{J} & c_0 & \longrightarrow & 0 \\ & & \Big\| & & \uparrow{\scriptstyle \underline{\tau}} & & \uparrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & C(\omega^\omega) & \longrightarrow & \mathrm{PB} & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Since $\operatorname{Ext}(X, C(\omega^\omega)) = 0$, the pullback sequence splits and so $\tau$ admits a bounded linear lifting $L\colon X \longrightarrow D(\omega^\omega; (\omega^\omega)')$. Place the functions $L(x_n)$ in Lemma 2.2.7 to get, for every $n$, integers $n \le n_1 < \cdots < n_k$ and signs $\varepsilon_i = \pm 1$ such that for all positive $\lambda_i$, we have

$$\|L\| \left\| \sum_{1\le i\le k} \varepsilon_i \lambda_i x_{n_i} \right\| \ge \left\| \sum_{1\le i\le k} \varepsilon_i \lambda_i L(x_{n_i}) \right\| \ge (1-\delta) \sum_{1\le i\le k} \lambda_i,$$

which proves the first part. A result of Beauzamy [34, I. 5. Proposition 1] implies that the canonical basis of the spreading model constructed over a normalised weakly null sequence is 1-unconditional, and this means that the spreading model constructed over the sequence $(x_n)$ is isomorphic to $\ell_1$. □

Ok, enough striking; we go to the ground and pound.

## When Is $\operatorname{Ext}(X, C(\omega^\omega)) = 0$ ?

Let $X$ be a separable Banach space. For every $N \in \mathbb{N}$, the space $C(\omega^N)$ is isomorphic to $c_0$, hence it is separably injective and $\operatorname{Ext}(X, C(\omega^N)) = 0$. On the other hand, the space $\omega^\omega$ can be represented as the one-point compactification of the disjoint union $\bigcup_N \omega^N$, hence $C_0\big(\bigcup_N \omega^N\big)$ is a hyperplane of $C(\omega^\omega)$ and

$$C(\omega^\omega) \simeq C_0\left(\textstyle\bigcup_N \omega^N\right) = c_0\left(\mathbb{N}, C(\omega^N)\right).$$

Thus, Corollary 5.2.6 yields $\operatorname{Ext}(X, C(\omega^\omega)) = 0$ if and only if $\operatorname{Ext}\left(X, C(\omega^N)\right) = 0$ uniformly on $N$, and the existence of non-trivial elements of $\operatorname{Ext}(X, C(\omega^\omega))$ therefore depends on quantitative aspects of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(\omega^N) & \longrightarrow & \ell_\infty(\omega^N) & \longrightarrow & Q_N & \longrightarrow & 0\\
 & & \| & & \uparrow & \nwarrow & \uparrow & & \\
0 & \longrightarrow & C(\omega^N) & \longrightarrow & \cdot & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

The upper sequence is the 'obvious' injective presentation of $C(\omega^N)$, and we have written $Q_N = \ell_\infty(\omega^N)/C(\omega^N)$ for the sake of simplicity. The lower sequence splits, and therefore every operator $X \longrightarrow Q_N$ admits a lifting to $\ell_\infty(\omega^N)$, as the dotted arrow reminds us.

We will focus on the ratio between the norms of the operators $X \longrightarrow Q_N$ and the norms of their liftings to $\ell_\infty(\omega^N)$. Since there is not much we can say about the structure of $Q_N$, we will proceed the other way a round (taking advantage, let us say it once more, of the fact that *all* such operators have liftings): we start with an operator $\tau\colon X \longrightarrow \ell_\infty(\omega^N)$ and obtain, after composition with the quotient map, an operator $\tau'\colon X \longrightarrow Q_N$. This operator might have a much

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smaller norm, and we then study the norms of its liftings to $\ell_\infty(\omega^N)$. All of them have the form $\tau - \tau''$, with $\tau'' \in \mathfrak{L}(X, C(\omega^N))$.

If $K$ is a compact space, operators $\tau\colon X \longrightarrow \ell_\infty(K)$ and (not necessarily continuous) bounded mappings $\varphi : K \longrightarrow X^*$ correspond one to each other in the obvious way's $(\tau x)(a) = \langle \delta_a, \tau x\rangle = \langle \varphi(a), x\rangle$. Identification of $a$ with the corresponding evaluation functional $\delta_a$ on $C(K)$ means that $\varphi(a) = \tau^*(\delta_a)$ can be interpreted as '$\varphi$ is the restriction of $\tau^*$ to $K$', and we can just write $\varphi(a) = \tau^*(a)$.

It is thus clear that $\|\tau\| = \sup_{a\in K} \|\tau^*(a)\|$. If $\tau'$ is the composition of $\tau$ with the quotient map $\ell_\infty \longrightarrow \ell_\infty/C(K)$ then $\|\tau'\| = \sup_{\|x\|\leq 1} \operatorname{dist}(\tau(x), C(K))$, and Lemma 2.2.2 immediately yields

$$2\|\tau'\| = \sup_{\|x\|\leq 1} \operatorname{osc}_K\langle \tau^*(\cdot), x\rangle.$$

We will informally refer to the right-hand member of the preceding equation as *the oscillation* of $\tau$. The operator $\tau$ takes values in $C(K)$ if and only if it has oscillation 0; equivalently, $\tau^*$ is weak*-continuous on $K$ (notice that the canonical inclusion $K \longrightarrow \ell_\infty(K)^*$ is not weak*-continuous).

Returning to $\omega^N$ and $\omega^\omega$, it is true that, using the constants $K^{(1)}[\cdot,\cdot]$ of (3.20), we have $\operatorname{Ext}(X, C(\omega^\omega)) = 0$ if and only if $\sup_{N\in\mathbb{N}} K^{(1)}[X, C(\omega^N)] < \infty$, but this does not fit very well with our current approach via operators. The following parameters provide more 'computable' forms for $K^{(1)}[X, C(\omega^N)]$:

**8.6.6** Let $\pi_N(X)$ (resp. $\sigma_N(X)$) be the smallest constant such that if

$$0 \longrightarrow C(\omega^N) \xrightarrow{\ \jmath\ } Z \xrightarrow{\ \rho\ } X \longrightarrow 0 \tag{8.32}$$

is an isometrically exact sequence of Banach spaces and $\varepsilon > 0$ then there is a linear projection $P$ through $\jmath$ with $\|P\| \leq \pi_N(X) + \varepsilon$ (resp. a linear section $S$ of the quotient map with $\|S\| \leq \sigma_N(X) + \varepsilon$).

It is clear that the sequences $(\pi_N(X))_N$, $(\sigma_N(X))_N$ and $(K^{(1)}[X, C(\omega^N)])_N$ are equivalent: actually $|\pi_N(X) - \sigma_N(X)| \leq 1$ for all $N$ by the discussion preceding Definition 2.1.6, while the argument in Corollary 3.3.8 implies that $\sigma_N(X) \leq 4K^{(1)}[X, C(\omega^N)]$ and the proof of Proposition 3.6.7 shows that $K^{(1)}[X, C(\omega^N)] \leq 2\sigma_N(X)$. Any other estimates the reader may find are welcome.

Given a separable Banach space $X$, we set

$$\rho_N(X) = \sup \operatorname{dist}(\tau, \mathfrak{L}(X, C(\omega^\omega))\,, \tag{8.33}$$

where the supremum is taken over all bounded operators $\tau : X \longrightarrow \ell_\infty(\omega^N)$ satisfying $d(\tau x, C(\omega^N)) \leq \|x\|$ for all $x \in X$. A brief reflection suffices to realise that $\rho_N(X)$ is the infimum of those constants $\varrho$ such that every operator

$X \longrightarrow Q_N$ admits a $\varrho$-lifting to $\ell_\infty(\omega^N)$. We must be Gandalf on this: you should not pass from here without assimilating this fact.

The sequences $(\rho_N(X))_N$ and $(\pi_N(X))_N$ are equivalent too:

**Lemma 8.6.7** *For any Banach space X, one has* $|\rho_N(X) - \pi_N(X)| \leq 1$.

*Proof* Consider an isometrically exact sequence as in (8.32), let $I\colon Z \longrightarrow \ell_\infty(\omega^N)$ be an extension of the identity of $C(\omega^N)$ with $\|I\| = 1$ and form the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(\omega^N) & \xrightarrow{\jmath} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I'} & & \\ 0 & \longrightarrow & C(\omega^N) & \longrightarrow & \ell_\infty(\omega^N) & \longrightarrow & Q_N & \longrightarrow & 0 \end{array}$$

in which $\|I'\| \leq 1$. If $L\colon X \longrightarrow \ell_\infty(\omega^N)$ is a lifting of $I'$ with $\|L\| \leq \rho_N(X) + \varepsilon$, then $I - L\rho$ is a projection along $\jmath$ of norm at most $1 + \rho_N(X) + \varepsilon$. Hence $\pi_N(X) \leq \rho_N(X) + 1$. The other inequality is clear after the following remark, which is nothing but *the* pullback trick in disguise:

**Claim** Let $\tau\colon X \longrightarrow \ell_\infty(\omega^N)$ be a linear map such that $\operatorname{dist}(\tau(x), C(\omega^N) \leq \|x\|$ for all $x \in X$. For every $\varepsilon > 0$, there is a linear map $\tau''\colon X \longrightarrow C(\omega^N)$ such that $\|\tau - \tau''\| \leq \sigma_N(X) + \varepsilon$. Moreover, if $\tau$ is bounded then $\tau''$ is bounded.

Indeed, let $Z[\tau]$ be the space $C(\omega^N) \times X$ normed by $\|(f, x)\| = \max\left(\|f - \tau(x)\|, \|x\|\right)$ and consider the sequence $0 \longrightarrow C(\omega^N) \longrightarrow Z[\tau] \longrightarrow X \longrightarrow 0$ with embedding $f \longmapsto (f, 0)$ and quotient map $(f, x) \longmapsto x$. The sequence is isometrically exact by the hypothesis on $\tau$: if $\|x\| < 1$ then there is $f$ such that $\|f - \tau(x)\| < 1$ and therefore the point $(f, x)$ is in the open unit ball of $Z[\tau]$. By definition, there is a section $S : X \longrightarrow Z[\tau]$ of the quotient map with $\|S\| \leq \sigma_N(X) + \varepsilon$. This $S$ must have the form $S(x) = (\tau''(x), x)$ for some linear map $\tau''\colon X \longrightarrow C(\omega^N)$, and since $\|S(x)\| = \|(\tau''(x), x)\| = \max\left(\|\tau''(x) - \tau(x)\|, \|x\|\right)$, it follows that $\|\tau'' - \tau\| \leq \sigma_N(X) + \varepsilon$. □

Consequently:

**Proposition 8.6.8** *Let X be a separable Banach space.* $\operatorname{Ext}(X, C(\omega^\omega)) = 0$ *if and only if* $\sup_N \rho_N(X) < \infty$.

Amir [8; 9] and Baker [28] proved that given an isometrically exact sequence $0 \longrightarrow C(\omega^N) \longrightarrow Z \longrightarrow X \longrightarrow 0$ with $X$ separable, there is always linear projection with norm at most $2N + 1$. Moreover, if $X = C(\omega^{N-1})$ then for every $\varepsilon > 0$, there is an exact sequence as above such that any projection has norm at least $2N + 1 - \varepsilon$. Corollary 2.2.8 yields $\sigma_N(c_0) \geq N$ (hence $\pi_N(c_0) \geq N - 1$ and

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$\rho_N(c_0) \geq N - 2$ by Lemma 8.6.7), while a more precise estimate [73, Theorem 3.5] yields $\pi_N(c_0) = 2N+1$, as in the Amir–Baker result, and thus $\rho_N(c_0) \geq 2N$. Let us continue with the general case.

When dealing with ordinals or with trees, as we are soon to do, we must not be stark and lose our heads (with unnecessary details): recall that despite its ordinal pedigree, the space $\omega^N$ was declared in Section 1.6 to be the only countable compact whose $N$th derived set is a singleton, no matter which peculiar representation of it we choose.

To simplify the analysis of the oscillation of the involved operators, we choose the following disguised $\sigma_N(2^{\mathbb{N}})$ form: $\omega^N$ is the set whose points are subsets of $\mathbb{N}$ with at most $N$ elements (including the empty set) with the topology inherited from $2^{\mathbb{N}}$. We write the elements of $\omega^N$ in increasing order: $a = (n_1, \ldots, n_k)$ with $n_1 < n_2 < \cdots < n_k$. We define an order on $\omega^N$ as follows: if $b = (m_1, \ldots, m_l)$ then $a \leq b$ means that $k \leq l$ and $n_j = m_j$ for $1 \leq j \leq k$. In particular, $\emptyset \leq b$ for all $b \in \omega^N$. Observe that this is a mere partial order, making our particular $\omega^N$ a tree, but not order isomorphic to any ordinal. Given $a$ as above, $a^- = (n_1, \ldots, n_{k-1})$ and $a^+ = \{b \colon a \leq b \colon |b| = |a| + 1\} = \{(n_1, \ldots, n_k, m) \colon m > n_k\}$. And in case of doubt, $\lim_{b\in a^+} b = a$ and $\lim a = \emptyset$ as $\min a \to \infty$.

**Definition 8.6.9** A map $\Upsilon \colon \omega^N \longrightarrow X^*$ is a weak*-null tree map if $\Upsilon(\emptyset) = 0$ and $\lim_{b\in a^+} \Upsilon(b) = 0$ in the weak*-topology for all $|a| < N$.

**Definition 8.6.10** An operator $\tau_\Upsilon \colon X \longrightarrow \ell_\infty(\omega^N)$ is said to be tree generated by a map $\Upsilon \colon \omega^N \longrightarrow X^*$ if

$$\langle \tau_\Upsilon x, a\rangle = \left\langle \sum_{b\leq a} \Upsilon(b), x \right\rangle.$$

Observe that all linear maps $\tau \colon X \longrightarrow C_0(\omega^N\backslash\{\emptyset\})$ are tree generated by weak*-null tree maps: set $\Upsilon(\emptyset) = 0$ and $\langle \Upsilon(a), x\rangle = \langle \tau x, a\rangle - \langle \tau x, a^-\rangle$. If $\tau_\gamma$ is tree generated, then since $\|\tau_\Upsilon x\|$ is essentially attained at the isolated points $|a| = N$ of $\omega^N$, we have $\|\tau_\Upsilon\| = \sup_{|a|=N} \left\|\sum_{b\leq a} \Upsilon(b)\right\|$.

Given a separable Banach space $X$, we introduce the parameter

$$\alpha_N(X) = \sup_{\Upsilon} \inf_{|a|=N} \left\| \sum_{b\leq a} \Upsilon(b) \right\|, \tag{8.34}$$

where the supremum is taken over all weak*-null tree maps $\Upsilon \colon \omega^N \longrightarrow B^*_X$.

**Lemma 8.6.11** *If $X$ is a Banach space with a monotone shrinking basis then*

$$\rho_{2N}(X) \leq 4\alpha_N(X) \leq 4\rho_N(X).$$

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*Proof* Let $(e_j)_j$ be the basis so that $x = \sum_j \langle e_j^*, x\rangle e_j$ for all $x \in X$, and let $P_n(x) = \sum_{j=1}^n \langle e_j^*, x\rangle e_j$ be the canonical projection. Since the basis is (monotone and) shrinking, $(P_n^*)$ is a sequence of (contractive) operators on $X^*$ that is pointwise convergent to the identity. We prove the first inequality. Let $\tau\colon X \longrightarrow \ell_\infty(\omega^{2N})$ be a linear operator with oscillation at most 2. Our plan is to find a linear $\ell\colon X \longrightarrow C(\omega^{2N})$ such that $\|\tau - \ell\| \le 4\alpha_N(X)$; or, what is the same, $\|\tau^*(a) - \ell^*(a)\| \le 4\alpha_N(X)$ for all $a$. Fix $\varepsilon > 0$. A compactness argument used in combination with the definition of oscillation yields for each finite-dimensional subspace $F \subset X$ and $a^-$ an index $\psi(F, a) \ge a^-$ such that for all $a^- \le b, c \le \psi(F, a)$, we have $\| (\tau^*(b) - \tau^*(c)) |_F\| = \sup_{x\in B_F} |\langle \tau^*(b) - \tau^*(c), x\rangle| \le 2 + \varepsilon$. Choose the spaces $P_n[X]$ as $F$ and let $\nu(a) = \max\{n\colon \psi(P_n[X], a^-) \ge a\}$. Thus, for $b \le a$, we have

$$\|P_{\nu(a)}^* (\tau^*(b) - \tau^*(a^-))\| \le 2 + \varepsilon. \tag{8.35}$$

Fix $\lambda$. We proceed by reverse induction to define what is a $\lambda$-acceptable set $\{f_1, \dots, f_k\} \subset B_X^*$ of cardinality $0 \le k \le N$. The set $\{f_1, \dots, f_N\} \subset B_X^*$ is $\lambda$-acceptable if $\|f_1 + \cdots + f_N\| \le \lambda$. For $k \le N - 1$, the set $\{f_1, \dots, f_k\} \subset B_X^*$ is *$\lambda$-acceptable* if there is a weak*-neighbourhood $V$ of 0 such that for each $f \in V \cap B_X^*$, the set $\{f_1, \dots, f_k, f\}$ is $\lambda$-acceptable. Our interest in this notion is that if $\lambda > \alpha_N(X)$ then $\emptyset$ is $\lambda$-acceptable. A collection of $k \le N$ block subspaces $\{G_1, \dots, G_k\}$ is *$\lambda$-good* if, for some $\mu < \lambda$, every set $\{x_1^*, \dots, x_k^*\}$ with $x_j^* \in G_j$ is $\mu$-acceptable. At this moment, the magical function $g\colon \mathbb{N} \longrightarrow \mathbb{N}$ appears.

**Claim** There is a function $g\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that if $k < N$ and $\{G_1, \dots, G_k\}$ is a $\lambda$-good family of block subspaces of $P_n^*[X^*]$ then for any block subspace $G_{k+1} \subset (\mathbf{1} - P_{g(n)})^*[X^*]$, the collection $\{G_1, \dots, G_k, G_{k+1}\}$ is $\lambda$-good.

*Proof of the claim* The family of block subspaces of $P_n^*[X^*]$ is finite; therefore, there is $\mu < \lambda$ such that every $\lambda$-good collection $\{G_1, \dots, G_k\}$ of block subspaces is $\mu$-good. Pick $\varepsilon > 0$ such that $\mu + N/\varepsilon < \lambda$, and choose in each block subspace $G$ an $\varepsilon$-net for $B_G$. This produces a finite collection $\mathscr{G}$ of $\mu$-acceptable sets so that whenever $\{G_1, \dots, G_k\}$ is a $\lambda$-good collection of block subspaces of $P_n^*[X^*]$ and $g_j \in B_{G_j}$, there then is $\{f_1, \dots, f_k\} \in \mathscr{G}$ with $\|g_j - f_j\| \le \varepsilon$ for all $1 \le j \le k$. Find $g(n)$ such that for every $x^* \in P_{g(n)}^*[X^*] \cap B_X^*$ and every $\{f_1, \dots, f_k\} \in \mathscr{G}$, the set $\{f_1, \dots, f_k, x^*\}$ is $\mu$-acceptable. A perturbation argument yields that for every $\lambda$-good family $\{G, \dots, G_k\}$ of block subspaces of $P_n[X^*]$ with $k < N$ and any block subspace $G \subset \mathbf{1} - P_{g(n)}^*[X]$, the collection $\{G_1, \dots, G_k, G\}$ is $(\mu + N/\varepsilon)$-good, hence $\lambda$-good. □

For the rest of the proof, set $\lambda = \alpha_N + \varepsilon$ and set $g$ to be the corresponding function. Define $\varphi\colon \omega^{2N} \longrightarrow \mathbb{N}$ by

$$\varphi(a) = \begin{cases} g(\emptyset), & \text{if} \quad a = \emptyset, \\ \varphi(a^-), & \text{if} \quad \nu(a) < g(\varphi(a^-)), \\ \nu(a), & \text{if} \quad \nu(a) \geq g(\varphi(a^-)). \end{cases}$$

To get $\ell\colon X \longrightarrow C(\omega^{2N})$, we define a weak*-continuous map $\ell^*: \omega^{2N} \longrightarrow X^*$ by setting $\ell^*(\emptyset) = \tau^*(\emptyset)$ and

$$\ell^*(a) = \sum_{\emptyset < b \leq a} \left(P_{\varphi(b)} - P_{\varphi(b^-)}\right)^* \tau^*(b^-) + \left(\mathbf{1} - P_{\varphi(a)}\right)^* \tau^*(a)$$

for $a \neq \emptyset$. The map $\ell^*$ is weak*-continuous because if $c \in a^+$,

$$\begin{aligned} \ell^*(c) - \ell^*(a) &= \left(P_{\varphi(c)} - P_{\varphi(a}\right)^* \tau^*(a) + \left(\mathbf{1} - P_{\varphi(c)}\right)^* \tau^*(c) - \left(\mathbf{1} - P_{\varphi(a)}\right)^* \tau^*(a) \\ &= \left(\mathbf{1} - P_{\varphi(c)}\right)^* \tau^*(c) - \left(\mathbf{1} - P_{\varphi(c)}\right)^* \tau^*(a) \\ &= \left(\mathbf{1} - P_{\varphi(c)}\right)^* \left(\tau^*(c) - \tau^*(a)\right), \end{aligned}$$

and thus

$$\begin{aligned} \lim_{c \in a^+} \langle \ell^*(c) - \ell^*(a), x\rangle &= \lim_{c \in a^+} \left\langle \left(\mathbf{1} - P_{\varphi(c)}\right)^* \left(\tau^*(c) - \tau^*(a)\right), x \right\rangle \\ &= \lim_{c \in a^+} \left\langle \tau^*(c) - \tau^*(a), (\mathbf{1} - P_{\varphi(c)})x \right\rangle = 0. \end{aligned}$$

Finally, we need to estimate $\|\tau - \ell\| = \sup_{a \in \omega^{2N}} \|\tau^*(a) - \ell^*(a)\|$. Given $a = (n_1, \ldots, n_k) \in \omega^{2N}$, set $m_0 = \varphi(\emptyset)$ and $m_j = \varphi(n_1, \ldots, n_j)$. If we look carefully at the family $(P^*_{m_1} - P^*_{m_0})[X^*], (P^*_{m_2} - P^*_{m_1})[X^*], \ldots, (P^*_{m_k} - P^*_{m_{k-1}})[X^*]$ then we detect that all the even and all the odd elements form one of those $\lambda$-good families in the claim. In other words, if $x^*_j \in (P^*_{m_j} - P^*_{m_{j-1}})[X^*]$ are chosen with $\|x^*_j\| \leq 1$ then $\|\sum_{j=1}^k x^*_j\| \leq 2\lambda$. We thus have

$$\begin{aligned} \tau^*(a) - \ell^*(a) &= \tau^*(a) - \sum_{\emptyset < b \leq a} \left(P_{\varphi(b)} - P_{\varphi(b^-)}\right)^* \tau^*(b^-) - \left(\mathbf{1} - P_{\varphi(a)}\right)^* \tau^*(a) \\ &= \sum_{\emptyset < b \leq a} P^*_{\varphi(b)} \tau^*(a) + \left(\mathbf{1} - P_{\varphi(a)}\right)^* \tau^*(a) \\ &\quad - \sum_{\emptyset < b \leq a} \left(P_{\varphi(b)} - P_{\varphi(b^-)}\right)^* \tau^*(b^-) - \left(\mathbf{1} - P_{\varphi(a)}\right)^* \tau^*(a) \\ &= \sum_{\emptyset < b \leq a} \left(P_{\varphi(b)} - P_{\varphi(b^-)}\right)^* \tau^*(a) - \sum_{\emptyset < b \leq a} \left(P_{\varphi(b)} - P_{\varphi(b^-)}\right)^* \tau^*(b^-) \\ &= \sum_{\emptyset < b \leq a} \left(P_{\varphi(b)} - P_{\varphi(b^-)}\right)^* \left(\tau^*(a) - \tau^*(b^-)\right). \end{aligned}$$

From (8.35), we get $\left\|(P_{\varphi(b)} - P_{\varphi(b^-)})^* \left(\tau^*(a) - \tau^*(b^-)\right)\right\| \leq 2 + \varepsilon$, hence

$$\|\tau^*(a) - \ell^*(a)\| = \left\| \sum_{\emptyset < b \leq a} \left(P_{\varphi(b)} - P_{\varphi(b^-)}\right)^* \left(\tau^*(a) - \tau^*(b^-)\right) \right\| \leq (4 + \varepsilon)\alpha_N(X).$$

We pass to prove the second inequality $\alpha_N \le \rho_N$. Let us inform the reader that $\mathcal{A} \subset \omega^N$ is a subtree if $a \in \mathcal{A}$ implies $a^- \in \mathcal{A}$.

**Lemma 8.6.12** *Let $\Upsilon : \omega^N \longrightarrow X^*$ be a weak*-null tree map. There is a subtree $\mathcal{A} \subset \omega^N$ that is order isomorphic to $\omega^N$ such that* $\lim_{\max a \to \infty} \Upsilon(a) = 0$ *in the weak* topology for all $a \in \mathcal{A}$.*

*Proof* Let $(V_n)$ be a countable base of weak*-neighbourhoods of 0 such that $V_{n+1} + V_{n+1} \subset V_n$ for all $n$. The subtree we need is $\mathcal{A} = \{a \in \omega^N :\ \text{if } \emptyset < b \le a \text{ then } \Upsilon(b) \in V_{\max b}\}$. □

Let $\Upsilon : \omega^N \longrightarrow B^*_X$ be a weak*-null tree map such that $\alpha_N(X) \le \left\| \sum_{b \le a} \Upsilon(b) \right\|$ $+\varepsilon$ for all $|a| = N$. Let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be any surjective map such that for each $k \in \mathbb{N}$, the set $\sigma^{-1}(k)$ is infinite. Say, $1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots$. We will set $\sigma\{n_1, \ldots, n_k\} = \{\sigma(n_1), \ldots, \sigma(n_k)\}$. To make this $\sigma$ a map $\omega^N \longrightarrow \omega^N$ in the representation we use, it is only necessary to work with sets $\{n_1, \ldots, n_k\}$ such that $\sigma(n_j) > \sigma(n_{j-1})$ whenever $n_j > n_{j-1}$, and so we will do to keep things tidy. We then consider the operator $\tau_{\Upsilon\sigma}$ tree generated by $\Upsilon\sigma$ (which is not a weak*-null tree map) for which $\tau^*_{\Upsilon\sigma}(a) = \sum_{\emptyset < b \le a} \Upsilon\sigma(b)$. The operator $\tau_{\Upsilon\sigma}$ has oscillation at most 1 at every point $x$: indeed, if $d \ge c \in a^+$ then

$$\begin{aligned}\limsup_{d \to a} \left| \langle \tau^*_{\Upsilon\sigma}(d) - \tau^*_{\Upsilon\sigma}(a), x \rangle \right| &= \limsup_{d \to a} \left| \left\langle \tau^*_{\Upsilon}(\sigma c) + \sum_{c < b \le d} \tau^*_{\Upsilon}(\sigma b), x \right\rangle \right| \\ &\le \left| \langle \tau^*_{\Upsilon}(\sigma c), x \rangle \right| + \left| \left\langle \lim_{d \to a} \sum_{c < b \le d} \tau^*_{\Upsilon}(\sigma b), x \right\rangle \right| \le 1 + 0\end{aligned}$$

by the additional property of Lemma 8.6.12 that we assume $\Upsilon$ enjoys. Therefore, by hypothesis, there exists a weak*-continuous map $\ell^* : \omega^N \longrightarrow X^*$ such that $\|\tau^*_{\Upsilon\sigma}(a) - \ell^*(a)\| \le \frac{1}{2}\rho_N(X)$ for all $a$. Now we just need to find some $|a| = N$ for which $\left\| \sum_{b \le a} \Upsilon\sigma(b) \right\|$ can be properly bounded by a multiple of $\rho_N(X)$. Begin with $\emptyset$ and find $m_1$ such that $\left\| (\mathbf{1} - P_{m_1})^* ( \Upsilon\, \sigma(\emptyset) - \ell^*(\emptyset)) \right\| \le \varepsilon$, which is possible since the FDD is shrinking; then pick $|c| = 1$ such that $\|P^*_{m_1}(\Upsilon(c))\| \le \varepsilon$, which is possible since $\Upsilon$ is weak*-null. Now pick $n_1 > m_1$ such that $\|(\mathbf{1} - P_{n_1})^*(\Upsilon(c))\| \le \varepsilon$. Among the infinitely many $|b| = 1$ with $\sigma(b) = c$, pick one $a_1$ such that $\|P^*_{n_1}(\ell^*(a_1) - \ell^*(\emptyset))\| \le \varepsilon$. This $a_1$ is our

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choice. Repeat the same construction inductively $N$ times until obtaining some $|a_N| = N$, and keep track of the pairs $(m_j, n_j)$:

$$\begin{aligned}
&\Big\| \sum_{k=1}^{N} \Upsilon\sigma(a_k) \Big\| \\
&\le \Big\| \sum_{k=1}^{N} (P_{n_k} - P_{m_k})^*(\Upsilon\sigma(a_k)) \Big\| + 2N\varepsilon \\
&\le \Big\| \sum_{k=1}^{N} ((P_{n_k} - P_{m_k})^*(\Upsilon\sigma(a_k)) + (P_{m_k} - P_{n_{k-1}})^*(\ell^*\sigma(a_k) - \ell^*\sigma(a_{k-1})) \Big\| + 4N\varepsilon \\
&\le \Big\| \sum_{k=1}^{N} (\Upsilon\sigma(a_k) + \ell^*\sigma(a_k) - \ell^*\sigma(a_{k-1})) \Big\| + 6N\varepsilon \\
&\le \|\Upsilon\sigma(a_N) - \ell^*(a_N) + \ell^*(\emptyset) - \Upsilon\sigma(\emptyset)\| + 6N\varepsilon \\
&\le \rho_N(X) + 2N\varepsilon.
\end{aligned}$$

□

The conclusion we get is that a Banach space $X$ with a shrinking basis satisfies $\operatorname{Ext}(X, C(\omega^\omega)) = 0$ if and only if $\sup_N \alpha_N(X) < \infty$.

Let $E$ be a Banach space. A tree map $\Upsilon : \omega^N \longrightarrow E$ is said to be weakly null if, for every $a \in \omega^N$, we have $\Upsilon(b) \longrightarrow 0 \longrightarrow$ weakly as $b \in a^+$. Given $\sigma > 0$, we define $N(E, \sigma)$ to be the least integer $N$ such that there exists a weakly null tree map $\Upsilon : \omega^{N+1} \longrightarrow E$ such that $\| \Upsilon(a)\| \le \sigma$ for all $a$ and $\| \sum_{b \le a} \Upsilon(b)\| > 1$ for $|a| = N$. We put $N(E, \sigma) = \infty$ if no such integer exists. We say that $E$ has a summable Szlenk index if there is $\sigma > 0$ such that $N(E, \sigma) = \infty$. This is not the 'original' definition, which is much funnier, but an equivalent formulation; see the equivalence between (i) and (ii) in [191, Theorem 4.10]. We see that $N(E, \sigma) = N$ means (dividing by $\sigma$) that

$$\sup_{\Upsilon} \inf_{|a|=N} \Big\| \sum_{b \le a} \Upsilon(b) \Big\| \le \frac{1}{\sigma}, \tag{8.36}$$

where the supremum runs *now* over the weakly null tree maps $\Upsilon : \omega^N \longrightarrow B_E$. By taking $E = X^*$, a comparison between (8.36) and (8.34) shows that $\sup_N \alpha_N(X) \le \lambda \Longrightarrow N(X^*, 1/\lambda) = \infty$ (and therefore $X^*$ has a summable Szlenk index). If $X$ is moreover reflexive, the converse is also true. We have:

**Proposition 8.6.13** *Let X be a Banach space with a shrinking basis.*

(a) *If* $\operatorname{Ext}(X, C(\omega^\omega)) = 0$ *then* $X^*$ *has a summable Szlenk index.*
(b) *If the basis is unconditional and* $\operatorname{Ext}(X, C(\omega^\omega)) = 0$ *then X is reflexive.*
(c) *If X is reflexive,* $\operatorname{Ext}(X, C(\omega^\omega)) = 0$ *if and only if* $X^*$ *has a summable Szlenk index.*

*Proof* By classical results of James, $X^*$ is separable. Part (a) has already been proved: $\mathrm{Ext}(X, C(\omega^\omega)) = 0 \implies \sup_N \rho_N(X) < \infty$ (Proposition 8.6.8) $\implies \sup_N \alpha_N(X) < \infty$ (Lemma 8.6.11) $\implies X^*$ has a summable Szlenk index.

(b) $X$ cannot contain $c_0$ (otherwise, since it is separable, it would contain it complemented, and then $\mathrm{Ext}(X, C(\omega^\omega)) \neq 0$). Thus, the basis is boundedly complete, and, being unconditional, $X$ must be reflexive [334, Theorem. 1.b.5].

(c) If $X$ is reflexive, the weak and weak* topologies of $X^*$ coincide, and the implications in the proof of (a) are all reversible. □

So far, the hunt for spaces $X$ such that $\mathrm{Ext}\,(X, C(\omega^\omega)) = 0$ but at the same time $\mathrm{Ext}(X, C(\Delta)) \neq 0$ required us to look at non-Schur spaces whose spreading models are all $\ell_1$. We see now that we can restrict our hunt further to spaces whose duals have a summable Szlenk index.

The property of having a summable Szlenk index goes back to [296] and can be considered a sophisticated way of saying that the space is close to being a subspace of $c_0$. Indeed, spaces uniformly homeomorphic to subspaces of $c_0$ have a summable Szlenk index [191]. But, fortunately, there are more: in [191, Remark p. 3911], the authors claim that the original Tsirelson space $\mathsf{T}^*$ has a summable Szlenk index, as is proved in [296, p. 196]. Furthermore, since $\mathsf{T}$ is reflexive, its basis is shrinking. In other (our) words,

**8.6.14** $\mathrm{Ext}\,(\mathsf{T}, C(\omega^\omega)) = 0$*, while* $\mathrm{Ext}\,(\mathsf{T}, C(\Delta)) \neq 0$.

There is no special difficulty adapting Lemma 8.6.11 to cover the case of spaces with shrinking FDD (see [73] for details): firstly, transform the FDD into a bi-monotone FDD with a renorming, then prove that $\rho_{2N} \leq 4\alpha_N$ and finally obtain $\alpha_N \leq \rho_N$ when the space has a (monotone) shrinking FDD. The interest all this has for us is it allows us to obtain a real, though modest, improvement of Proposition 8.6.13:

**Proposition 8.6.15** *Let X be a separable reflexive Banach space.*

(a) *If $X^*$ has a summable Szlenk index then* $\mathrm{Ext}(X, C(\omega^\omega)) = 0$.
(b) *If X has a FDD and* $\mathrm{Ext}(X, C(\omega^\omega)) = 0$ *then $X^*$ has a summable Szlenk index.*

*Proof* (a) We appeal to the full force of the Johnson–Rosenthal decomposition 5.3.1 to represent $X$ in the form $0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$ with both $A, B$ having shrinking FDD. Since having a summable Szlenk index passes to subspaces and quotients, if $X^*$ has a summable Szlenk index then so have $A^*$ and $B^*$, hence $\mathrm{Ext}(A^*, C(\omega^\omega)) = 0, \mathrm{Ext}(B^*, C(\omega^\omega)) = 0$ and thus also $\mathrm{Ext}(X^*, C(\omega^\omega)) = 0$.

(b) Since $X$ is reflexive, the FDD must be shrinking. Thus, $\operatorname{Ext}(X, C(\omega^{\omega})) = 0$ implies that both $\rho_N(X)$ and $\alpha_N(X)$ are uniformly bounded so that $X^*$ has a summable Szlenk index. □

## 8.7 The Astounding Story behind the CCKY Problem

The paper [73] ended with a comment on a problem that had been ricocheting around our heads since [79, final problem], which we will call the CCKY problem: *show that if $K$ is a non-metrisable compact then* $\operatorname{Ext}(C(K), c_0) \neq 0$. Since $\operatorname{Ext}(c_0(\aleph_1), c_0) \neq 0$ by everything said in Section 2.2, while $\operatorname{Ext}(\ell_\infty, c_0) \neq 0$ by the construction in 2.12.9, it is clear that $\operatorname{Ext}(X, c_0) \neq 0$ for every Banach space containing a complemented copy of either $c_0(\aleph_1)$ or $\ell_\infty$. Compacta $K$ for which $C(K)$ does not contain $c_0(\aleph_1)$ are characterised by the *countable chain condition* (*ccc* in short): every family of pairwise disjoint open sets is countable; see [413, Theorem 4.5]. No characterisation is currently known for compacta $K$ such that $C(K)$ contains / does not contain $\ell_\infty$. Thus, the list of compacta $K$ for which the CCKY problem was known *by then* to have an affirmative answer is:

(a) Eberlein compacta,
(b) Valdivia compacta failing the *ccc*,
(c) $C(K)$ contains $\ell_\infty$,
(d) $C(K)$ admits a continuous injection into $C(\mathbb{N}^*)$ but not into $\ell_\infty$,
(e) $K$ is an ordinal space.

Assertion (a) follows from the following general statement [103, Theorem 3.4]:

**Lemma 8.7.1** *Every non-separable Banach space admits a non-WCG extension by $c_0$.*

*Proof* We assume that $X$ is WCG, since the result is trivial otherwise. According to [400, pp. 336–337], the space $X$ admits a Markuševič basis $(x_\gamma, f_\gamma)_{\gamma\in\Gamma}$, which means a biorthogonal system in $X \times X^*$ for which $(x_\gamma)_{\gamma\in\Gamma}$ separates the points of $X^*$ and $(f_\gamma)_{\gamma\in\Gamma}$ separates the points of $X$. One may assume without loss of generality that $(f_\gamma)_{\gamma\in\Gamma}$ is bounded. Thus, the map $T\colon X \longrightarrow c_0(\Gamma)$ defined by $Tx = (f_\gamma(x))_{\gamma\in\Gamma}$ is easily checked to be an operator with dense range. Consider any non-trivial sequence $0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\Gamma) \longrightarrow 0$ in which $X$ is not WCG, as in Section 2.2. The pullback space in the diagram

$$\begin{CD} 0 @>>> c_0 @>>> X @>>> c_0(\Gamma) @>>> 0\\ @. @| @A\underline{T}AA @ATAA @.\\ 0 @>>> c_0 @>>> \mathrm{PB} @>>> X @>>> 0 \end{CD}$$

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cannot be WCG because $\underline{T}$ has dense range by Lemma 2.1.8. □

Thus, the lower sequence cannot split because $X$ is WCG, so (a) is true.

We prove (b): compacta failing *ccc* contain $c_0(\aleph_1)$. Use [16, Theorem 1.2] – *if* $c_0(\Gamma) \subset C(K)$ *for a Valdivia compact* $K$ *then there is a subset* $J \subset \Gamma$ *such that* $|J| = |\Gamma|$ *and* $c_0(J)$ *is complemented in* $C(K)$ – to get a complemented copy of $c_0(\aleph_1)$ inside $C(K)$, which is enough.

Assertion (c) is clear since the copy of $\ell_\infty$ is necessarily complemented.

To get (d), form the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & C(\mathbb{N}^*) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \imath} & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & \mathrm{PB} & \longrightarrow & C(K) & \longrightarrow & 0 \end{array}$$

in which $\imath$ is the continuous injection claimed in the hypothesis. Instances of (d) appear when $K$ contains a dense set of weight at most $\aleph_1$ but $C(K)$ is not a subspace of $\ell_\infty$, by Parovičenko's first theorem mentioned in Section 1.6, example 6, or else $C(K)$ spaces with non-weak*-separable dual, but admitting continuous injections into $C(\mathbb{N}^*)$, such as $C[0, \omega_1]$.

The uncountable ordinal cases of (e) can be reduced to $[0, \omega_1]$.

Now the story goes that [73, p. 4539–4540] claims that Corson compacta can also play the role of Eberlein compacta just using 'similar arguments'. However, Correa and Tausk noticed that a 'similar argument' cannot work (an explanation of why can be found in [93, p. 115]), amended the situation in [136, Theorem 3.1] and obtained a result of general interest in passing:

**Proposition 8.7.2** $\mathrm{Ext}(X, c_0) \neq 0$ *for every Banach space* $X$ *admitting a biorthogonal system* $(x_{n,\gamma}, f_{n,\gamma})_{n\in\omega,\gamma\in\mathfrak{c}}$ *such that* $(f_{n,\gamma}(x)) \in c_0(\omega \times \mathfrak{c})$ *and*

$$\sup \left\| \sum_{i=1}^{k} x_{n_i,\gamma_i} \right\| < \infty,$$

*where the* sup *is taken over all finite sets of* $\omega \times \mathfrak{c}$.

*Proof* The idea is to obtain an operator $\tau\colon X \longrightarrow \ell_\infty/c_0$ that cannot be lifted to $\ell_\infty$, which is clearly enough to conclude that the lower pullback sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\ & & \| & & {\scriptstyle \underline{\tau}}\uparrow & & {\scriptstyle \tau}\uparrow & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & \mathrm{PB} & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

does not split. The operator $\tau$ will appear as a composition $\tau = uv$ with $v\colon X \longrightarrow c_0(\omega \times \mathfrak{c})$ and $u\colon c_0(\omega \times \mathfrak{c}) \longrightarrow \ell_\infty/c_0$, where $v$ is obviously $v(x) = (f_{n,\gamma}(x))$, while $u$ is $u(e_{n,\gamma}) = 1_{A_{n,\gamma}} + c_0$ for $(A_{n,\gamma})_{n\in\omega,\gamma\in\mathfrak{c}}$, an almost disjoint family

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of subsets of $\mathbb{N}$ with the following property: for every family $(B_{n,\gamma})_{n\in\omega,\gamma\in\mathfrak{c}}$ with $B_{n,\gamma} \subset A_{n,\gamma}$ cofinite, we have $\sup_{m\in\mathbb{N}} \left|\{n \in \omega : m \in \bigcup_{\gamma\in\mathfrak{c}} B_{n,\gamma}\}\right| < \infty$. The existence of such a family is shown in [136, Lemma 2.1]. Assume that $\tau$ admits a lifting $T$. The set $B_{n,\gamma} = \{m \in A_{n,\gamma} : \delta_m T(x_{n,\gamma}) \geq \frac{1}{2}\}$ is cofinite in $A_{n,\gamma}$, so for each $k \geq 1$, there exists $p \in \omega$, $n_1, \ldots, n_k \in \mathbb{N}$ pairwise distinct and $\gamma_1, \ldots, \gamma_k \in \mathfrak{c}$ such that $p \in B_{n_i,\gamma_i}$ for $i = 1, \ldots, k$. Therefore, we get a contradiction with

$$\frac{k}{2} \leq \delta_p\left(\sum_{i=1}^{k} x_{n_i,\gamma_i}\right) \leq \sup\left\|\sum_{i=1}^{k} x_{n_i,\gamma_i}\right\| \|T\|. \qquad \square$$

A simple way to satisfy the conditions of Proposition 8.7.2 in a $C(K)$ space is to have a bounded weak*-null biorthogonal system $(x_{n,\gamma}, f_{n,\gamma})_{n\in\omega,\gamma\in\mathfrak{c}}$ (a biorthogonal system $(x_i, f_i)_{i\in I}$ is bounded if $\sup_i (\|x_i\|, \|f_i\|) < \infty$ and weak*-null if $(f_i(x)) \in c_0(I)$ for all $x \in X$) such that $x_{n,\alpha}x_{m,\beta} = 0$ provided $(n,\alpha) \neq (m,\beta)$ in $\omega \times \mathfrak{c}$. This situation occurs under the following condition, which is satisfied by Corson compacta [136, Lemmata 2.8 and 3.2] under CH: there is a sequence $(F_n)_{n\in\omega}$ of closed subsets of $K$ and a bounded biorthogonal weak*-null system $(x_{n,\gamma}, f_{n,\gamma})_{n\in\omega,\gamma\in\mathfrak{c}}$ in $C(K)$ such that $F_n \cap \overline{\bigcup_{n\neq m} F_m} = \emptyset$ and $\operatorname{supp} f_{n,\gamma} \subset F_n$ for all $n \in \omega$ and all $\gamma \in \mathfrak{c}$. Therefore,

**Proposition 8.7.3** [CH] *If $K$ is a non-metrisable Corson compact then*

$$\operatorname{Ext}(C(K), c_0) \neq 0.$$

In the meantime, one more offspring emerged. During the final stages of the writing of [22], Avilés asked: if a $C(K)$ space is itself a twisted sum of $c_0$ and $c_0(I)$ and $c_0$, does it admit a non-trivial twisted sum with $c_0$?

The intended purpose of [93] was to answer with a consistent yes. In a sense, the paper proves more than announced: any space that is a twisted sum of two $c_0(I)$ can be twisted against $c_0$ under CH; the same is true for the newly obtained twisted sums, and for the new twisted sums and so on. The proof waddles between homology and cardinal set theory, with not much left to $C(K)$-spaces. Its formulation in $C(K)$ terms could, however, be as follows:

**Proposition 8.7.4** [CH] *If $K$ is a non-metrisable compact space of finite height then* $\operatorname{Ext}(C(K), c_0) \neq 0$.

The proof goes as follows. Since $K$ must be infinite, $K' \neq \emptyset$. If $K^{(2)} = \emptyset$ then $K'$ is finite, $C(K)$ is isomorphic to a finite product $c_0(I_1) \times \cdots \times c_0(I_n)$ and the assertion is true. So our first serious concern is with compacta $K$ having $K^{(2)} \neq \emptyset$ and $K^{(3)} = \emptyset$. The natural exact sequence $0 \longrightarrow C_0(K\backslash K') \longrightarrow C(K) \longrightarrow C(K') \longrightarrow 0$ becomes $0 \longrightarrow c_0(I) \longrightarrow C(K) \longrightarrow c_0(J) \longrightarrow 0$. Now, if $J$ is countable, the sequence splits, and $C(K)$ is isomorphic to $c_0(\mathfrak{c})$, so the conclusion follows. Otherwise, the following result applies:

**Lemma 8.7.5** *If $X$ fits into an exact sequence* $0 \longrightarrow c_0(I) \longrightarrow X \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$ *then* $\operatorname{Ext}(X, c_0) \neq 0$.

*Proof* If $|I| \leq \mathfrak{c}$, there is a pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0(\mathfrak{c}) & \longrightarrow & X_0 & \longrightarrow & c_0(\mathfrak{c}) & \longrightarrow & 0 \\ && \downarrow{\scriptstyle \imath} && \downarrow && \| && \\ 0 & \longrightarrow & c_0(I) & \longrightarrow & X & \longrightarrow & c_0(\mathfrak{c}) & \longrightarrow & 0 \end{array} \tag{8.37}$$

and the same if $|I| \geq \mathfrak{c}$ using Lemma 3.9.4. Apply homology with target $c_0$ to the lower exact sequence in (8.37) to get an exact sequence

$$\mathfrak{L}(c_0(I), c_0) \xrightarrow[\text{morphism}]{\text{connecting}} \operatorname{Ext}(c_0(\mathfrak{c}), c_0) \longrightarrow \operatorname{Ext}(X, c_0).$$

If $\operatorname{Ext}(X, c_0) = 0$, the connecting morphism is surjective, which by composition with the map $\imath$ in (8.37) in turn yields a surjective map $\mathfrak{L}(c_0(\mathfrak{c}), c_0) \longrightarrow \operatorname{Ext}(c_0(\mathfrak{c}), c_0)$, something that cardinal arithmetics will show to be impossible. Indeed, $\left|\mathfrak{L}(c_0(\mathfrak{c}), c_0)\right| < \left|\operatorname{Ext}(c_0(\mathfrak{c}), c_0)\right|$. To prove this, fix a Banach space $Y$ and observe that $|\mathfrak{L}(Y, c_0)| \leq |\mathfrak{L}(\ell_1, Y^*)|$: since $\mathfrak{L}(\ell_1, Y^*)$ is the set of bounded sequences of $Y^*$, there are $|(Y^*)^{\mathbb{N}}|$ countable subsets of $Y^*$ and each of them admits $\mathfrak{c}$ bounded sequences, we get

$$|\mathfrak{L}(Y, c_0)| \leq \left|\mathbb{R} \times (|Y^*|^{\aleph_0})^{\aleph_0}\right|.$$

Therefore, if $|Y^*| \leq \mathfrak{c}$, as is the case when $Y = c_0(\mathfrak{c})$, one gets $|\mathfrak{L}(Y, c_0)| \leq \mathfrak{c}^{\aleph_0} = \mathfrak{c}$. On the other hand, Marciszewski and Pol show in [355, Proposition 7.4] (see Proposition 8.7.18 below) that there exist $2^{\mathfrak{c}}$ non-equivalent exact sequences $0 \longrightarrow c_0 \longrightarrow \cdot \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$; i.e. $|\operatorname{Ext}(c_0(\mathfrak{c}), c_0)| \geq 2^{\mathfrak{c}}$. □

It is easy now to believe, even if this sounds like the shoulder on which mathematicians come to cry after making a gaffe, that the ideas above can be inductively continued. Full details are in [93].

The point that is relevant in this tale is that no CH has been used so far, and the reader has our word that it was not used for the rest of the proof in [93]. So the question is unavoidable: does that mean that we have proved that any non-metrisable finite height compact $K$ is such that $\operatorname{Ext}(C(K), c_0) \neq 0$? No. The reason is cardinal arithmetics. What has been proved is that any non-metrisable compact $K$ with height at most 3 and such that $|K'| \geq \mathfrak{c}$ has $\operatorname{Ext}(C(K), c_0) \neq 0$;

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at the end of the day, the proof worked because $\mathfrak{c}^{\aleph_0} < 2^{\mathfrak{c}}$. But the inequality $\aleph^{\aleph_0} < 2^{\aleph}$ does not necessarily hold for all cardinals. Indeed, see [221, Theorem 5.15]: assuming GCH, if $\aleph$ has cofinality greater than $\aleph_0$ then $\aleph^{\aleph_0} = \aleph$, but $\aleph^{\aleph_0} = 2^{\aleph}$ if $\aleph$ has cofinality $\aleph_0$. In particular, $\aleph_1^{\aleph_0} < 2^{\aleph_1}$ does not necessarily hold. On the other hand, even the Marciszewski–Pol argument could not work for $\aleph_1$ without CH: if $\mathfrak{c} < 2^{\aleph_1}$ then the same proof as in [355] yields that there are $2^{\aleph_1}$ different exact sequences $0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\aleph_1) \longrightarrow 0$. However, if $2^{\aleph_1} = \mathfrak{c}$ then the method in [355] does not decide. Summing up, the proof works when $\mathfrak{c}$ is the first step, something that happens under CH. In fact, Proposition 8.7.4 can be formulated as a theorem in ZFC [25, Theorem 6.2]:

**8.7.6** $\operatorname{Ext}(C(K), c_0) \neq 0$ *for every compact space of finite height and weight at least* $\mathfrak{c}$.

In this scenario, Marciszewski and Plebanek [354] show that *that* first step cannot actually be done for $\aleph_1 < \mathfrak{c}$ under Martin's axiom. Let us briefly sketch, but we are just a hunchback digging on a hill of gold, how their ideas go. The key property behind the Marciszewski–Plebanek construction is the following:

**Definition 8.7.7** Let $K$ be a compact space. A countable discrete extension of $K$ is a compact space $L$ containing (a homeomorphic copy of) $K$ whose complement is countable and discrete. A compact space $K$ has the $*$-extension property if, whenever $L$ is a countable discrete extension of $B^*_{C(K)}$ (not $K$!), the canonical embedding $\delta\colon C(K) \longrightarrow C(B^*_{C(K)})$ lifts to $C(L)$ through the restriction map.

The kernel of the restriction map arising from any countable discrete extension is isometric to $c_0$. We have:

**Proposition 8.7.8** *If $K$ has the $*$-extension property then* $\operatorname{Ext}(C(K), c_0) = 0$.

*Proof* We show that every exact sequence $0 \longrightarrow c_0 \xrightarrow{\imath} X \xrightarrow{\rho} C(K) \longrightarrow 0$ actually fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & C(L) & \xrightarrow[\text{restriction}]{r} & C(B^*_{C(K)}) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & X & \longrightarrow & C(K) & \longrightarrow & 0 \end{array} \tag{8.38}$$

*Construction of $L$*: there is no loss of generality assuming that $\imath\colon c_0 \longrightarrow X$ is an isometric embedding and $\rho\colon X \longrightarrow C(K)$ is an isometric quotient. Then let

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$x_n^*$ be Hahn–Banach extensions of the coordinate functionals of $c_0$ through $\imath$ so that

$$M = \rho^*[B^*_{C(K)}] \cup \{x_n^* : n \in \mathbb{N}\},$$

equipped with the weak*-topology is a countable and discrete extension of $\rho^*[B^*_{C(K)}]$ since all accumulation points of $\{x_n^* : n \in \mathbb{N}\}$ vanish on $c_0$. Now we form a countable discrete extension of $B^*_{C(K)}$ taking

$$L = B^*_{C(K)} \cup \{x_n^* : n \in \mathbb{N}\},$$

topologised to be homeomorphic with $M$ via the obvious bijection $h\colon L \longrightarrow M$ given by $h(\mu) = \rho^*(\mu)$ and $h(x_n^*) = x_n^*$. This yields Diagram (8.38), whose lower sequence must split. □

The lower sequence in (8.38) splits if and only if there is an extension operator $E\colon C(K) \longrightarrow C(L)$, i.e. a lifting of $\delta$. Since we can assume without loss of generality that $X$ is the pullback space, $f \longmapsto (Ef, f)$ is a linear continuous selection for the lower sequence and thus $(g, f) \longmapsto (g - Ef, 0)$ defines a projection $X \longrightarrow c_0$. This means that whenever $g \in C(L)$ is such that $rg = \delta f$, $g - Ef \in c_0$, namely $\lim_n(g(n) - Ef(n)) = 0$. Since $Ef(n) = \langle E^*(\delta_n), f\rangle$, it turns out that a lifting $E$ exists if and only if there is a bounded sequence $(\nu_n) \in C(K)^*$ such that

$$\lim_{n\to\infty} (g(n) - \nu_n(f)) = \lim_{n\to\infty} (Ef(n) - \nu_n(f)) = 0 \tag{8.39}$$

for every $f \in C(K)$ and every $g \in C(L)$ such that $rg = \delta f$.

The next natural step towards the solution of the CCKY problem is to find a way to arrive at a bounded sequence $(\nu_n)$ as above. So, the authors place the action in the duality between Boolean algebras $\mathfrak{A}$ and their Stone compacta $\mathrm{ult}(\mathfrak{A})$ as described in Note 4.6.1. Concatenation of the functors Boolean algebras ⇝ Stone compacta ⇝ $C(K)$-spaces yields the correspondence $a \rightsquigarrow a^\circ \rightsquigarrow a^{\circ\circ}$, taking Boolean homomorphisms to continuous functions and then to operators. It is clear that when an arrow at a 'lower level' (Boole, compact) has a (left, right) inverse then the same is true for the induced arrows at 'higher levels' (compact, Banach), but not the converse. In particular, if $a\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ and $b\colon \mathfrak{B} \longrightarrow \mathfrak{A}$ are Boolean morphisms such that $ab = \mathbf{1}_{\mathfrak{B}}$ then $a^{\circ\circ}b^{\circ\circ} = \mathbf{1}_{C(\mathrm{ult}(\mathfrak{B}))}$. An especially important case is that of a Boolean algebra $\mathfrak{A} \subset \mathcal{P}(\mathbb{N})$ containing the finite subsets: if $\rho\colon \mathfrak{A} \longrightarrow \mathfrak{A}/\mathrm{fin}(\mathbb{N})$ is the natural quotient map, then $\rho^{\circ\circ} = r$ is the restriction operator $C(\mathrm{ult}(\mathfrak{A})) \longrightarrow C(\mathrm{ult}(\mathfrak{A}/\mathrm{fin}(\mathbb{N})))$. It turns out then that if $s$ is a right inverse for $\rho$, $s^{\circ\circ}$ is an extension operator for $r$.

We will set $M(\mathfrak{A}) = C(\mathrm{ult}(\mathfrak{A}))^*$ to both simplify notation and stress the fact that elements $C(\mathrm{ult}(\mathfrak{A}))^*$ and bounded additive functions $\mu\colon \mathfrak{A} \longrightarrow \mathbb{R}$ can be identified via $\mu(A) = \langle \mu, 1_A \rangle$. It will also simplify the notation to write $M_1(\mathfrak{A})$ for the unit ball of $M(\mathfrak{A})$ . Given a subalgebra $\mathfrak{B} \subset \mathfrak{A}$, we define a seminorm for bounded functions $\varphi\colon \mathfrak{A} \longrightarrow \mathbb{R}$ by

$$\| - \varphi\|_{\mathfrak{B}} = \sup_{B\in\mathfrak{B}} |\varphi(B)|,$$

which induces, via the identification above,

$$\mathrm{dist}_{\mathfrak{B}}(\varphi, M_1(\mathfrak{A})) = \inf_{\mu\in M_1(\mathfrak{A})} \|\varphi - \mu\|_{\mathfrak{B}}.$$

**Definition 8.7.9** A Boolean algebra $\mathfrak{A}$ has the approximation property ($\mathfrak{a}.\mathfrak{p}.$) if, given any sequence $(f_n)$ of functions $f_n\colon \mathfrak{A} \longrightarrow [-1, 1]$ such that for any finite subalgebra $\mathfrak{B} \subset \mathfrak{A}$,

$$\lim_n \mathrm{dist}_{\mathfrak{B}}(f_n, M_1(\mathfrak{A})) = 0,$$

there is a bounded sequence $(\nu_n) \subset M(\mathfrak{A})$ such that $\lim(f_n(a) - \nu_n(a)) = 0$ for all $a \in \mathfrak{A}$.

The approximation property of the Boolean algebra $\mathfrak{A}$ is crying out: the associated Stone compact $\mathrm{ult}(\mathfrak{A})$ enjoys the $*$-extension property!

**Lemma 8.7.10** *If $K$ is a totally disconnected space whose Boolean algebra of clopen sets has the $\mathfrak{a}.\mathfrak{p}.$ then $K$ has the $*$-extension property.*

*Proof* Let $\mathfrak{A} = \mathfrak{cl}(K)$ so that $K = \mathrm{ult}(\mathfrak{A})$. For each $A \in \mathfrak{A}$, consider the evaluation map $M_1(\mathfrak{A}) \longrightarrow [-1, 1]$ that sends $\mu$ to $\mu(A)$. Given a countably discrete extension $L$ of $M_1(\mathfrak{A})$, obtain a continuous extension $\theta_A\colon L \longrightarrow [-1, 1]$, and then form the sequence of functions $f_n\colon \mathfrak{A} \longrightarrow [-1, 1]$ given by $f_n(A) = \theta_A(n)$. Since $L$ is a countably discrete extension of $M_1(\mathfrak{A})$, all accumulation points of the elements of the countable discrete addition are in $M_1(\mathfrak{A})$, thus $\lim_n \mathrm{dist}_{\mathfrak{B}}(f_n, M_1(\mathfrak{A})) = 0$ precisely because $\mathfrak{B}$ is finite. So, there is a bounded sequence $(\nu_n)$ such that $\lim(\nu_n(A) - f_n(A)) = 0$ for every $A \in \mathfrak{A}$; that is,

$$\lim_{n\to\infty} (\nu_n(A) - \theta_A(n)) = 0,$$

which implies (8.39) for functions $f = 1_A$ with $A \in \mathfrak{A}$: if $rg = 1_A$ then necessarily $\lim(g(n) - \theta_A(n)) = 0$ and thus

$$\lim (g(n) - \nu_n(1_A)) = \lim (g(n) - \theta_A(n) + \theta_A(n) - \nu_n(A)) = 0.$$

Finally, if (8.39) holds for all functions $1_A$ with $A \in \mathfrak{A}$ then it holds for all functions $f \in C(\mathrm{ult}(\mathfrak{A}))$. □

What else has to be done? Oh, yes, to construct a Boolean algebra $\mathfrak{A}$ with the $\mathfrak{a.p.}$! And this is where Martin's axiom, which is a statement about certain partially ordered sets, has a leading role. A strong antichain (downwards) in a partially ordered set $P$ is a subset in which no two elements have a common lower bound. $P$ is said to satisfy the countable chain condition (*ccc*) if every strong antichain is countable. A subset $D \subset P$ is *dense* if, for every $p \in P$, there is $d \in D$ such that $d \leq p$; a subset $F \subset P$ is said to be a *filter* (on $P$) if it is directed ($\forall f, g \in F\ \exists\, h \in F\colon f, g \leq h$) and downwards closed (if $g \leq f$ and $f \in F$ then $g \in F$). Consider the following statement for a cardinal $\aleph_0 \leq \aleph \leq \mathfrak{c}$:

**8.7.11** $\mathsf{MA}(\aleph)$ Given a partially ordered set $P$ satisfying the *ccc*, for every collection $\mathcal{D}$ of dense subsets of $P$ such that $|\mathcal{D}| \leq \aleph$, there is a filter on $P$ meeting all the elements of $\mathcal{D}$.

Assertion $\mathsf{MA}(\aleph_0)$ is a theorem in $\mathsf{ZFC}$ (the Rasiowa–Sikorski lemma), while $\mathsf{MA}(\mathfrak{c})$ is false. Martin's axiom $\mathsf{MA}$ is the statement that $\mathsf{MA}(\aleph)$ holds for all $\aleph < \mathfrak{c}$. As Levy mentions [321, p. 280], $\mathsf{MA}$ is a formal consequence of $\mathsf{CH}$ that, instead of necessarily denying the existence of cardinals between $\aleph_0$ and $\mathfrak{c}$, asserts that, if they exist, they behave like $\aleph_0$; moreover, it adds no information about the value of $\mathfrak{c}$, and it is therefore consistent with any reasonable specification of which value for $\mathfrak{c}$ one assumes. We only need Martin's axiom at the first level, namely $\mathsf{MA}(\aleph_1)$:

**Lemma 8.7.12** [$\mathsf{MA}(\aleph_1)$] *Let $\mathcal{M}$ be an almost disjoint family of subsets of $\mathbb{N}$ of size $\aleph_1$. The Boolean algebra $\mathfrak{A}_{\mathcal{M}}$ generated by the sets in $\mathcal{M}$ and the finite sets has the* $\mathfrak{a.p.}$

The proof is by no means simple: Marciszewski and Plebanek introduce [354, Definition 4.6] the complicated notion of the local extension property of order $r$ (LEP($r$)) for Boolean algebras, which turns out to be the fulcrum on which the lever of $\mathsf{MA}$ is placed.

And with this we arrive at the awful truth: the space that Marciszewski and Plebanek show as impossible to twist against $c_0$ is the same space of continuous functions on $\triangle_{\mathcal{M}}$ we know so well from 2.2.10 and which was the first one we twisted against $c_0$ in Lemma 8.7.5 to answer Avilés question: the Johnson–Lindenstrauss space $\mathsf{JL}_\infty$ of Diagram (2.37) – the reason is that the Stone compact corresponding to $\mathfrak{A}_{\mathcal{M}}$ is $\triangle_{\mathcal{M}}$. We have:

**Theorem 8.7.13** *Given an almost disjoint family $\mathcal{M}$ of size $\aleph_1$,*

- [$\mathsf{CH}$] $\operatorname{Ext}(C(\triangle_{\mathcal{M}}), c_0) \neq 0$.
- [$\mathsf{MA}(\aleph_1)$] $\operatorname{Ext}(C(\triangle_{\mathcal{M}}), c_0) = 0$.

Funny, isn't it? Thus, the CCKY problem cannot be solved in ZFC! And more nails have been hammered into the CCKY problem's coffin: [25; 134; 135; 136; 137; 354]. So, all's well that ends well? Well, Avilés, Marciszewski and Plebanek decided this was the ideal place to be an apple tree [25]:

**Theorem 8.7.14** [CH] *If $K$ is a non-metrisable compact,* $\operatorname{Ext}(C(K), c_0) \neq 0$.

The proof contains a stab at the heart of [93] in the form of an incredibly clever counting lesson plus some general results of independent interest:

**Proposition 8.7.15** *If $X$ is a Banach space of density $\aleph_1$ such that $|X^*| < 2^{\aleph_1}$ then* $\operatorname{Ext}(X, c_0) \neq 0$.

This result depends only on the following general estimate [25, Lemma 4]: *if $K$ is a compact space of weight $\aleph_1$ then $|C(\mathbb{N}^*, K)| \geq 2^{\aleph_1}$*. The proof then goes smoothly once the implication ✓ is established in the following chain:

$$\dim(X) \leq \aleph_1 \Rightarrow \operatorname{weight}(B_X^*) \leq \aleph_1 \Rightarrow |C(\mathbb{N}^*, B_X^*)| \geq 2^{\aleph_1} \overset{\checkmark}{\Rightarrow} \operatorname{Ext}(X, c_0) \neq 0.$$

This implication is simultaneously impossible to figure out and easy to see in light of the following definition:

**Definition 8.7.16** Let $K$ be a compact subspace of another topological space $H$. We say that a countable discrete extension $L$ of $K$ can be realised in $H$ if there is a homeomorphic embedding $L \longrightarrow H$ making a commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & L \\ & \searrow \quad \swarrow & \\ & H & \end{array}$$

Avilés, Marciszewski and Plebanek [25, Theorem 3.1 and Corollary 4.3] then show the following:

**8.7.17** *Let $X$ be an infinite-dimensional Banach space. The following are equivalent:*

(i) $\operatorname{Ext}(X, c_0) = 0$.
(ii) *Every countable discrete extension of $B_X^*$ can be realised in* $(X^*, \text{weak*})$.
(iii) *Every weak*-continuous function $\mathbb{N}^* \longrightarrow X^*$ extends to a weak*-continuous function $\beta\mathbb{N} \longrightarrow X^*$.*

It is then clear that implication ✓ holds: $\operatorname{Ext}(X, c_0) = 0$ is impossible since otherwise, using (iii),

$$2^{\aleph_1} \leq |C(\mathbb{N}^*, B_X^*)| \leq |X^*|^{\aleph_0} = |X^*| < 2^{\aleph_1}.$$

We continue to sketch the proof of Theorem 8.7.14. The first task is to settle the case in which $K$ has weight $\aleph_1 = \mathfrak{c}$. This requires the counting lesson mentioned above but also the pièce de résistance [25, Corollary 5.6]: [CH] *if* $\operatorname{Ext}(C(K), c_0) = 0$ *then there is a finite set* $F \subset K$ *such that for every closed subspace* $L \subset K$ *of weight* $\mathfrak{c}$ $L \setminus F$ *is locally metrisable*. Having the case $\mathfrak{c}$ in hand, the authors appeal to Juhász's theorem [239]: [CH] *if K has weight greater than* $\mathfrak{c}$ *then there is a compact subspace of K of weight* $\mathfrak{c}$. This, and a lot of know-how, will result in the final contradiction: if $\operatorname{Ext}(C(K), c_0) = 0$ then $K$ must be metrisable. Ok this is the end of the line (see Note 8.8.5 if it is not immediately clear why).

Or maybe not. Koszmider posed [298] five problems on the spaces $C(\triangle_{\mathcal{M}})$:

(1) Is there a separable Banach space $X$ not isomorphic to $c_0$ whose only non-trivial decompositions are of the form $c_0 \times X$?
(2) [ZFC] Are there almost disjoint families $\mathcal{M}$ such that every decomposition of $C(\triangle_{\mathcal{M}})$ into two factors has one separable factor?
(3) [MA] Is it true that if $|\mathcal{M}| = |\mathcal{N}| < \mathfrak{c}$, then $C(\triangle_{\mathcal{M}}) \simeq C(\triangle_{\mathcal{N}})$?
(4) [MA] Is it true that if $|\mathcal{M}| < \mathfrak{c}$ then $C(\triangle_{\mathcal{M}})$ is isomorphic to its square?
(5) [ZFC] Are there two almost disjoint families $\mathcal{M}$ and $\mathcal{N}$ of the same cardinality such that $C(\triangle_{\mathcal{M}})$ and $C(\triangle_{\mathcal{N}})$ are not isomorphic?

These problems are relevant to our discussion. Koszmider himself gives a partial solution to (2) by showing that under either CH or MA, there is a family $\mathcal{M}$, constructed ad hoc, such that if $C(\triangle_{\mathcal{M}}) = A \oplus B$ with $A$ and $B$ infinite-dimensional, then $A \simeq c_0$ and $B \simeq C(\triangle_{\mathcal{M}})$, or vice versa. Argyros and Raifkotsalis [20] showed the existence of separable spaces $\mathsf{AR}(p)$ for $1 \leq p < \infty$ such that if $\mathsf{AR}(p) = A \oplus B$ then $A \simeq \ell_p$ and $B \simeq \mathsf{AR}(p)$ (or vice versa). Problem (5) is about how many different $C(\triangle_{\mathcal{M}})$ spaces exist: it was solved by Marciszewski and Pol [355], who showed that there exist $2^{\mathfrak{c}}$ non-isomorphic spaces with $|\mathcal{M}| = \mathfrak{c}$; we already used this during the proof of Proposition 8.7.4. The proof in [355], or at least a large share of its three lines, could well be considered implicit. An explicit proof has been produced in [74]. Question (3) is the same question (5), but under MA plus $|\mathcal{M}| < \mathfrak{c}$. Koszmider's questions have surprising answers:

**Proposition 8.7.18**

(a) *There exist* $2^{\mathfrak{c}}$ *almost disjoint families* $\mathcal{M}$ *of size* $\mathfrak{c}$ *such that the Banach spaces* $C(\triangle_{\mathcal{M}})$ *are pairwise non-isomorphic.*
(b) *Under* MA($\aleph$)*, all the spaces* $C(\triangle_{\mathcal{M}})$ *with* $|\mathcal{M}| = \aleph$ *are isomorphic and isomorphic to their squares.*

*Proof* Part (a) is the result of Marciszewski and Pol [355] already mentioned. Part (b) is a formal consequence of Theorem 8.7.13: Let $\mathcal{M}, \mathcal{N}$ be almost disjoint families of size $\aleph < \mathfrak{c}$. Since $\operatorname{Ext}(C(\triangle_{\mathcal{M}}), c_0) = 0$ and $\operatorname{Ext}(C(\triangle_{\mathcal{N}}), c_0) = 0$, the two exact sequences in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & C(\triangle_{\mathcal{M}}) & \longrightarrow & c_0(\aleph_1) & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & C(\triangle_{\mathcal{N}}) & \longrightarrow & c_0(\aleph_1) & \longrightarrow & 0 \end{array}$$

are semi-equivalent; thus, the diagonal principle in Theorem 2.11.6 yields

$$C(\triangle_{\mathcal{M}}) \simeq c_0 \times C(\triangle_{\mathcal{M}}) \simeq c_0 \times C(\triangle_{\mathcal{N}}) \simeq C(\triangle_{\mathcal{N}})$$

since it is plain that $C(\triangle_{\mathcal{M}}) \simeq c_0 \times C(\triangle_{\mathcal{M}})$. The last assertion is clear since for every $\aleph_0 \leq \aleph \leq \mathfrak{c}$, there exist families $\mathcal{M}$ of size $\aleph$ such that $C(\triangle_{\mathcal{M}})$ is isomorphic to its square. □

We can polish the idea behind the argument above as follows:

**Lemma 8.7.19** *Given two exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & Z & \longrightarrow & c_0(\aleph) & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & Z' & \longrightarrow & c_0(\aleph) & \longrightarrow & 0 \end{array}$$

*if* $\operatorname{Ext}(Z, c_0) = 0$ *and* $\operatorname{Ext}(Z', c_0) = 0$ *then* $Z \simeq Z'$.

*Proof* Working as above, one gets $Z \times c_0 \simeq Z' \times c_0$, and we just need to check that both $Z, Z'$ contain $c_0$ complemented, which is obvious since both quotient maps are invertible on every separable subspace of $c_0(\aleph)$. □

## 8.8 Notes and Remarks

### 8.8.1 Homogeneous Zippin Selectors

One might wonder which additional properties a Zippin selector could enjoy and also which classes of Banach spaces could play the role of $\mathscr{C}$-spaces throughout this chapter. Both questions are somehow connected since there are simple correspondences between certain types of Lindenstrauss spaces and certain types of Zippin selectors.

Consider homogeneous Zippin selectors (i.e. $\omega(\lambda y^*) = \lambda(y^*)$ for $|\lambda| \leq 1$) and positively homogeneous Zippin selectors (only for $0 \leq \lambda \leq 1$). They correspond to two well-known types of Lindenstrauss spaces. A $\mathscr{G}$-space is a Banach space $X$ for which there exists a compact space $K$ and a set of

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triples $\{k_\alpha^1, k_\alpha^2, \lambda_\alpha\}_{\alpha\in A}$ with $k_\alpha^1, k_\alpha^2 \in K$ and $\lambda_\alpha \in \mathbb{K}$ such that $X = \{f \in C(K)\colon f(k_\alpha^1) = \lambda_\alpha f(k_\alpha^2) \quad \forall\alpha \in A\}$. An $\mathscr{M}$-space is a $\mathscr{G}$-space with $\lambda_\alpha \geq 0$ for all $\alpha$, equivalently, a sublattice of a $\mathscr{C}$-space.

**Lemma** *An embedding is $\mathscr{G}$-trivial (resp. $\mathscr{M}$-trivial) if and only if it admits a homogeneous (resp. positively homogeneous) Zippin selector.*

*Proof* We prove the case of $\mathscr{G}$-spaces, and the other is similar. Form the $\mathscr{G}$-space $G(B_Y^*) = \{f \in C(B_Y^*)\colon f(\lambda y^*) = \lambda f(y^*),\ \forall|\lambda| \leq 1,\ \forall y^* \in B_Y^*\}$ and observe that the natural embedding $\delta\colon Y \longrightarrow G(B_Y^*)$ has the universal property that every $\mathscr{G}$-valued operator on $Y$ factors through $\delta$. Now, let $\jmath\colon Y \longrightarrow X$ be an embedding. There is a correspondence between homogeneous Zippin selectors for $\jmath$ and extensions $D$ of $\delta$ through $\jmath$ as in the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad \jmath \quad} & X \\ & {\scriptstyle\delta}\searrow \quad \swarrow{\scriptstyle D} & \\ & G(B_Y^*) & \end{array} \tag{8.40}$$

given by $D(x)(y^*) = \langle\omega(y^*), x\rangle$. The universal property of $\delta$ yields, then, the extension of any $\mathscr{G}$-valued operator defined on $Y$. □

The following result was instrumental in the proofs of Propositions 8.5.1 and 8.5.7.

**Proposition** *Let $Y$ be a separable Banach space and let $\jmath\colon Y \longrightarrow X$ be an embedding. If $\jmath$ admits a Zippin selector then it admits a homogeneous Zippin selector.*

*Proof* Everything rests on Benyamini's magical result [38] that separable $\mathscr{G}$-spaces are actually isomorphic to $\mathscr{C}$-spaces. However, to control the bound of the homogeneous selector, we need to go inside [38, Proof of the Theorem] where it is shown that if $G$ is a separable $\mathscr{G}$-space then there is a metric compactum $K$ and an operator $u\colon G \longrightarrow C(K)$ whose range is 1-complemented and such that for all $g \in G$, one has $\frac{1}{2}\|g\| \leq \|u(g)\| \leq \frac{3}{2}\|g\|$. Now, if $\jmath$ admits a $\lambda$-Zippin selector then $\jmath$ is $\mathscr{C}$-trivial, and there is a commutative diagram as (8.40) with $\|D\| \leq 3\lambda$ that yields a homogeneous $3\lambda$-Zippin selector. □

Therefore, most of the material of this chapter for $\mathscr{C}$-spaces could be adapted for $\mathscr{G}$-spaces just multiplying by 3 here and there. Does this deliver all answers with it? For that, you're going to need a bigger boat: Kalton shows in [274, Proposition 4.5] that, for $1 < p < \infty$, the canonical embedding $\delta\colon \ell_p \longrightarrow C(B_{\ell_p}^*)$, which obviously admits a 1-Zippin selector (and therefore a homogeneous 3-Zippin selector), does not admit a homogeneous $\lambda$-Zippin

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selector for any $\lambda < (1 + (q-1)q^{-p})^{1/q}$, where $q^{-1} + p^{-1} = 1$. Moreover, in the non-separable case, the existence of a Zippin selector does not guarantee at all the existence of a homogeneous one: in [39], Benyamini constructs a non-separable $\mathscr{M}$-space $M$ that is not complemented in any $\mathscr{C}$-space. The embedding $M \longrightarrow C(B_M^*)$ is $\mathscr{C}$-trivial but not $\mathscr{M}$-trivial and therefore cannot be $\mathscr{G}$-trivial. Moreover, as we already know, an embedding can be $\mathscr{C}$-trivial without being $\mathscr{L}_\infty$-trivial: in fact, the identity on the Gurariy space $\mathsf{G}$ cannot be extended through the canonical embedding $\mathsf{G} \longrightarrow C(B_{\mathsf{G}}^*)$ since $\mathsf{G}$ is not complemented in any $\mathscr{C}$-space [22, Theorem 3.34]. It is an open question whether there exists a global approach to the extension problem for arbitrary Lindenstrauss-valued operators.

### 8.8.2 Lindenstrauss-Valued Extension Results

We will show in Section 10.6 that $\mathscr{C}$-valued extension results do not automatically pass to $\mathscr{L}_\infty$-valued results, except the Johnson–Zippin theorem 8.6.2, of course. Let us focus here on results for operators with values in Lindenstrauss spaces. The proof of the Lindenstrauss–Pełczyński theorem presented in Theorem 8.2.2 is a slightly edited version of the original proof in [330]. In that paper, the authors suggest that a version for Lindenstrauss spaces should also hold and give some hints about how to proceed: by using a generalisation of Edwards' separation theorem for Lindenstrauss spaces [319, Theorem 2.1] instead of the naive Hahn–Tong insertion trick. The version of the separation theorem that best suits our needs seems to be the following due to Olsen; see [371, Theorem 4.1] or [316, Theorem 1 on p. 220]:

**Lemma** *A Banach space $\mathcal{L}$ is a Lindenstrauss space if and only if, for every lower semicontinuous, concave function $G\colon B_{\mathcal{L}}^* \longrightarrow \mathbb{R}$ such that $G(e^*) + G(-e^*) \geq 0$ for all $e^* \in B_{\mathcal{L}}^*$, there exist $\xi \in \mathcal{L}$ such that $\langle \xi, e^* \rangle \leq G(e^*)$ for every $e^* \in B_{\mathcal{L}}^*$.*

We are ready to prove:

**Theorem** *Every operator from a subspace of $c_0$ to a Lindenstrauss space has a $1^+$-extension to $c_0$.*

*Proof* Let $\mathcal{L}$ be a Lindenstrauss space. The proof treads in the footprints of that of Theorem 8.2.2, with $B_{\mathcal{L}}^*$ in the role of the underlying compact space. Lindenstrauss spaces are exactly the $\mathscr{L}_{\infty,1+}$-spaces, and so each separable subspace of $\mathcal{L}$ is contained into a separable Lindenstrauss subspace. Thus we may assume that $\mathcal{L}$ is separable and that $\|\tau\| = 1$. We prove that, for each $\lambda > 1$ and for each $x \in c_0 \setminus H$, $\tau$ can be extended to an operator on $H + [x]$ having norm at most $\lambda$. This amounts to showing that there exists $\xi \in \mathcal{L}$ such that $\|\xi - \tau y\| \le \lambda\|y - x\|$ for all $y \in H$. This $\xi$ has to satisfy

$$\langle e^*, \tau y\rangle - \lambda\|y - x\| \le \langle e^*, \xi\rangle \le \langle e^*, \tau y\rangle + \lambda\|y - x\| \tag{8.41}$$

for all $e^* \in B^*_{\mathcal{L}}$ and all $y \in H$. We define two functions on $B^*_{\mathcal{L}}$ as follows:

$$G(e^*) = \inf_{y\in H}\Big(\langle e^*, \tau y\rangle + \lambda\|y - x\|\Big),$$
$$F(e^*) = \sup_{y\in H}\Big(\langle e^*, \tau y\rangle - \lambda\|y - x\|\Big).$$

It is clear that $F(e^*) = -G(-e^*)$ and so $\xi$ satisfies (8.41) if and only if $\langle e^*, \xi\rangle \le G(e^*)$ for every $e^* \in B^*_{\mathcal{L}}$. Keeping an eye on Lemma 8.8.2, notice that $G$ is concave since it is a pointwise infimum of affine functions. Form its lower semicontinuous envelope $G_{\text{lsc}}$, which is also concave. Thus, to apply Olsen's lemma, we only have to see that $G_{\text{lsc}}(e^*) + G_{\text{lsc}}(-e^*) \ge 0$ for every $e^*$ in the ball of $\mathcal{L}^*$. Assuming, on the contrary, that there exists some $e^* \in B^*_{\mathcal{L}}$ such that $-G_{\text{lsc}}(-e^*) > G_{\text{lsc}}(e^*)$, then there also exist sequences $(s_n), (t_n)$ converging to $e^*$ in $B^*_{\mathcal{L}}$ such that $\lim_n F(s_n) > \lim_n G(t_n)$ and, therefore, there exist points $y_n, z_n \in H$ such that

$$\lim_n\Big(\langle s_n, \tau(y_n)\rangle - \lambda\|x - y_n\|\Big) > \lim_n\Big(\langle t_n, \tau(z_n)\rangle + \lambda\|x - z_n\|\Big).$$

Now, switch to the part of the proof of Theorem 8.2.2 starting at (8.2), with $e^*$ replacing $\delta_s$, until reaching a contradiction. □

The $\mathscr{C}$-extensibility of $\ell_1$ also passes without modification to Lindenstrauss-space extensibility:

**Proposition** *Every operator from $\ell_1$ to a Lindenstrauss space admits $1^+$ extensions to any separable superspace.*

*Proof* Let $\mathcal{L}$ be a Lindenstrauss space, and let $\tau\colon \ell_1 \longrightarrow \mathcal{L}$ be an operator. We can assume that $\mathcal{L}$ is separable by the same argument as in the previous proof. Johnson and Zippin [234] proved that there is an isometric quotient map $Q\colon C(\Delta) \longrightarrow \mathcal{L}$. Pick a $1^+$-lifting $t$ of $\tau$ through $Q$ and then a $1^+$-extension $T$ of $t$. The operator $QT$ is a $1^+$-extension of $\tau$. □

### 8.8.3 The Last Stroke on the Extension of $\mathscr{C}$-Valued Lipschitz Maps

As we have already mentioned, the material contained in Section 8.3 originates in Kalton's studies about the extension of $\mathscr{C}$-valued Lipschitz maps, which we have avoided in its full generality as much as possible thus far. Oh well, in for a penny, in for a pound: it is time to explain why the non-linear

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context yields even better results. Suppose we are given a metric space $X$, not necessarily normed, and a Lipschitz map $\tau\colon Y \longrightarrow C(K)$, where $Y$ is a subset of $X$ and $K$ is a metrisable compact space. If we want to extend $\tau$ to $X$ to one more point $x \in X \setminus Y$, we typically generate two bounded functions $g, h\colon K \longrightarrow \mathbb{R}$, depending on $x$, in such a way that $g$ is upper semicontinuous, $h$ is lower semicontinuous and $g \leq h$ in order to then use the Hahn–Tong sandwich theorem to insert a continuous $f$ between $g$ and $h$. We choose this $f$ as the value of the extension of $\tau$ at $x$. If, moreover, we could almost preserve the Lipschitz constant of the extension then we could proceed through an enumeration of a dense subset of $X \setminus Y$ to get a global extension to $X$, as in Theorem 8.3.10. Otherwise, when no almost-isometric preservation of the Lipschitz constant can be done, one instead needs to control the Lipschitz constant of the extension by means of the distances between the semicontinuous functions entering into the Hahn–Tong theorem. All this is treated by Kalton in a rather unexpected way. Before proceeding, let us make our lives easier by recalling that, in our current separable setting, all problems about extensions of $\mathscr{C}$-valued maps can be reduced to the case where the underlying compactum is the Cantor set, as Lemma 1.6.2 clearly explains. Now, consider SUB($\Delta$), the subset of those pairs $(g, h) \in \ell_\infty(\Delta) \times \ell_\infty(\Delta)$ such that $g$ is upper semicontinuous, $h$ is lower semicontinuous and $g \leq h$. We measure distances in SUB($\Delta$) just using the restriction of the norm:

$$d((g_1, h_1); (g_2, h_2)) = \max\left(\|g_1 - g_2\|_\infty, \|h_1 - h_2\|_\infty\right).$$

We have (we skip the proof):

**Proposition** *There is a contraction $\theta\colon$ SUB($\Delta$) $\longrightarrow C(\Delta)$ such that, for every $(g, h) \in$ SUB($\Delta$), we have $g \leq \theta(g, h) \leq h$.*

The task now at hand is to give a metric characterisation of those pairs $(Y, X)$ for which every $\mathscr{C}$-valued contraction on $Y$ has a $\lambda$-Lipschitz extension to $X$ (for fixed $\lambda$!). It is clear from Lemma 8.3.3 that $(Y, X)$ satisfies condition $\Sigma_1(\lambda)$ if and only if every contraction $Y \longrightarrow c$ admits a $\lambda$-Lipschitz extension to one more point, no matter which point one chooses. It seems like magic that this suffices to get a global, coherent extension with the same bound, but it does:

**Theorem** *If $Y \subset X$ is a separable subset of a metric space and $\lambda \geq 1$, the following are equivalent:*

(i) *Every $\mathscr{C}$-valued Lipschitz map on $Y$ admits a $\lambda$-extension to $X$.*
(ii) *The pair $(Y, X)$ verifies condition $\Sigma_1(\lambda)$.*

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*Proof* The implication (i) $\Longrightarrow$ (ii) is contained in Lemma 8.3.3. To prove the converse, it suffices to consider the case in which the target space is $C(\Delta)$. So, let $\tau\colon Y \longrightarrow C(\Delta)$ be a contraction. We define two $\lambda$-Lipschitz maps $\tau^-, \tau^+\colon X \longrightarrow \ell_\infty(\Delta)$ by means of

$$\tau^-(x) = \bigvee_{y\in Y} (\tau(y) - \lambda d(y,x)1_\Delta),$$
$$\tau^+(x) = \bigwedge_{z\in Y} (\tau(z) + \lambda d(z,x)1_\Delta),$$

where the order refers to $\ell_\infty(\Delta)$, a complete lattice. Clearly, $\tau^-(x) \le \tau^+(x)$ for every $x \in X$, and $\tau^-(y) = \tau^+(y) = \tau(y)$ for $y \in Y$. Now, with an eye on Proposition 8.8.3, we define $G, H\colon X \longrightarrow \ell_\infty(\Delta)$ by $G(x) = \tau^-(x)^{\text{usc}}$ and $H(x) = \tau^+(x)_{\text{lsc}}$. It is easy to check that these are again $\lambda$-Lipschitz and that $G(y) = H(y) = \tau(y)$ for $y \in Y$. The core of the argument is contained in the following:

**Claim** For every $x \in X \setminus Y$, one has $G(x) \le H(x)$.

*Proof of the claim* Following the proof of Lemma 8.3.3, assume that there are $x \in X \setminus Y$, $s \in \Delta$ and $\varepsilon > 0$ such that $G(x)(s) > H(x)(s) + 2\varepsilon$. Then there are sequences $(s_n), (t_n)$ in $\Delta$, both converging to $s$, such that $\tau^-(x)(s_n) > \tau^+(x)(t_n) + 2\varepsilon$. For each $n$, we may select $y_n$ and $z_n$ in $Y$ such that $\tau(y_n)(s_n) - \lambda d(y_n, x) > \tau(z_n)(t_n) + \lambda d(z_n, x) + 2\varepsilon$. Applying condition $\Sigma_1(\lambda)$ to the sequences $(y_n), (z_n)$ and $\varepsilon$, we get $u \in Y$ such that $d(u, y_n) + d(u, z_n) \le \lambda\big(d(x, y_n) + d(x, z_n)\big) + \varepsilon$ for infinitely many $n$s. Let us set $\eta = \tau(u)$ and see what happens. We have $\tau(y_n) \le \eta + d(u, y_n)$ and $\tau(z_n) \ge \eta - d(u, z_n)$ and, in particular,

$$\tau(y_n)(s_n) \le \eta(s_n) + d(u, y_n) \qquad \text{and} \qquad \tau(z_n)(t_n) \ge \eta(t_n) - d(u, z_n).$$

Combining these, we obtain that for infinitely many $n$,

$$\begin{aligned}\eta(s_n) - \eta(t_n) &\ge \tau(y_n)(s_n) - d(u, y_n) - \tau(z_n)(t_n) - d(u, z_n)\\ &> \lambda\big(d(x, y_n) + d(x, z_n)\big) + 2\varepsilon - d(u, y_n) - d(u, z_n) \ge \varepsilon,\end{aligned}$$

which contradicts the continuity of $\eta$ at $s$. □

Thus, the $\lambda$-Lipschitz map $x \in X \longmapsto (G(x), H(x)) \in \ell_\infty(\Delta) \times \ell_\infty(\Delta)$ actually takes values in $\text{SUB}(\Delta)$. Composing with the contraction $\theta\colon \text{SUB}(\Delta) \longrightarrow C(\Delta)$ provided by Proposition 8.8.3, we obtain the required $\lambda$-Lipschitz extension of $\tau$, concluding the proof. □

This provides a complete characterisation of the separable subsets $Y$ of a metric space $X$ for which all $\mathscr{C}$-valued Lipschitz maps admit $\lambda$-extensions. Of

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course, it includes the case in which $X$ is a Banach space. Compare to Theorem 8.3.10. A version of Lindenstrauss' classic comes as a bonus:

**Corollary** *Every separable $\mathscr{C}$-space is an absolute 3-Lipschitz retract.*

### 8.8.4 Property $(M)$ and $M$-Ideals

Properties $(L)$ and $(L^*)$ are the ugly mates of properties $(M)$ and $(M^*)$ introduced by Kalton in his study of $M$-ideals [264]. A Banach space $X$ is said to have property $(M)$ if every weakly null type on $X$ is a function of the norm, and $X$ is said to have property $(M^*)$ if every weak*-null type on $X^*$ is a function of the norm. It is clear that $c_0$ and the spaces $\ell_p$ for $1 < p < \infty$ have properties $(M)$ and $(M^*)$. The Yellow Book [209] incorporates most of the discoveries of [264], although not the ultimate connections between properties $(M^*)$, $(M)$ and $M$-ideals of compact operators, which came later [370; 287; 322]. Namely

- A Banach space $X$ has property $(M^*)$ if and only if it has the metric compact approximation property (obvious meaning) and $\mathfrak{K}(X)$ is an $M$-ideal in $\mathfrak{L}(X)$ – see [264, Theorem 2.4] and [287] for separable $X$ and [370; 322] for the general case.
- Property $(M^*)$ implies $(M)$ for separable spaces [264, Proposition 2.5], and Property $(M)$ implies $(M^*)$ for separable spaces containing no copy of $\ell_1$ [287, Theorem 2.6].

The paper [287] contains a bunch of examples (and counterexamples) concerning these properties too. Regarding the topic of this chapter, separable spaces having property $(M^*)$ have the almost isometric $\mathscr{C}$-extension property [273, Theorem 7.5]. One could actually develop Section 8.4 focusing on properties $(M)$ and $(M^*)$ instead of $(L)$ and $(L^*)$. In some sense, this is what Kalton did in [264, Section 4] and [273, Section 3]. We decided to present the results in their $L$-version only because the $M$-version of Proposition 8.4.2 (which is a particular case of [273, Proposition 3.4]) is much harder to prove.

### 8.8.5 Set Theoretic Axioms and Twisted Sum Affairs

Cardinal axiomatics have made an essential irruption in homological affairs. We list the places where they played or will play a role, if not for better, at least for good:

- The CCKY-problem has different solutions under $\mathsf{CH}$ or $\mathsf{MA}$.
- Under $\mathsf{CH}$, there are $2^{\aleph_1}$ non-isomorphic spaces $C(\triangle_{\mathcal{M}})$ for $|\mathcal{M}| = \aleph_1$. All of them are isomorphic under $\mathsf{MA}$.

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- Under $\mathsf{CH}$, every 1-separably injective Banach space contains $\ell_\infty$, while under $\mathsf{MA} + \aleph_2 = \mathfrak{c}$, there exists a 1-separably injective $\mathscr{C}$-space that does not contain $\ell_\infty$ [24] .
- Under $\mathsf{CH}$, universal separable injectivity is not a 3-space property (Theorem 10.5.6 (c)).
- Under $\mathsf{CH}$, there exist non-trivial sequences $0 \longrightarrow C(\mathbb{N}^*) \longrightarrow C(\mathbb{N}^*) \longrightarrow C(\mathbb{N}^*) \longrightarrow 0$ (Theorem 10.5.6 (b)). In particular, $\operatorname{Ext}(C(\mathbb{N}^*), C(\mathbb{N}^*)) \neq 0$ [23]. It is apparently unknown whether the same holds in $\mathsf{ZFC}$.
- Under $\mathsf{CH}$, there is just one space of separable universal disposition and dimension $\mathfrak{c}$, the Kubiś space $\mathsf{F}_1$. Under different axiomatics (see after Corollary 7.3.5), there is a continuum of mutually non-isomorphic spaces of that type.

## Sources

The material in this chapter is classical, old and new, all at once. It is classical because some parts can be already found in Semadeni's book [430]; it is as old as the 1970s [330] and as new as 2007 because a large part of it develops Kalton's extraordinary series of four papers [271; 272; 273; 274]. Weak*-continuous selectors are among the extremely nice contributions of Zippin [463; 464; 465], who coined the term *global approach*. The four 'elementary' examples in Proposition 8.1.3 have different origins. Example (a), embeddings into $\ell_p$, is from Zippin, although the simple proof we present was the idea of Yost and appeared in [76]. Example (b), the embedding $\ell_p \longrightarrow \ell_p(\varphi)$, is a particular case of the $\mathscr{C}$-extensibility of the spaces $\ell_p(\varphi)$, although the argument presented is original. Example (c) and its proof are a reworking of some parts of Kalman's paper [245]; however, the use of triangularisations to prove Kalman's theorem was suggested to us by Francisco Santos. The Lindenstrauss–Pełczyński theorem is one of the beautiful contributions in [330]. As mentioned in the text, Zippin [464; 465] produced his own proof; the homologically flavoured proof in Theorem 8.2.1 is from [76], and the Lindenstrauss-valued extension à la Lindenstrauss–Pełczyński in Section 8.8.2 is from [129]. The result remained isolated until Johnson and Zippin [236] obtained an extension to subspaces of $c_0(I)$ and later in [237] to weak*-closed subspaces of $\ell_1$, which is the Johnson–Zippin theorem. It involves $\mathscr{L}_\infty$-valued operators and is, in fact, a characterisation of them [118, Proposition 3.1]: *$X$ is an $\mathscr{L}_\infty$-space if and only if every $X$-valued operator defined on a weak*-closed subspace of $\ell_1$ can be extended to $\ell_1$*. The Ext-version of this result is Proposition 5.2.9 (b). The paper [76] contains the first example showing that $\mathscr{L}_\infty$-spaces cannot replace $\mathscr{C}$-spaces in the Lindenstrauss–Pełczyński theorem

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and answers Zippin's problem 6.15 in [466]. The existence of such 'rare' $\mathscr{L}_\infty$-spaces sparked the theory of Lindenstrauss–Pełczyński spaces, to be developed in Section 10.6. The nice unexpected example in 8.2.3 has been taken from [236]. The concept of a type on a Banach space was introduced by Krivine and Maurey [307]. That $\ell_2$ is not extensible and all that comes with it was Kalton's response (see [271, Section 4]) to the final comments in [109]. In this paper, the connection between the $\mathscr{C}$-extensibility property of $X$ and the $X$-automorphic character of $C[0,1]$ was established (Proposition 7.4.15), something that Kalton reformulates as [271, Theorem 4.1] or else as [274, Proposition 2.5]. The connection between the two properties was established while studying [237, Problem 4.2]: *if E is a reflexive subspace of X, does every $\mathscr{C}$-valued operator on E extend to X?* After that initial impetus, Kalton's imagination ran free. He considered the question of when $\operatorname{Ext}(X, C[0,1]) = 0$ in [267], obtaining Proposition 8.6.4 and much more, and then again [113] following a different approach. The problem was revisited in [73] considering also the case $C(\omega^\omega)$. The reader is warned that the reading of [73] is difficult at some points, and quite difficult for the rest of the time; it is advisable to have [191] at hand. In [73], the parameter $\alpha_N$ defined in (8.34) is rather associated with $X^*$, or with some of its subspaces, and the role of $X$ is to define the weak* topology in $X^*$. In the text, we have treated $\alpha_N$ as a constant associated with $X$ that is computed on its dual. Very recently, Causey, Fovelle and Lancien have shown that having a summable Szlenk index is a 3-space property [131]. Zippin's problem 8.6.1 still remains open. All adventurous episodes of the astounding story have been told during Section 8.7. But there remain outrageous possibilities: Proposition 8.7.2 has been generalised in [25, Proposition 8.2] to:

**Proposition** *Let X be a Banach space and let $c_n, d_n > 0$ be two sequences such that $\lim n\, d_n\, c_n^{-1} = \infty$. Suppose that, for every n, there exist $\Phi_1, \dots, \Phi_n \subset B_X^*$ and $\Psi_1, \dots, \Psi_n \subset B_X$ such that*

- $\left\|\sum_{i=1}^n x_i\right\| \le c_n$ *for any choice of vectors* $x_i \in \Psi_i$;
- *for every i and* $x^* \in \Phi_i$, *there is* $x \in \Psi_i$ *such that* $x^*(x) > d_n$;
- *the sets* $\Phi_F$ *are pairwise disjoint and* $|\Phi_i| \ge \mathfrak{c}$;
- $\Phi_1 \cup \cdots \cup \Phi_n$ *is discrete, and its weak*-closure is its one-point compactification.*

*Then there is a non-trivial twisted sum of $c_0$ and X.*

What the proposition says, in its somewhat technical formulation, is that biorthogonality is not essential in Proposition 8.7.2. The assumption CH on Proposition 8.7.3 can be relaxed to MA; see [25, Corollary 4.2]. The problem of whether the assertion holds in ZFC is, however, open. What the whole story teaches us is that it is by now clear that ‘there are nonmetrisable compacta $K$ for which the question of whether $\mathrm{Ext}(C(K), c_0) \neq 0$ is undecidable within the usual axioms of set theory’ [25]. Which compacta those are and for which ones there is a plain answer in ZFC is an entire world of research. Many more results about good-natured compacta not mentioned in the astounding tale can be found in [25; 134; 135; 136; 137; 354]. A few additional steps towards 8.7.6 are worth mentioning: Correa [135] proves it under MA and, in private communications long before [25] was available, she informed us that the result was true in ZFC for metrisable height 3 compacta. Property LEP($r$) has been reconsidered in [137] for finite height compacta. The results in Sections 8.8.1 and 8.8.2 are from [129], and those in Section 8.8.3 are from [272] and [273].