6
Equivariant Borel Conjecture

6.1 Motivation

Remember our founding myth, whereby we pretended that Borel obtained his topological rigidity conjecture starting from Mostow rigidity:

Theorem 6.1 *Suppose that M and M' are closed irreducible Riemannian manifolds covered by G/K for some semisimple Lie group G that is not* $\mathrm{PSL}^2(\mathbb{R})$*. Then, any isomorphism $\phi\colon \pi_1 M \to \pi_1 M'$ is induced by a unique isometry $\phi\colon M \to M'$.*

We have concentrated so far on the existence of this isometry and its topological analogues, but now let's consider the implications of uniqueness.

It is worth noting that the uniqueness is not replaced by "a contractible space of choices" even in the case that M is locally symmetric but not semisimple. For instance, when $M \cong M'$ are isometric flat tori, ϕ is an arbitrary translational isometry. Thus the space of equivalences *is* a torus. This actually is more reasonable, because as Borel had noted:

Proposition 6.2 *If M is an aspherical complex, then the identity component of* $\mathrm{Aut}(M)$ *(the space of self-homotopy equivalences of M) is aspherical with abelian fundamental group $\cong Z(\pi)$ (the center of the fundamental group).*

This has two consequences: (1) rigidity will be somewhat stronger in situations that avoid center (or even normal abelian subgroups); and (2) one should not, in any case, want more topological rigidity than occurs homotopy-theoretically.

Alas, we have seen that the most obvious topological variant, the contractibility of the space of homeomorphisms, say, if the fundamental group is centerless, is unfortunately rarely true in high dimensions. On the other hand, we have also seen that the "cubist variant" of contractibility – namely, uniqueness up to

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pseudo-isotopy (and higher "block" analogues of this statement) – are well-founded conjectures (e.g. are consequences of the Borel conjecture itself, albeit for other groups).

Thus, we concentrate on problems and statements that are visible at the level of individual manifolds, rather than families – the essential difference between the block and fibered worlds. This is critical. I can't overemphasize this difference between the smooth category and the PL and topological categories.

In the smooth category, understanding *objects* more complicated than manifolds requires some understanding of families. For example, submanifolds have tubular neighborhoods that are unstable vector bundles. Maps are often thought of as "singular fibrations" like Lefshetz pencils, and one considers "Whitney stratified spaces" and so on.

In the PL situation, it is more natural to break things up over simplices in the base (i.e. not over individual points). Over vertices, one has a fiber F, and over edges, one has something isomorphic to $F \times \mathrm{I} = [0, 1]$, but not with any particular projection map to I, and more generally over a simplex Δ one has a space isomorphic to $\Delta \times F$ (compatibly with the face relations of the simplex, but not with respect to anything going on over points – see Figure 6.1). It results in a "cubist" decomposition of the space. (See Figure 6.2 for an example of how a typically smooth object becomes polyhedral in a cubist perspective.) Analyzing such an object is never more complex than analyzing a manifold with boundary – since those are all that occur inductively. Spaces of such block bundles are effectively understood using blocked surgery.

We will be most interested in the topological category, where there are obstructions in algebraic K-theory to this structure. What exists is an even more smeared out structure, where *no point or edge* (or lower-dimensional sub-object) is given a particular pre-image. This is the content of the "teardrop neighborhood theorem" (Hughes *et al.*, 2000) and we will discuss it Chapter 7. As in Chapter 5, we will start by ignoring the constraints of K-theory to form intuitions, and then following it up by discussing the inevitable changes that K-theory necessitates.

To return to our story, one discovers, as a consequence of Mostow rigidity that:

Corollary 6.3 *If $M = K\backslash G/\Gamma$ with G semisimple, M irreducible and not a hyperbolic surface, then* $\mathrm{Out}(\Gamma)$*, the outer automorphism group of* Γ*, is finite and isomorphic to the isometry group of M. Furthermore, the action is unique up to conjugacy by isometry.*

In the excluded case of surfaces, one does not have the finiteness, but one still has the theorem (the "Nielsen realization conjecture") of Kerckhoff, and proved

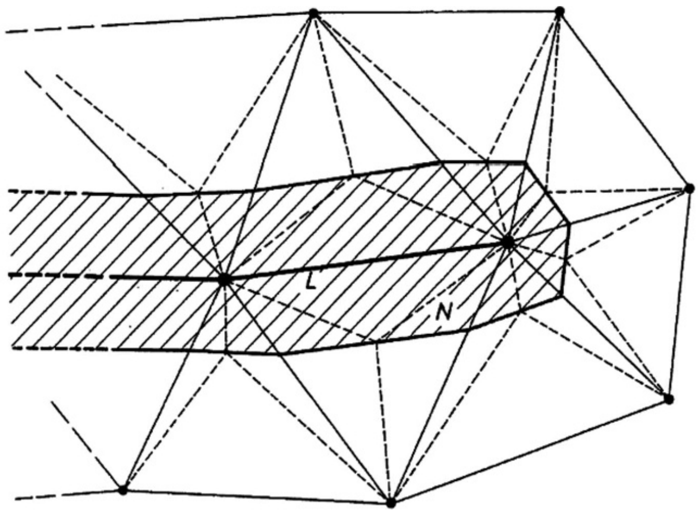

Figure 6.1 Regular neighborhood of a PL-manifold with boundary is an example of a block bundle – note the absence of fibers over many points of the submanifold. (Reproduced from Rourke and Sanderson (1982) with permission of Springer.)

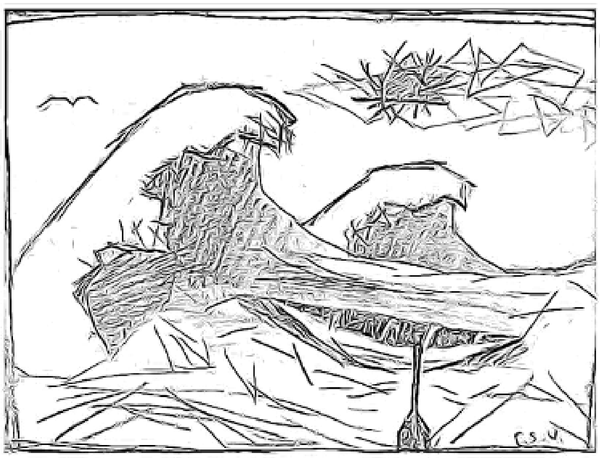

Figure 6.2 PL wave. (Courtesy Esther Segal-Weinberger.)

several times since,[1] that finite subgroups of Out(Γ) act on M. A consequence of the proof, by hyperbolic geometry, is that the action is unique up to topological

[1] See §6.11 at the end of the chapter.

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conjugacy – from the moduli space point of view, these actions are the fixed set of an action on Teichmüller space, and this fixed point set is contractible.

Thus, our interest in this and the next chapters is called to:

Problem 6.4 (Nielsen realization problem) If M is an aspherical manifold with centerless fundamental group, can one realize finite subgroups of $\mathrm{Out}(\pi_1 M)$ by group actions?

Problem 6.5 If G acts on M and M' is a compact group action on an aspherical manifold, and $f: M' \to M$ is an equivariant homotopy equivalence, then is f equivariantly homotopic to an equivariant homeomorphism?

We shall see that the answer to both problems is negative, but we shall also see that there are many interesting problems raised by their study. We shall study Problem 6.5 first and then return to Problem 6.4 in Chapter 7.[2]

Remarkably enough, the Novikov version of this conjecture does not yet have any known counterexamples despite the counterexamples to the rigidity statements. (Moreover, these Novikov statements have additional interesting applications to closed manifolds, even without group actions.)

6.2 Trifles

This section is devoted to several examples of group actions that show different kinds of phenomena that are present for different kinds of actions. We will ultimately focus on the topological category, and, for the sake of rigidity, carefully confine our attention to the type of group actions we allow: by the end of this section we will see that, if we are not somewhat picky, many lattices have infinitely many (or even uncountably many) co-compact properly discontinuous (C^0) actions on Euclidean space.

The picture of a smooth (compact) group action is kind of simple: the manifold decomposes according to orbit types. Each orbit type defines a stratum. They are essentially principal bundles over their quotient spaces, which are manifolds. These strata have neighborhoods that are equivariant vector bundles. They are put together in a reasonably comprehensible way.

This is proved using the elementary Riemannian geometry of any invariant metric (and such a metric can be obtained by averaging any particular metric over the group) and playing around with the exponential map (see Bredon, 1972). It has many straightforward consequences: e.g., the quotient of the set

[2] Actually, these two questions aren't the usual two sides of a coin that we usually look for in existence and uniqueness: the actions demanded in Nielsen would not – in a relative version – be enough to give us an "equivariant h-cobordism."

of free points for a group action on a compact manifold is a (noncompact, if the action is not free) manifold that has a canonical compactification as a manifold with boundary. The boundary is essentially the set of points of distance ε from the singular points for some ε smaller than the normal injectivity radius of the fixed set.

Equivariant vector bundles are studied via detailed understanding of the topology of various Lie groups. Unfortunately, this does not reduce to K-theory because the bundles involved will be unstable, even if the codimension(s) of the fixed set (and other strata) is(are) very high. For example, if F is the fixed set, the equivariant normal bundle decomposes canonically into bundles associated to the irreducible representations of G. Obviously, the sub-bundle corresponding to the trivial representation is trivial, but some other representation might occur with a very low multiplicity, forcing an unstable bundle as part of the data.

This picture is essentially correct in the PL situation, except that the fixed set need not be a manifold: if we insist that it *is*, by fiat, then the rest follows, except that, instead of vector bundles, there are block bundles. The proof of this is even simpler: one writes down formulas for the neighborhoods, just like in ordinary regular neighborhood theory (see Rourke and Sanderson, 1968a,b,c). As we mentioned in §6.1, the liberation of the "structure group" from a compact Lie group to the complicated space of "block automorphisms of the fiber" is actually a blessing when it comes to rigidity, which will become clearer as we proceed.

In any case, it still is the case that the quotient of the set of free points for a group action on a compact manifold is a (noncompact, if the action is not free) manifold that has a canonical compactification as a manifold with boundary.

In the topological case, this is not true.

Our first task is to give a bunch of examples of what group actions on some simple manifolds look like and how the different categories compare to each other. For instance, although there are only finitely many smooth structures on a compact topological manifold (except in dimension 4), this is not at all the case equivariantly. We will see very crude and also some subtle differences between these categories.

In the beginning there were linear actions. The orthogonal group acts on Euclidean space, preserving unit spheres. Every subgroup therefore acts linearly on the sphere, and the most obvious thing to do is to try to compare arbitrary actions to linear ones.

And, indeed, this works quite well[3] (smoothly and PL) in dimension ≤ 3.

Let us consider, as a starting place, $\mathbb{Z}_k$-actions and S^1-actions on the disk

[3] See §6.11.

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that are semi-free and equivariantly contractible. Semi-free means that there are only two possible isotropy groups – the whole group and the trivial group. Equivariantly contractible is equivalent to asserting that the map to a point is an equivariant homotopy equivalence, which is equivalent to asserting that, restricted to all fixed sets (including the trivial and whole group), the map is a homotopy equivalence, which, in our situation, just means that we are assuming *a priori* that the fixed set, henceforth denoted F, is contractible.

This *is* a nontrivial restriction. Smith theory would tell us that F is $\mathbb{Z}/p$-acyclic (or $\mathbb{Z}$-acyclic for the case of $\mathcal{S}^1$-actions). In a moment we will see that any $\mathbb{Z}$-acyclic manifold V^n is the fixed set of a smooth $\mathcal{S}^1$-action on a disk $\mathcal{D}^{n+2}$ unless $n = 3$.

But let's start with the slightly simpler case where we start with a manifold V (of dimension at least 5) that is contractible (so the action will be of the desired sort). In that case, $V \times [0, 1]$ is homeomorphic to the disk. (This is a straightforward application of the h-cobordism theorem.[4]) Thus *a fortiori* $V \times \mathcal{D}^i$ is a disk, possessing an action of $\mathrm{O}(i)$ on it with fixed set V. Restricting to a semi-free action on $\mathcal{D}^i$ gives us an appropriate semi-free action on $\mathcal{D}^{n+i}$ with fixed set V.

(Note that if we restrict this action to the boundary, we get action on S^{n+i-1} with fixed set ∂V; by a theorem of Kervaire (1969), the boundaries of contractible manifolds in dimension ≥ 4 are exactly the integral homology spheres in the PL and topological categories – in the smooth category, there is a unique differential structure on the sphere that one must connect sum with to get it to be a boundary. In short, *every homology sphere of dimension* $d > 3$ *is the fixed set of a semi-free* $\mathcal{S}^1$*-action on the sphere* $\mathcal{S}^{d+2}$.)

The question of which integral homology sphere – or mod p homology spheres – are fixed sets of *smooth* semi-free $\mathcal{S}^1$- or $\mathbb{Z}_p$-actions is more subtle and studied by Schultz in a remarkable series of papers[5] (see, e.g., Schultz, 1985, 1987).

Example 6.6 (A PL action whose fixed set is not a manifold) Take a homology sphere that bounds a contractible manifold. If we consider the action on the sphere with that as the fixed set, then we can cone (suspend) the action. It gives a PL-action on the disk (sphere) whose fixed set is a polyhedron with a (two) singular point(s) (but one can shrink an arc connecting them to a point, to get a new action on the sphere with one singular point). This action on the disk is equivariantly contractible.

Remark 6.7 Some information is given by Smith theory. In this semi-free

[4] It's a contractible manifold with simply connected boundary, which must be a disk.

[5] As well as what happens for dimension 3.

case, one knows that the fixed set, F, is a homology manifold (with coefficients in $\mathbb{Z}$ for the circle, and $\mathbb{Z}/|G|$ for a finite group G). In the case of $\mathcal{S}^1$-actions, the converse holds and any acyclic PL homology manifold is the fixed set of a semi-free PL-action.[6] The proof of this is an induction in the spirit of Cohen (1970).

Now let us address the uniqueness question. How many actions are there with a given fixed set? The following is a slight variant of an old theorem of Rothenberg and Sondow (1979).

Theorem 6.8 *If the codimension of* F $> \dim(G) + 2$*, then the smooth semi-free G-actions on a disk with the contractible fixed set* F *and given local representation (which we assume is a free representation of dimension equal to the codimension of* F*) are a one-to-one correspondence with* $\mathrm{Wh}(\pi_0(G))$.

(The normal representation is determined by differentiating the action of G at a fixed point.) Near the fixed set, bundle theory and the tubular neighborhood tell us that the action is a product action. Then the rest of the proof is an application of the h-cobordism theorem.

Remark 6.9 The condition on codimension is important, so that we can use homological methods to control the homotopy theory – in other words, we want to be able to conclude that complements are simply connected.

It is not very hard to construct "exotic" actions, even smoothly, with codimension-2 fixed-point sets, on the sphere or the disk once dim > 3. These are called "counterexamples to the Smith conjecture"(see Giffen, 1966). Here's a sketch of a construction in dimension 5 and higher based on the Poincaré conjecture,[7] and making use of a nontrivial knot K, so that π_1 of the knot complement is $\mathbb{Z}$.

Consider a free action of $\mathbb{Z}_p$ on the sphere with an invariant codimension-2 subsphere $\mathcal{S}$ (which might even be assumed unknotted for simplicity). It is easy enough to see that $\mathcal{S}\#K\#K\#K \cdots K$ (with p copies) is invariant under the action as well. Now do "surgery on this action," i.e. remove this invariant knot, and glue in $\mathcal{S} \times \mathcal{D}^2$ with the action that is trivial on the $\mathcal{S}$ direction and semi-free on the $\mathcal{D}^2$ direction. (The reader can check that this is possible.) This gives a $\mathbb{Z}_p$-action on the sphere (here we use the Poincaré conjecture) whose fixed set is connected of p copies of K, and thus is an example.[8]

[6] However, not necessarily in *all* even codimensions because of the contribution of Rochlin's theorem (as in Schultz, 1987), but in codimensions that are a multiple of 4, this is OK.

[7] Giffen worked in the smooth category, and gave some examples in dimension 4 because he was also able to avoid use of the Poincaré conjecture in that elegant paper.

[8] It is a theorem of Levine (1965) that a knot in a high-dimensional sphere is trivial iff its complement has the homotopy type of a circle; as a consequence, one can't unknot a knot by taking its connected sum with another knot.

In the PL category, we don't necessarily have a bundle structure. Nevertheless, for locally linear actions, the theorem is correct – because local linearity implies enough homogeneity to give ("simple") block structures, which are determined by maps into classifying spaces.

At this point there is a subtlety related to torsions (the "simple" in the previous paragraph) and therefore ultimately to decorations in L-theory (see §5.5) as we now explain:

Example 6.10 (PL neighborhoods) A basic fact about the smooth category is that the neighborhood theory near the fixed points is bundle theory, and therefore homotopical in nature: the germ neighborhoods of $\mathrm{F} \times [0, 1]$ are just those of the product of F with $[0, 1]$).

However, suppose F is a point, and we start with a semi-free linear action of $\mathbb{Z}_p$ on a disk $\mathcal{D}^n$ with 0 as the isolated fixed point. The quotient of the boundary of an invariant neighborhood of 0 is a lens space. Now suppose that $p \geq 5$, so $\mathrm{Wh}(\mathbb{Z}_p)$ is nontrivial. Erect an h-cobordism on this lens space. We can take universal covers, and cone the two boundaries separately to obtain a nonlinear action on $\mathcal{S}^n$ that has fixed set $\mathcal{S}^0 = 0 \cup \infty$ (with obvious conventions). Near 0 the action is linear, but near ∞ it is not: the Whitehead torsion of the homotopy equivalence from this quotient to the linear lens space is nontrivial. (Exercise, but see §5.5.3 if you need a hint.)

Now consider the cone on this action.

We obtain a PL $\mathbb{Z}_p$-action, whose fixed set is an interval I. However, the germ neighborhood is not trivial. For then the "normal representations" at the two fixed points would have to be the same.

Moral: In the PL category one has to do one of two things. Either assume a local model: this is perhaps not so unreasonable if one recalls that assuming the fixed set is a manifold is an assumption – not guaranteed, as Example 6.6 shows.

Or, alternatively, one can work up to concordance: view two neighborhoods of F as equivalent if there is a neighborhood of $\mathrm{F} \times [0, 1]$ which restricts to each on the boundaries. (Better, we should allow F to change, and allow h-cobordisms into the equivalence relation on the blocks over F. A neighborhood of F will be equivalent to a neighborhood of F' if they are h-cobordant, and there is a neighborhood of the h-cobordism that restricts to each.) Both of these theories give rise to block bundle theories and have classifying spaces. The relation between these theories is established in Cappell and Weinberger (1991a) and is determined by "Rothenberg classes" that lie in the cohomology of F.

In any case, the smooth (Rothenberg–Sondow) examples we discussed do not become PL equivalent.

Example 6.11 The neighborhoods of 0 and ∞ in the previous example are topologically equivalent.[9] Thus, in the topological category there cannot be uniqueness of "closed regular neighborhoods of fixed points" as there is in the PL category.

This is a simple consequence of the h-cobordism theorem. Since L and L' are h-cobordant, they are diffeomorphic after crossing with S^1 (as torsions multiply by $\chi(S^1) = 0$.) Passing to infinite cyclic covers gives a diffeomorphism $L \times (-\infty, \infty) \to L' \times (-\infty, \infty)$. Taking covers and extending to a point at $-\infty$ gives an equivariant homeomorphism between the two open neighborhoods.

Example 6.12 With a little more care one can see that all the smooth actions with fixed set F and given normal representation are topologically equivalent. This is surely plausible, as we've seen that torsion does not obstruct, and the torsion is all there is in the smooth category.

The result actually follows from the following beautiful fact, due to Stallings, whose proof goes back to Euler and Eilenberg, and then to Mazur and Stallings (Stallings, 1965b)[10] (and oft exploited since).

Proposition 6.13 (Stallings) *If* $(W, \partial_+, \partial_-)$ *is an h-cobordism (of dimension greater than* 4*), then* $W - \partial_- \cong \partial_+ \times [0, \infty)$.

Let V be the h-cobordism with $\tau(V, \partial_-) = -\tau(W, \partial_+)$. We have $W \cup V \cong \partial_+ \times [0, 1]$, by the h-cobordism theorem, and similarly, $V \cup W \cong \partial_- \times [0, 1]$. Then

$$\begin{aligned}\partial_+ \times [0, \infty) &\cong (W \cup V) \cup (W \cup V) \cup \cdots \\ &\cong W \cup (V \cup W) \cup (V \cup W) \cup \cdots \\ &\cong W \cup \partial_- \times [0, 1] \\ &\cong W - \partial_-.\end{aligned}$$

Thus, we've now seen infinitely many PL-inequivalent topologically-equivalent smooth actions for reasons that can be attributed to K-theory.

Topological actions that cannot be triangulated exist for many reasons, of various degrees of subtlety.

9 This example is a variation on the trick used by Milnor in his disproof of the general *hauptvermutung*.

10 Our scholarship is inadequate to the task of justifying this folklore description of the history of the series $1 - 1 + 1 - 1 + 1 - \cdots$.

Example 6.14 We can take an infinite connect sum of $\mathcal{S}^1$-actions on $\mathcal{S}^n$ with fixed set Σ (a non-simply connected homology sphere). This will give an action on $\mathcal{S}^n$ with fixed set the one-point compactification of the infinite connected sum $\Sigma\#\Sigma\#\Sigma\#\cdots$. This fixed set is not even an ANR – its fundamental group is uncountable!

Example 6.15 Bing (1959) gave an example of a non-manifold X whose product with $\mathbb{R}$ is $\mathbb{R}^4$, and so that $X^+ \times \mathbb{R} \cong \mathcal{S}^3 \times \mathbb{R}$. Now we know that such examples abound (i.e that there are uncountably many different such spaces in every dimension[11]). In any case, $X^+ \times \mathcal{S}^1$ is a manifold whose quotient under a circle action is a non-manifold.

Bing also gave uncountably many $\mathbb{Z}_k$ actions on $\mathbb{R}^3$ whose fixed sets are different non-locally flat $\mathbb{R}$s.

In this chapter we will often assume that the fixed sets (and quotient spaces of the free parts) of our topological actions are ANRs or even compact topological manifolds.

Example 6.16 For this example, we will make use of Siebenmann's proper h-cobordism theorem Siebenmann (1970b). For W a paracompact manifold, Siebenmann defined $\mathrm{Wh}^p(W)$, which classifies proper h-cobordisms with one boundary component W. This group can frequently be computed, if W is "not too wild." For our purposes, we just note for L a compact manifold

$$\mathrm{Wh}^p(L \times \mathbb{R} \cong K_0(\pi_1(L)).$$

If one takes a prime with nontrivial class group, we can start with a linear action, and erect an h-cobordism with given element of K_0 as its torsion, and, with a one-point compactification, obtain an action with fixed set an interval, but that does not have *any* invariant closed tubular neighborhoods.

(The reader with some number theory experience can use the analogue of the Milnor duality theorem in this setting to give examples with fixed set a circle by arranging for the "other end" to be trivial and then gluing the ends together.)

These kinds of examples occur very naturally when one studies possible fixed sets of group actions. Certain lens spaces (with odd-order fundamental groups) occur as fixed sets of, for example, Q_8 actions only if one allows actions where

[11] These can be obtained by shrinking quite general decompositions to a point, and ultimately using Edwards's theorem. We refer the reader to Daverman (2007) for a discussion of this beautiful area of topology.

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there are no invariant closed tubular neighborhoods.[12] (See Figure 6.3 for a picture decribing how such a construction works.)

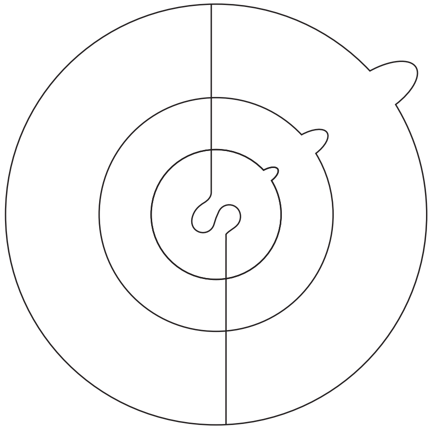

Figure 6.3 The actions are created on an ascending union of mod 2 homology balls on which PL-actions can be constructed, the mod 2 homology being present to get around a finiteness obstruction. The action on the sphere is the one-point compactification of the union.

It is a remarkable fact, discovered by Quinn (1979, 1982a,b,c, 1986, 1987b), that actions which have locally flat fixed point sets (and manifold quotients of pure strata[13]) – actions that have come to be called "tame" – have some topological homogeneity: if one has an arc entirely within a pure stratum of this action (i.e. where the isotropy group does not change along the curve) then there is an equivariant isotopy covering this. This means that if an action is locally linear at one point, it will be locally linear over the whole stratum. As a result, the seeming singularities that arise at one-point compactifications and related constructions often are not there at all topologically.

In some sense the differences between Top and PL (when one has local triangulability of fixed sets and quotients) can be attributed to K-theory (and the Kirby–Seibenmann obstruction). Of course, the non-ANR situation is a serious problem in general for the topological category. The smooth category differs from these categories for other local reasons as well.

[12] A three-dimensional lens space with odd-order fundamental group $\mathbb{Z}_n$ is a PL fixed set iff n is $\pm 1 \bmod 8$. This is due to the Swan homomorphism that associates to n the projective module given by the kernel of the reduction of the augmentation map $\mathbb{Z}Q_8 \to \mathbb{Z} \to \mathbb{Z}_n$ (the one-dimensional free module $\mathbb{Z}Q_8$). (For any G, this defines – nontrivially – a homomorphism $\mathbb{Z}/|G|^* \to K_0(\mathbb{Z}\mathbb{Q})$.) See Example 6.18 below and Example 6.28 in §6.6.

[13] We obviously only need to add this as a hypothesis when a positive-dimensional group acts (and then we do because of Bing-type examples as above).

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Example 6.17 Let's think about $\mathbb{Z}_p$-actions. In the smooth case, the fixed set F will be smooth and the neighborhood of F has the structure of an $\mathbb{R}[\mathbb{Z}_p]$-module (with no trivial piece). So, for $p = 2$, the neighborhood is essentially an arbitrary vector bundle.

In the PL (and Top is essentially equivalent in this case) case, we can analyze the situation using "blocked surgery." For the orientation reversing situation, $\mathrm{BPL}_{2k-1}(\mathbb{Z}_2) \cong \mathrm{BAut}(\mathbb{RP}^{2k-1}) \otimes \mathbb{Z}[1/2]$ (as the L-spaces $\mathrm{L}_{2k}(\mathbb{Z}_2, -)$ are 2-torsion). Thus, the map $\mathrm{BO}(2k-1) \to \mathrm{BPL}_{2k}(\mathbb{Z}_2)$ loses most of the rational Pontrjagin classes that are present in the smooth case!

On the other hand, for even codimensions, when the action is orientation-preserving, the PL space is much richer because $L_{2k}(\mathbb{Z}_2, +)$ is rationally a product of $\mathrm{BO} \times \mathrm{BO}$ or its second loop space (depending on k).

However, the reader should not jump to any conclusions. The stabilization map $\mathrm{BPL}_{2k}(\mathbb{Z}_2) \to \mathrm{BPL}_{2k+2}(\mathbb{Z}_2)$ (and these are approximately both isomorphic to $\mathrm{BO} \otimes \mathbb{Q}$ aside from a few homotopy groups coming from the BAut) factors through $\mathrm{BPL}_{2k+1}(\mathbb{Z}_2)$ which is essentially trivial.

Consequently, although the maps $\mathrm{BO}(n) \to \mathrm{BO}(n+1)$ become highly connected with n, the equivariant PL versions *never* stabilize (even rationally).[14]

This is closely related to the failure of equivariant transversality in the PL and topological categories (see Madsen and Rothenberg, 1988a,b, 1989).

For p odd, there is another interesting difference between the categories. For the smooth category, one gets a decomposition of the vector bundle according to irreducible representations of $\mathbb{Z}_p$. In PL (and Top) there is no analogue of this.

In the smooth category, the structure of the neighborhoods is thus very dependent on what the local representation is: if, for example, the normal representation is a sum of n distinct irreps, then the bundle is equivalent to a sum of n-complex line bundles, while if it is untypical, the bundle is a $\mathrm{U}(n)$ bundle. In the PL (and topological situations) the classifying spaces for the neighborhoods is pretty insensitive to the type of the local representation (e.g. to whether it is untypical or not).

Example 6.18 (Converses to Smith theory) The possible fixed sets of a group action are not arbitrary. Smith theory, and its generalizations, give connections between the group action, the homotopy type of the space acted upon, and the fixed set. In the extreme of a contractible space, the phenomena are rather stark and were pioneered by Lowell Jones (1971) and Oliver (1976b), respectively.

[14] Nor do the topological versions of these spaces for the very same reason (although the details necessary to rigorously verify this depends on Quinn's theory of controlled ends – or something equivalent).

For G a p-group, the fixed set must be mod p-acyclic. This condition is essentially necessary and sufficient (although for complicated G, in the PL case, there is a $K_0(\mathbb{Z}G)$ obstruction – see Assadi, 1982). To be really concrete, if p is prime, then a manifold M is the fixed set of a semi-free PL locally linear $\mathbb{Z}/p^n$-action on a high-enough-dimensional sphere iff M is a mod p homology sphere of the same parity of dimension as that of the sphere.[15] In view of this, if the fixed set is not unique in its homotopy type, we can easily get equivariantly homotopy-equivalent actions on the sphere by realizing these different manifolds as fixed sets.

For G a non-p-group, the homotopy types of possible fixed sets are determined by a number n_G called the Oliver number. In this case F is a fixed set on a finite contractible complex (and hence homotopy-equivalent to the fixed set of some action on a disk) iff $\chi(\mathrm{F}) \equiv 1 \bmod (n_G)$. When $n_G = 1$ then every finite complex F is *actually* the fixed set of a G-action on some disk (such that F is embedded in the interior of the disk!).

Putting together these methods of construction with a few variations and the results we will explain later in this chapter, one obtains the following result (which answers a question of Borel).

Theorem 6.19 (Trichotomy; Cappell *et al.*, 2015) *Let G be a real Lie group, and suppose that the dimension d of G/K is at least* 5[16] *and suppose Γ is a uniform lattice in G. Then the number of properly discontinuous actions of Γ on $\mathbb{R}^d$ is either* 1, $\aleph_0$, *or c (the continuum). In the last case, there are (a continuum of) examples that are not locally rigid (e.g. arbitrarily C^0-close to the left action of Γ on G/K – indeed that are degenerations of this action).*

This trichotomy is determined by the nature of the singular set[17] of the isometric action of Γ on G/K. One has rigidity if the action is free (i.e. if Γ is torsion-free) and sometimes if the singular set is zero-dimensional,[18] but if the singular set is positive-dimensional then the number of actions is always c (and it's never uncountable unless the singular set is positive-dimensional).

15 Indeed, if M embeds in the sphere and we are even codimension other than 2, it is the fixed set of PL locally linear action (see Weinberger, 1985b, 1987; Cappell and Weinberger, 1991a).

16 The paper gives information in low dimensions as well.

17 This is the set of points whose isotropy is nontrivial.

18 $\aleph_0$ nonrigidity holds iff (the action has discrete singular set and) d is 2 mod 4 (and greater than 2), and Γ contains an element of order 2. (As a comment whose significance will only become clear in the next section, Γ then automatically has at least two conjugacy classes of involutions, and, indeed, Γ contains an infinite dihedral group.)

Moral: For the equivariant version of the Borel conjecture, we shall assume that our actions are “tame”: that the fixed sets are nice submanifolds and we shall also not allow codimension-1 and codimension-2 situations.[19]

In addition, we should assume that the G-action on M makes it into an “equivariant Eilenberg–Mac Lane space.” This should have been obvious for reasons of functoriality (this is like the assumption of low $\mathbb{Q}$-rank in the proper Borel conjecture – we should not have made that mistake twice!). In any case, isometric actions on locally symmetric manifolds are Eilenberg–Mac Lane in the appropriate sense (as will be clear in a moment). This eliminates the converses to Smith theory examples (Example 6.18).

To make this condition clearer, recall that a $K(\pi, 1)$ is the terminal object in a category that includes some connected space, and only 1-equivalences (i.e. maps for which one can uniquely lift all maps of 1-complexes into the target). This same notion makes sense equivariantly. It boils down to – see tom Dieck (1979), Lück (1987), and May (appendix to Rosenberg and Weinberger, 1990) – all components of all fixed sets of all subgroups being aspherical.[20]

The smooth category, even more transparently than for the original Borel category, is not suitable for the equivariant version. The PL category also has no chance – there are too many K-theoretic obstructions. The K-theory that enables topological actions to exist that don't have closed equivariant neighborhoods haven't yet been implicated as an obstacle – but we shall have to study this more carefully. Depending on formulation, Nil is a problem or it is not. It will cause a perturbation in our understanding.

The topological category will be a reasonable one for studying the problem. Without assuming tameness, examples such as Examples 6.6 and 6.14 are unavoidable, and one cannot hope for equivariant homeomorphisms.

The need to avoid the low-codimension situation is because of the failures of the Smith conjecture. With codimension 2, one loses too much information on moving to closed strata from pure ones.

6.3 h-Cobordisms

In the case of closed manifolds, the Borel conjecture boils down to two statements: one about the vanishing of Whitehead groups, i.e. that h-cobordisms are products; and the second a statement about L-groups, that a certain assembly

[19] It is not impossible to incorporate codimension-1 and codimension-2 phenomena in an “isovariant Borel conjecture” – see Chapter 13 of Weinberger (1994). However, although the isovariant conjecture is “more true,” the equivariant one is “more interesting.”

[20] However, we will see that even this assumption does not save the day even for “equivariant Novikov conjectures” in the appendix to §6.7. Nevertheless, till then we will use the current guess as our guide till we are forced to abandon this as too naive.

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map is an isomorphism[21] – which surgery then translates into the statement that homotopy-equivalent manifolds are h-cobordant, and therefore homeomorphic.

In the smooth and PL locally homogeneous categories, the h-cobordism theorem is quite straightforward:[22]

$$\mathrm{Wh}^G(M) \cong \bigoplus \mathrm{Wh}(\pi_1(M^H/(\mathrm{N}H/H))) :$$

here $\mathrm{N}H/H$ is the normalizer of H divided by H and the sum is over conjugacy classes of subgroups; we use the convention that π_1 of a disconnected set is the sum of the π_1s of the components. Thus, on the right-hand side, we have a sum of the fundamental groups of all components of all strata. (Note that the group that acts on the stratum fixed by H is $\mathrm{N}H/H$.)

The proof is a straightforward induction on the strata. Once one has a product structure on a stratum, the structure of neighborhoods (e.g. the tubular neighborhood theorem) extends it to a neighborhood, and then one uses the torsion on the complement and a relative form of the h-cobordism theorem to extend it to the outside.

Although the Whitehead group has a straightforward decomposition, the involution does *not* preserve the terms of the decomposition. It does at the level of an associated graded of a filtration, but not on the nose. To give an example, suppose that $G = \mathbb{Z}_2$ acts on a closed manifold W with codimension-1 fixed set. Then W/G is a manifold with boundary F:

$$\mathrm{Wh}^G(W) \cong \mathrm{Wh}\big(\pi_1(W/G)\big) \oplus \mathrm{Wh}(\pi_1(\mathrm{F})).$$

The involution, thought of as a matrix, has one non-diagonal term corresponding to $\pm$ (depending on conventions) the inclusion map $\pi_1(\mathrm{F}) \to \pi_1(W/G)$. So if $\pi_1(\mathrm{F}) \to \pi_1(W)$ is an isomorphism, the Tate cohomology $H^*(\mathbb{Z}_2; \mathrm{Wh}^G(M)) = 0$ – which would not be the case if the involution had actually preserved the pieces.

Note that these Whitehead groups can be quite large. If we consider a disk $\mathcal{D}$ (of dimension greater than 2) with a linear G-action (with no low-codimension situations), then[23]

$$\mathrm{Wh}^G(\mathcal{D}) \cong \bigoplus \mathrm{Wh}(\mathrm{N}H/H),$$

[21] As noted at the end of Chapter 5, the statement about Whitehead groups can also be viewed as the bottom part of an isomorphism of assembly maps in algebraic K-theory.

[22] We do not allow any low-dimensional strata, or assume that the h-cobordism is assumed to be a product on those. Needless to say, when we work with h-cobordisms that are trivialized on a union of strata, we get the same answer, except that those strata do not come up in the right-hand side.

[23] Recall our convention that if we do not include the boundary in the notation, then we are working relative to the boundary. If we were not working relative to the boundary, then we would include the boundary into our notation, $\mathrm{Wh}^G(\mathcal{D}, \partial\mathcal{D})$.

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where the sum is over conjugacy classes of isotropy subgroups (G always occurs as the isotropy of 0). This can be quite a large finitely generated group. (One can compute its rank using representation theory – it is the number of real irreducible representations that are not rational.)

Exercise 6.20 Use the PL Whitehead group to give equivariantly homotopy equivalent G-actions on an aspherical manifold that are not equivariantly PL homeomorphic.

Now, for a more typical situation, let's consider $\mathrm{Wh}^G(\mathcal{S}^1 \times \mathcal{D})$, where G acts trivially on the circle. In that case, we get the analogous decomposition:

$$\begin{aligned}\mathrm{Wh}^G(\mathcal{S}^1 \times \mathcal{D}) &\cong \bigoplus \mathrm{Wh}(Z \times \mathrm{N}H/H)\\ &\cong \bigoplus \mathrm{Wh}(\mathrm{N}H/H) \bigoplus K_0(\mathrm{N}H/H) \bigoplus \mathrm{Nil}_{\pm}(\mathrm{N}H/H)\end{aligned}$$

by the Bass–Heller–Swan formula. For higher tori, we can iterate this formula. The Nil terms, when nonzero, give us infinitely generated torsion terms.[24]

However, as we saw in §6.2, in the topological case this formula is not quite right. It is not so hard to see that

$$\mathrm{Wh}^{G,\mathrm{top}}(\mathcal{D}) \cong 0$$

as a consequence of Siebenmann's thesis – which, while giving a condition for a manifold to be the interior of a manifold with boundary, gives, *inter alia*, a condition for a manifold V to be $\partial V \times [0, \infty)$ (rather analogous to the h-cobordism theorem – except that there is no K-theory obstruction[25]).

The situation is rather different for $\mathrm{Wh}^{G,\mathrm{top}}(\mathcal{S}^1 \times \mathcal{D})$. The $\bigoplus \mathrm{Wh}(\mathrm{N}H/H)$ terms go away for the same reason as before. The $\bigoplus K_0(\mathrm{N}H/H)$ also go away. We can explain this as follows.

The $K_0(\mathrm{N}H/H)$ term corresponds to h-cobordisms that are isomorphic to their own 2-fold covers. So when doing a $1 - 1 + 1 - 1 + \cdots$ trick, one can have each term represent the 2-fold cover of its predecessor. When one does this, the homeomorphism produced in the limit will actually be convergent along the circle.

This leaves only the Nil terms; that is, the correct answer

$$\mathrm{Wh}^{G,\mathrm{top}}(\mathcal{S}^1 \times \mathcal{D}) \cong \bigoplus \mathrm{Nil}(\mathrm{N}H/H).$$

Nil, which obstructs the fundamental theorem of algebraic K-theory holding for non-regular rings, also prevents too naive a form of the equivariant Borel conjecture from holding.

24 Recall that if A is a nilpotent matrix, then $\mathbf{I} + tA$ is a typical element in the Nil term: $\mathbf{I} + t^i A$ contains an infinite number of linearly independent elements (distinguished by which covers they transfer nontrivially to). See Bass and Murthy (1967) for some examples.

25 Compare to Stallings's result, Proposition 6.13.

Needless to say, we haven't proved any of this. The proof (see Quinn, 1988; Steinberger, 1988) uses the controlled h-cobordism. What occurs on a pure stratum is not merely a proper h-cobordism – that can be analyzed via Siebenmann. It has control with respect to the lower stratum as one goes to ∞. As controlled K-theory is a homology theory, the

$$\bigoplus \mathrm{Wh}(\mathrm{N}H/H) \bigoplus K_0(\mathrm{N}H/H) \bigoplus \mathrm{Nil}_{\pm}(\mathrm{N}H/H)$$

that one sees in the interior is modified by

$$H_*\big(S^1; \bigoplus \mathrm{Wh}(\mathrm{N}H/H)\big) \cong \bigoplus \mathrm{Wh}(\mathrm{N}H/H) \bigoplus K_0(\mathrm{N}H/H),$$

leaving the Nil terms left over.

Exercise 6.21 Give an example of a closed aspherical manifold where there are equivariant homotopy equivalences not equivariantly homotopic to homeomorphisms because of a nontrivial Nil group.

For small groups like $\mathbb{Z}_p$ this doesn't make a difference, but even for $\mathbb{Z}_n$ or $\mathbb{Z}_p \times \mathbb{Z}_p$, the group $\mathrm{Wh}^{\mathrm{top}}$ can be large (because of K_{-1} bubbling up from a fixed set of dimension 2 or because of Nil). In any case, these issues can be handled by assuming it away in the equivariant Borel conjecture:

Conjecture 6.22 (Modified equivariant Borel conjecture) *Suppose $G \times M \to M$ is a tame action, and that it is an equivariantly aspherical manifold. If $f: M' \to M$ is an equivariant topologically simple homotopy equivalence, then f is equivariantly homotopic to a homomorphism.*

A negative aspect of this modification is that it loses the vanishing of Whitehead groups that is part and parcel of the usual Borel conjecture.

On the other hand, it is a possibly true (*prima facie*) rigidity statement.[26]

The better resolution of this difficulty is the Farrell–Jones conjecture, that makes a prediction of the structure of $\mathrm{Wh}^{G,\mathrm{top}}(M)$ in terms of the space of equivariant G-submanifolds of M of dimension 1. For example, if the action is semi-free, and suppose the fixed set contains no higher-rank abelian subgroups, the relevant Whitehead group should just depend on the $\mathrm{Nil}(G)$, parameterized by the conjugacy classes of maximal cyclic subgroups of $\pi_1(\mathrm{F})$.[27]

We shall return to this later.

6.4 Cappell's UNil Groups

We are not out of the woods yet in understanding the equivariant Borel conjecture because of a remarkable phenomenon discovered by Cappell (1973, 1974a,b).

[26] Since we will see that it is indeed false, perhaps it would be better to say frequently true.
[27] This description tacitly assumes no infinitely divisible elements.

This is a beautiful story worth telling in its own right, not just as an adjunct to the Borel story – so we shall delay the application to the equivariant Borel conjecture for a couple of sections and discuss a part of Cappell's work in its original context.

We are now back in the world of manifolds, with no group actions. For simplicity we will assume that all manifolds here are orientable and will only deal with one special splitting problem: Cappell's work is much more general.

We begin with a theorem of Browder (and its easy generalization by Wall, 1968).

Theorem 6.23 *Suppose M is a closed manifold (of dimension greater than 5) and V is a codimension-1 submanifold, dividing M into two parts, $M_\pm$, so that $\pi_1(V) \to \pi_1(M_+)$ is an isomorphism. Then any homotopy equivalence $f\colon M' \to M$ can be homotoped to one where f is transverse to V, and $f|_{f^{-1}(V)}$ is a homotopy equivalence.*

Corollary 6.24 *In the* PL *and* Top *categories, the question of whether a manifold is a connected sum only depends on the homotopy type of the manifold* if *it's simply connected (or one of the summands is).*

Let V be the separating subsphere in the connect sum decomposition, and make use of the Poincaré conjecture to assert that $f^{-1}(V)$ is also a sphere.

An analogue of this corollary (ignoring the possibility of taking a summand that is a counterexample to the Poincaré conjecture) without the simple connectivity was first proved in dimension 3. Stallings showed that a 3-manifold is a connected sum of non-simply connected pieces iff its fundamental group is a nontrivial free product (see, e.g., Hempel, 1976).

In dimension 4 this corollary is now known not to be true in PL (because of Donaldson's work), and in dimension 5 it is possible to fix the argument and establish the result. In general, we have the following theorem of Cappell (1974b, 1976a):

Theorem 6.25 *For manifolds whose fundamental groups have no 2-torsion, being a connected sum is homotopy-invariant. However, there are infinitely many manifolds homotopy-equivalent to $\mathbb{RP}^{4k+1}\#\mathbb{RP}^{4k+1}$ that are not connected sums.*

It is the second part of this theorem that will imply, for example, that the equivariant Borel conjecture still fails for certain involutions on the torus.

We now know, thanks to unpublished work of Connolly and Davis, that connected sum is homotopy invariant for *all* orientable manifolds of dimension 0 and 3mod 4.[28] It turns out that *this* positive result is a consequence of results proved about the equivariant Borel conjecture (or, perhaps better, the Farrell–Jones conjecture).

But let's do things in order.

The Browder–Wall splitting theorem is a consequence of the π–π theorem. Consider $M' \times [0,1]$ and glue on to $M' \times 1$ a normal Z, ∂Z cobordism of $f^{-1}(M_-, V)$ to a homotopy equivalence. This normal cobordism exists by the π–π theorem. We now have another surgery problem,

$$M' \times [0,1] \bigcup_{f^{-1}(M,V)} (Z, \partial Z) \to M \times [0,1] \bigcup_{M \times 1} M_- \times [1,2] \cong M \times [0,2],$$

relative to M_-. We can view this as a π–π problem, because by Van Kampen's theorem, the map $\pi_1 M_+ \to \pi_1 M$ is an isomorphism. When we solve it, we obtain an h-cobordism to the solution of the splitting problem. If we glue on an h-cobordism on the part mapping to $M_+ \times 2$ with negative torsion (of the above h-cobordism), we turn it into an s-cobordism – i.e. we have produced the desired s-cobordism.

To study the connected sum problem, it is easy enough enough to construct a normal cobordism of the homotopy equivalence to a split one.[29] The real problem is to somehow understand elements in the cokernel of $L(G) \times L(H) \to L(G * H)$ which will measure the difficulty in taking a normal cobordism to a connected sum, and modify it (via a Wall realization) to one where the surgery obstruction vanishes, and can be turned into a homotopy.

Cappell showed, on the one hand, that one can give a complete analysis in terms of an analogue of Nil, based on the bimodule $(\mathbb{Z}; \mathbb{Z}[G-e], \mathbb{Z}[H-e])$. These are bimodules with involution as in the Milnor duality formula. In this case, we essentially are dividing $\mathbb{Z}[G-e]$ (and $\mathbb{Z}[H-e]$) into pieces that are $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ terms that are preserved or interchanged by the involution. When there is no 2-torsion, we are in the situation where everything is of the form $(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z})$ where there is nothing. However, terms of the form $(\mathbb{Z}, \mathbb{Z})$ give a very large group.

In this case, Cappell wrote down quite explicit elements in $L_2(\mathbb{Z}[\mathrm{D}_\infty])$, where D_∞ is the infinite dihedral group. He showed that these are nontrivial by mapping to the finite dihedral groups D_{2n} (for n odd). More precisely he

[28] Their work, as well as work of Banagl and Ranicki, show that there are non-connected sums homotopy-equivalent to $\mathbb{RP}^{4k-1} \times S^3 \# \mathbb{RP}^{4k-1} \times S^3$ as well – so the phenomenon does arise in both of these sets of dimensions. The case of 3mod 4 was proved by Cappell in his original paper.

[29] Here we only need to use functoriality to build a map $L(G * H) \to L(G) \times L(H)$ that is almost a retraction of the maps induced by inclusion $L(G) \times L(H) \to L(G * H)$. (Why is it not a retraction? How far off is it?)

considered the Arf invariant in $L_2(\mathbb{Z}) = \mathbb{Z}_2$ and by first going to an odd-fold cover and then taking the Arf invariant (i.e., mapping to $L_2(\mathbb{Z})$). If splitting were possible, i.e. if the L-group were as small as predicted, these two elements would be the same. More precisely, $L_2(\mathbb{Z}) \to L_2(Z[\mathrm{D}_\infty])$ would be an isomorphism, in which case passing to a finite-fold cover would just multiply the element by the index of the cover. Since the Arf element is of order 2, we would expect equality under odd-fold covers. But we do not obtain this by explicit calculation.

Some more details: recall that $L_{2k}(\pi)$ is built out of $(-1)^k$ symmetric quadratic forms over $\mathbb{Z}\pi$. Let $\pi = Z_2 * \mathbb{Z}_2$, generated by involutions g and h. Here $gh = t$ is the translation in the usual view of D_∞ as the affine isomorphisms of $\mathbb{Z}$ (and g is then $x \to -x$ and h is $x \to 1 - x$).

Cappell's elements γ_k are defined on a two-dimensional quadratic form – generated by (e, f), with $\lambda(e, e) = \lambda(f, f) = 0$ and $\lambda(e, f) = 1$ (i.e. looking hyperbolic, i.e. trivial, from the λ point of view) and with $\mu(e) = g$ and $\mu(f) = t^k g t^{-k}$. Note that γ_k is essentially γ_1 pushed forward from a subgroup of index k. It is obvious that the augmentation, sending g and h to the trivial element, takes γ_k to the standard element of $L_2(\mathbb{Z})$ which has nontrivial Arf invariant. On passing to a large cover, one checks that the Arf invariant becomes trivial and we have a non-split example.

One can check that for different ks one gets different elements by examining the various transfers and augmentations and thus obtains Cappell's result that $L^2(\mathbb{Z}[\mathrm{D}_\infty])$ contains an infinite $\bigoplus \mathbb{Z}_2$.

Now we know the full structure of this L-group[30] (and explicitly, not just as an abstract statement), namely:

$$
\begin{aligned}
L_0(\mathbb{Z}[\mathrm{D}_\infty]) &= \mathbb{Z}^3,\\
L_1(\mathbb{Z}[\mathrm{D}_\infty]) &= 0,\\
L_2(\mathbb{Z}[\mathrm{D}_\infty]) &= \bigoplus \mathbb{Z}_2,\\
L_3(\mathbb{Z}[\mathrm{D}_\infty]) &= \bigoplus \mathbb{Z}_2 \bigoplus \mathbb{Z}_4.
\end{aligned}
$$

All unlabeled sums are infinite. They show that L-groups of nice small lattices can be infinitely generated. It is not shocking that they give rise to an infinitely generated group of counterexamples to the equivariant Borel conjecture – as we shall see in §6.5.

The results about homotopy invariance of connected sums for oriented manifolds of dimension 0 and 3 mod 4 for general fundamental groups is a consequence of first, the work of Cappell on the algebraic nature of the obstruction mentioned above, and second, some specific calculations that Connolly and

30 Thanks to work of Banagl and Ranicki, and of Connolly and Davis.

Davis did – using the methods of Farrell and Jones[31] – so that the calculations done for the dihedral group end up sufficing for all groups. The first point is that the $\mathbb{Z}$-bimodule with involution basically only sees the number of elements of order 2 and the remaining number of elements. Unlike $L(\mathbb{Z}\pi)$ – that depends on the ring structure of π (and the involution) – UNil really looks at much less. It turns out that with cleverness one can reduce to the cases of $\mathbb{Z}_2 * \mathbb{Z}_2$ and, say, $\mathbb{Z}_2 * \mathbb{Z}_3$. The second case is algebraically tough[32] – however, the Farrell–Jones conjecture will reduce it to the former case,[33] and this group is a retract of the fundamental group of a two-dimensional hyperbolic orbifold, so it's a case that can be handled by the ideas of Farrell and Jones (1993a) and Bartels *et al.* (2014a,b).

6.5 The Simplest Nontrivial[34] Examples

Let us consider the simplest special case of the equivariant Borel conjecture, when M is a G-manifold with singular set of dimension 0.[35] In that case, the quotient is a non-manifold with just isolated singularities (corresponding to the nontrivial isotropy). This case doesn't require any controlled (or stratified) topology to analyze. As in §6.2, the key issues are all susceptible to analysis by means of proper topology and then one-point compactifying. We shall see that the $\mathrm{Wh}^{\mathrm{top}}$ theory is apt to have an especially simple form, because of the discreteness of the singular set, but that despite this there can be failures on very concrete manifolds, because of the nonvanishing of UNil.

Let W be the nonsingular part of M/G. We should then consider the proper topology of W. Its fundamental group Γ fits into an exact sequence $1 \to \pi \to \Gamma \to G \to 1$ where $\pi = \pi_1 M$.

The singular points correspond to subgroups of finite order in Γ. The isotropy group acts on the normal sphere to the fixed point, and then embeds a "space

[31] The more recent paper of Bartels and Lück (2012a) – which also is a further development of the Farrell–Jones ideas – includes enough examples to suffice for this purpose. It has the advantage of being published, while the work of Connolly and Davis has the advantage of only requiring ideas of negative curvature.

[32] The fundamental group contains a nonabelian free group of rank 2. However, the only virtually cyclic subgroups inside of it are dihedral – which, as we will later see, gives vanishing in the relevant dimensions.

[33] The basic example of this is the reduction of $\mathbb{Z} * \mathbb{Z}_2$ to $\mathbb{Z}_2 * \mathbb{Z}_2$. There are two conjugacy classes of maximal infinite dihedral groups each of which contributes its UNil elements to $L(\mathbb{Z} * \mathbb{Z}_2)$.

[34] By which we mean non-free – so that new complications arise that are not part of the ordinary Borel conjecture.

[35] This case is considered in detail in Connolly *et al.* (2015) which gives full justification of the somewhat heuristic descriptions given here. Recall that the singular set is the set of points where the isotropy group is nontrivial.

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form" (i.e., manifold[36] quotient of the sphere) near an end of W corresponding to the deleted neighborhood of this point in M/G.

Indeed, a little reflection[37] shows that Γ acts on the universal cover of M, and that its singular set is discrete – all isotropy finite, injecting into G, and that by taking Γ orbits, the singular points are in a one-to-one correspondence with conjugacy classes of maximal finite subgroups of Γ. The key nontrivial observation here is that Smith theory (see, e.g., Bredon, 1972) guarantees that the fixed set of each element of prime (power) order (acting on the universal cover) is a single point, by our discreteness condition.

Moreover, these maximal finite subgroups are disjoint (except for the identity element), and (since M is compact) there can only be finitely many (conjugacy classes) of them. We shall call them $G_1, G_2, \ldots, G_k$ (or maybe use some other indexing set).

The Whitehead theory, according to Siebenmann (1970b), then fits into an exact sequence:

$$\bigoplus \mathrm{Wh}(G_i) \to \mathrm{Wh}(\Gamma) \to \mathrm{Wh}^p(W) \to \bigoplus K_0(G_i) \to K_0(\Gamma) m$$

which suggests – and one can correctly do so – extending the sequence both to the left and right and make Wh^p into a relative group. Indeed, for this special class of groups, i.e. where all elements of finite order lie in a unique conjugacy class of subgroup of finite order, the Farrell–Jones K-theory conjecture boils down to the statement[38] that:

(*) For such groups $\bigoplus \mathrm{Wh}(G_i) \to \mathrm{Wh}(\Gamma)$ and $\bigoplus K_0(G_i) \to K_0(\Gamma)$ are isomorphisms.

The reader might suspect (correctly) that the injectivity of these maps is (part of) a Novikov conjecture statement. In any case, conjecturally, these proper Whitehead groups vanish.

Needless to say, we don't expect

$$\bigoplus L(H_i) \to L(\Gamma)$$

to be an isomorphism; after all, that is not what happens in the torsion-free setting when there are no G_i!

[36] Because we are assuming the discreteness of the singular set.
[37] Using discreteness of singular sets!
[38] As was observed by Connolly, Davis, and Khan.

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To simplify[39] the discussion,[40] let's ignore the issues of algebraic K-theory (i.e. how to decorate the L-groups), and use the $-\infty$ decoration here:

$$\bigoplus L_n^{-\infty}(G_i) \to L_n^{-\infty}(\Gamma) \to L_n^{-\infty}(W) \to \bigoplus L_{n-1}^{-\infty} \to L_{n-1}^{-\infty}(\Gamma).$$

After all, this is the sequence one would get for manifolds with boundary, and after crossing with a circle, we can put a boundary on these manifolds, and, furthermore, it will be essentially unique (certainly after crossing with another one!).

Henceforth we will just write L for this decorated version, which means that final results will have to take the change of decoration into account.

Let us now combine this with the surgery exact sequence:

$$\to S^{\mathrm{p}}(W) \to H_n^{\mathrm{lf}}\big(W; L(e)\big) \to L_n^{\mathrm{p}}(W) \to .$$

Conjecturing that $S^{\mathrm{p}}(W)$ vanishes (or, better, is contractible, if we spacify) would boil down to the statement that $H_n^{\mathrm{lf}}\big(W; L(e)\big) \to L_n^{\mathrm{p}}(W)$ is an isomorphism.

However, we would like to improve this since W is not an invariant of Γ, although its proper L-group is the cofiber of $\bigoplus L_{n-1}(G_i) \to L_{n-1}(\Gamma)$, which patently is. The idea is[41] to recognize that M with its G-action is the equivariantly canonical object that naturally arises, and on it there is a natural cosheaf of spectra which is $L(G_m)$, the L-group of the isotropy group at that point.

An analogous point is this: suppose we are interested in $S^{\mathrm{p}}(M - A)$ where M is a compact manifold and A is a subspace,[42] then the normal invariants would *seem* to be the invariants of the hard-to-understand object $H_n^{\mathrm{lf}}\big(M - A; L(e)\big)$. However, thanks to excision, this group is isomorphic to the relative group $H_n\big(M, A; L(e)\big)$. This latter group has much better functoriality. If the codimension of A is at least 3, then (modulo K-theoretic issues) $L_n^{\mathrm{p}}(M - A) \cong L_n(\pi_1 M, \pi_1 A)$, also a group with much better functoriality.

39 It is a little tricky trying to relate proper L-groups to more ordinary ones of groups (or rings). For a noncompact manifold with a simply connected end, the proper (h-)theory will be the reduced L^h-group of the interior. If W were $N \times \mathbb{R}$, then it would be $L^{\mathrm{p}}(N)$ (with a shift, and here "p" means the algebra is based on projective modules, rather than on free modules). A key important case is the π–π case, where the proper L-group vanishes.

40 Actually, the ideas of tangentiality that we discussed in §4.6 could allow us to put a boundary on, and work with, ordinary surgery of manifolds with boundary (recognizing the non-uniqueness of the boundary that we have put on). But we will not burden our discussion with this.

41 This idea seems to have first been enunciated in algebraic K-theory by Quinn (1985b) in thinking about the K-groups of crystallographic groups. I was led to it by thinking about what an equivariant Novikov conjecture should say, and being inexorably led to the equivariant K-theory as the home of the equivariant signature operator – which also is essentially the same modification.

42 Note that one does not need a submanifold.

As a consequence, although the space $M-A$ (importantly its homotopy type) depends on the exact embedding of A in M, $S^{\mathrm{p}}(M-A)$ actually only depends[43] on the homotopy class of the inclusion map $A \to M$.

Back to our situation, we can write the homology group as $H_n\big(M/G; L(G_m)\big)$. By doing this, rather than having a point with $L(e)$ as the relevant coefficient at the singularity, we put an $L(G_m)$ *there*, which replaces the $L(G_m)$ modification to $L(\Gamma)$ that takes place in the proper L-group. In short, the sequence becomes[44]

$$\to L_{n+1}(\Gamma) \to S^G(M) \to H_n(M/G; L(G_m)) \to L_n(\Gamma).$$

There are a number of ways of making this precise. On the face of it, the middle term doesn't quite make sense – G_m is actually a conjugacy class of subgroups, rather than a subgroup. From a stratified point of view, it should be viewed as the local fundamental group of the pure stratum near the point; in Davis and Lück (1998) a general theory is developed perfectly adapted to group action purposes, and essentially one uses the *einsatz*:

$$L(G_m) = L^G(G/G_m).$$

Note that G/G_m is a sensible thing to look at: it is the orbit corresponding to the given point in the orbit space.

The good news is that, with these modifications, one actually has a valid calculation (with the $-\infty$ decoration) for G-actions on M, aspherical actions such that there are no codimension ≤ 2 situations (note that the singular set below is the set of all points whose isotropy group is nontrivial):

$$\to L_{n+1}(\Gamma) \to S^G(M, \text{ rel sing }) \to H_n\big(M/G; L(G_m)\big) \to L_n(\Gamma),$$

and

$$H_n(M/G; L\big(G_m)\big) \cong H_n^{\Gamma}\big(\underline{\mathrm{E}\Gamma}; L(?)\big),$$

where $\underline{\mathrm{E}\Gamma}$ denotes the universal space for proper Γ actions. More precisely, $\underline{\mathrm{E}\Gamma}$ is a space that has a proper action of Γ and furthermore, given any X with proper Γ action, there is an equivariant map $X \to \underline{\mathrm{E}\Gamma}$; moreover, this map is unique up to homotopy. (Note the similarity to $\mathrm{E}\Gamma$ which has a similar characterization, except that only free Γ spaces are used.) We can characterize $\underline{\mathrm{E}\Gamma}$ can be characterized as a proper Γ-space so that for all finite subgroups G of Γ the fixed set $\underline{\mathrm{E}\Gamma}^G$ is contractible.

Once we reach this point, the formula for $\mathrm{Wh}^{\mathrm{top}}$ in the general case has an entirely similar description. There is a very interesting point here – not visible

[43] This is indeed true, although, obviously, the above heuristic does not give a proof of this.
[44] We ignore here the tacit use of the fact that $S(*)$ is trivial.

when M is equivariantly aspherical, but of great use in trying to apply the equivariant h-cobordism theorem to more general G-manifolds.

The first point is that $\mathrm{Wh}^{\mathrm{top}}$ (even relative to singularities) depends on more than the fundamental group of the top stratum. This is pretty clear: for a G-action on $M \times X$ for a free simply connected G-manifold X, the top stratum has the same fundamental group as that for M – indeed, it is all top stratum, since $M \times X$ has a free action, so its Whitehead group is $\mathrm{Wh}(\Gamma)$. However, this point can be absorbed in the statement that this Whitehead group depends on the "equivariant fundamental group," which will then include the fact that, for all nontrivial subgroups of G, the fixed set is nonempty.

The second point is that it does not depend on more than this (i.e., the equivariant fundamental groupoid). This is truly remarkable and is a consequence of a theorem of Carter (and one of Hopf):

Let's think about an example – say M a manifold with a semi-free G-action with fixed set F. We have the sequence

$$H_0\big(\mathrm{F}; \underline{\mathrm{Wh}}(G)\big) \to \mathrm{Wh}(\Gamma) \to \mathrm{Wh}^{\mathrm{top},G}(M\mathrm{rel}\mathrm{F}) \to H_0(\mathrm{F}; \underline{K_0}(G)) \to K_0(\Gamma).$$

This is because the h-cobordism that we are considering on the top pure stratum is controlled over $\mathrm{F} \times [0,1]$ and controlled K-groups form a homology theory (see §§4.8 and 5.5.1).[45] To unpack this a little bit, the term $H_0\big(\mathrm{F}; \underline{\mathrm{Wh}}(G)\big)$ can be computed using an Atiyah–Hirzebruch spectral sequence whose E^2 term involves things like $H_0(\mathrm{F}; \mathrm{Wh}(G)), H_1(\mathrm{F}; K_0(G)), H_2(\mathrm{F}; K_{-1}\big(G)\big)$ and so on.

The statement we made about the Whitehead group only depending on the equivariant "fundamental group" is therefore surprising because H_2 and higher all depend on F, not just its fundamental group. The reason that this statement is true is because of two facts:

(1) $K_{-i}(\mathbb{Z}G) = 0$ for $i > 1$ (Carter's vanishing theorem, Carter, 1980).
(2) $H_2(X) \to H_2(\pi_1(X))$ is surjective for all X (Hopf's theorem).

Naturality tells us that the part of $(\mathrm{Wh}\Gamma)$ coming from $H_2(\mathrm{F}; K_{-1}(\mathbb{Z}G))$ factors through $H_2\big(\pi_1(X); K_{-1}(\mathbb{Z}G)\big)$, but Hopf tells us that it actually always hits all of it.

Carter's theorem is the result of computation – there is no known purely conceptual explanation for this vanishing. Indeed, given current knowledge, one could conjecture[46] that its statement is true for all groups, not just finite ones.

[45] The infinite processes we can use to kill elements of the Whitehead group need to be controlled over F, which puts a condition on them (i.e. they are not arbitrary elements of $\pi_1(\mathrm{F}) \times G$ – as we discussed in seeing that Nils enter).

[46] And this is a part of the Farrell–Jones conjecture that we will discuss later.

Hopf's theorem is quite simple: the Eilenberg–Mac Lane space $K(\pi_1(X), 1)$ can be obtained by attaching 3-cells and higher to X. (This argument was not actually available to Hopf, and, indeed, he needed to give a definition of $H_2(\pi_1(X))$ without having the concept of an Eilenberg–Mac Lane space!)

Now, let us turn to the construction of some counterexamples to the equivariant Borel conjecture as we have rephrased it: some equivariant simple homotopy equivalences to G acting on M which are not equivariantly homotopic to a homeomorphism. Indeed, the original examples of Cappell can be turned into such examples, albeit with nondiscrete singular set. Using the calculations of Connolly and Davis (2004), we can get similar examples with a discrete singular set, but for non-orientation-preserving actions.

Consider an affine involution on a torus with fixed points: such is necessarily of the form $\mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3$ where the first torus has a trivial action, the second is a product of the "complex conjugation action" (thinking of the circle as unit complex numbers) some number of times, and the final torus is the interchange of pairs of factors. We just use this to set notation.

Let Γ denote the group $\pi_1 \rtimes \mathbb{Z}$. This is the group that acts on the universal cover $\mathbb{R}^d$. Note that Γ always contains a subgroup isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. (Frequently it contains a subgroup like this that splits off, in which case, more transparently, $L(\mathbb{Z}_2 * \mathbb{Z}_2)$ is a split summand of $L(\Gamma)$.) This will be the source of nonrigidity.

Suppose that we are in the situation of an action of type $\mathbb{T}_2$, i.e. with isolated fixed points. In that case, indeed $\mathbb{Z}_2 * \mathbb{Z}_2$ is a split summand of Γ. (Here the $\mathbb{Z}_2$s must be given the orientation character of the action of the involution on M.)

We can act on the proper structures $S^{\mathrm{p}}((M^d - \mathrm{F}/\mathbf{Z}_2))$ by any element of $L_{d+1}(\Gamma)$. We shall use the nontrivial elements of $L_{d+1}(\mathbb{Z}_2 * \mathbb{Z}_2)$ coming from UNil. Cappell's elements live in $L_2(\mathbb{Z}_2 * \mathbb{Z}_2)$ with an orientation-preserving action, so it would be necessary to cross with a circle to act by these, but the elements constructed in Connolly and Davis (2004) can be used even in the isolated fixed-point situation.

We claim that the result of such an action produces a new proper structure, and indeed a new equivariantly homotopy-equivalent action. The proof of this is a straightforward application of functoriality:[47] the point being that these elements of $L_{d+1}(\mathbb{Z}_2 * \mathbb{Z}_2)$ survive the map $L_{d+1}(\mathbb{Z}_2 * \mathbb{Z}_2) \to L_{d+1}(\mathbb{Z}_2 * \mathbb{Z}_2, \mathbb{Z}_2 \coprod \mathbb{Z}_2)$.

Actually, we can produce similar actions on hyperbolic manifolds. In order to do so, we note a key trick for showing that UNil groups split the L-groups, a trick that is similar to other transfer devices used in Chapter 4. Suppose, for concreteness, we have an involution on a hyperbolic manifold M. Suppose that γ is an invariant geodesic for the involution. Then there is a cover of M

[47] The original proof of this was by a counting argument, and just showed that this proper structure set was an infinitely generated group, but did not control individual elements.

associated to this subgroup of the fundamental group, and the involution lifts to that cover.

Note that there is a normal exponential isomorphism Exp: $N\gamma \to \mathbb{H}/Z$, where $\mathbb{H}$ is the hyperbolic space, and Z is the group acting by translation associated to the geodesic γ. The normal bundle $N\gamma$ can be split as a product of trivial Euclidean bundles according to the eigenspace decomposition of the involution. The inverse of this map is Lipschitz (because of nonpositive curvature) and produces a map $S^{G,\mathrm{Bdd}}(\mathbb{H}/Z) \to S^{G,\mathrm{Bdd}}(\mathbb{R}^d/Z)$. We can split off the trivial summand, and then get a map map $\to S^{G,\mathrm{Bdd}}(\mathbb{R}^k/Z)$. where the action on the $\mathbb{R}^k$-direction is antipodal. This last structure set can easily be computed: at the infinity from the $\mathbb{R}^k$-direction, we have arbitrary control (by rescaling, since we are now in Euclidean space), and the action is free. The two fixed points can be deleted, at the cost of allowing proper control in those directions, but that mods out by $L_{d+1}(\mathbb{Z}_2 \coprod \mathbb{Z}_2)$. In any case, the UNil elements do survive.

The methods of constructing hyperbolic manifolds using quadratic forms explained in Chapter 2 give an ample supply of involutions to which to apply this construction.

With more effort, we are even led to speculate (and this is a theorem modulo the Farrell–Jones conjecture that we will get to later) that, in this case, the equivariant structure set is a sum of contributions associated to invariant unions of closed geodesics. (Free unions, though, contribute nothing, as indeed do ones where only odd-order isotropy arises. Indeed, a bit of thought reduces to geodesics that are invariant under some nontrivial involution.)

6.6 Generalities about Stratified Spaces

In §6.5 we dealt mainly with the situation of an isolated singular set, so that the quotient spaces of these group actions could be thought of as noncompact manifolds with some ends compactified by gluing in points. What happens when the singular set is higher-dimensional? One approach is via considering the quotients as *stratified spaces*.

For the purposes of the rest of this chapter, we will see that there is a reasonable classification theory for stratified spaces with respect to *stratified homeomorphism*, i.e. within a (simple) stratified homotopy type. Unfortunately, this rarely[48] will coincide with what one is interested in, in the situation of

[48] Or, fortunately, this occasionally will coincide with what we are interested in.

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equivariant homotopy types. Consequently, our later sections will deal with the implications of the tension between "stratified" and "equivariant."

A *stratified space* X is a space with a filtration $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0$ by closed subsets (called *strata*, *stratum* in the singular). We shall always assume that, for each i, the *pure stratum* $X^i = X_i - X_{i-1}$ is an i-dimensional (ANR homology) manifold. Beyond this, there are various theories about how the strata are demanded to fit together. We shall use the notion of homotopically stratified spaces, introduced by Quinn (1988) (see also Hughes, 1996). The precise general definition need not bother us here – instead, we shall give some examples that are and some that are not.

Example 6.26 The one-point compactification, M^+, of a noncompact manifold M, is sometimes a stratified space for us, and sometimes not. The obvious stratification consists of a bottom zero-dimensional stratum consisting of the added point, and the remaining points form the top pure stratum.

The usual strong stratifications (such as Whitney stratifications) would require the existence of a compactification of M to a manifold with boundary: M^+ could then be viewed as the result of shrinking this boundary to a point, or gluing on the cone of the boundary to this compactification.

Unfortunately, this manifold with boundary is *not* a topological invariant of this situation (even when it exists). There is an indeterminacy associated to $\mathrm{Wh}(\pi_1\partial)$.

Our assumption is that M is a *tame* in the sense of Siebenmann (1965). This is equivalent to the condition that $M \times S^{-1}$ has a compactification as a manifold with boundary. Essentially this condition means that complements of sufficiently large compact sets can be "pulled" closer from infinity. We refer to Siebenmann, and the predecessor work of Browder and Livesay (1973) for more information, and, in particular, how to recognize this.

However, many noncompact manifolds are not allowed (to be the nonsingular part of a compact stratified space): for example, a typical infinite cover of a compact manifold (such as infinite abelian covers of a surface of genus greater than 1), or any manifold with infinitely generated fundamental group or homology.

Example 6.27 A manifold with a *nice* submanifold (W, M) can be viewed as a two-strata space, with bottom stratum M, and the ambient manifold being the top stratum. The condition of tameness follows from the condition that M is *locally flat* in W, i.e. that each point in M has a neighborhood in W which is isomorphic to $(\mathbb{R}^w, \mathbb{R}^m)$; this does not guarantee that M has a topological bundle neighborhood.

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There are some other submanifolds that are not locally flat, but still give us homotopically stratified spaces – they all have a nice homogeneity property. The basic source is the Cannon–Edwards theorem that the second suspension of any homology sphere Σ is a topological sphere[49] (see Daverman, 2007). As a result, if $W = M \to c\Sigma$ (or, more generally, the mapping cylinder of any Σ (block) bundle over M), one obtains a topological manifold in which M is embedded in a quite nontrivial, yet homogeneous, way. This is a rather exotic embedding from the conventional point of view, yet it gives a reasonable homotopically stratified space.

Example 6.28 If G is a finite group acting on M, a sufficient condition for M/G (stratified by orbit types) to be homotopically stratified is that, for $H \subset K$, the embedding of $M^K \subset M^H$ should be a locally flat embedding of manifolds. For this situation, the homogeneity property is quite remarkable: it includes some of the one-point compactification examples mentioned above! As in Example 6.26, there does not have to be a closed invariant "regular neighborhood" of the fixed set. The following result gives an example of how such actions occur naturally in converses to Smith theory.

Theorem 6.29 (See Weinberger, 1985a) *A submanifold Σ of the sphere is the fixed set of a (locally linear[50]) Q_8-action iff Σ has codimension a multiple of* 4 *and is a $\mathbb{Z}_2$-homology sphere. The top stratum of the quotient can be compactified as a manifold with boundary iff*

$$\prod \#\mathrm{tor}(H_i(\Sigma;\mathbb{Z})) \equiv \pm 1 \bmod 8.$$

In this case, there is always a PL *locally linear action.*

The group Q_8 can be replaced by any other group that can act freely on the sphere, but then the conclusions have to be modified. (The simplest modification is that for cosmetic reasons we wrote down a product of numbers that really should be an alternating product.) This is the simplest case where there is a nontrivial restriction on the homology of the fixed set that follows from algebraic K-theory. The actions can always be made PL locally linear in the complement of a point – indeed, that is a natural feature of their construction – the numerical obstruction is a Wall finiteness obstruction that doesn't arise in the noncompact setting. The local linearity at ∞ is a remarkable consequence of general features of homotopically stratified spaces (see Quinn, 1970).

[49] The deep part is that it's a manifold at all. Identifying the manifold with a sphere then follows from the Poincaré conjecture.

[50] If you wish.

Example 6.30 (Supernormal spaces) These are spaces modeled by a strengthening of the condition of normality that occurs in algebraic geometry. We mention them because their theory is more elementary than the general situation, but is quite beautiful and is a good place to start.

A stratified space X is *supernormal* if each *pure* stratum is dense in the corresponding closed stratum,[51] and near each point x of X_k and any $\varepsilon > 0$, there is a $\delta > 0$ such that any 1-manifold[52] in X^r for $r > k$, within δ of x, is null-homotopic within a ball of radius ε.

Note that any manifold with boundary can be thought of as a supernormal stratified space with two strata.

For an embedding of closed manifolds (W, M), supernormality is (by a nontrivial theorem; see Daverman and Venema, 2009) exactly the condition that one is in codimension greater than 2 and the submanifold is locally flat. If M is a subpolyhedron, supernormality (from the point of view of the top stratum) would follow if the Hausdorff codimension is greater than 2.

The Whitehead group for supernormal spaces[53] is just $\bigoplus \mathrm{Wh}(\pi_1(X_i))$ (i.e. just like the PL situation).

The surgery theory describes the *structure sets* (which are actually groups) $S(X) = \{(X', f)\colon f\colon X' \to X$ is a stratified homotopy equivalence up to stratified s-cobordism$\}$.[54] If Y is a union of strata of X, then we can also form $S(X \operatorname{rel} Y)$ which is defined in the same way, but we insist that $f|_{Y'}$ is already a homeomorphism.

A *stratified map* $f\colon X' \to X$ is a map that preserves *pure* strata, i.e. so that $f(X'^r) \subset X^r$. A *stratified homotopy equivalence* is a stratified map $f\colon X' \to X$ for which there is a stratified "inverse" $g\colon X \to X'$ such that the composites fg and gf are both stratified homotopic to the identity.

Theorem 6.31 (Cappell and Weinberger, 1991a) *If X is supernormal of dimension $n > 4$, with Σ its singularity set (i.e., the complement of the top pure stratum), then $S(X \operatorname{rel} \Sigma) \cong S^{\mathrm{alg}}(X)$, where $S^{\mathrm{alg}}(X)$ denotes the fiber of the assembly map – i.e. what surgery would predict had X been a closed manifold:*

$$\to L_{n+1}(\pi_1 X) \to S^{\mathrm{alg}}(X) \to H_n(X; L) \to L_n(X).$$

[51] This is just a convenience to ensure that our picture of the singularity set to correspond to the largest proper closed stratum.

[52] We cannot just use loops because we want to require normality in this definition. Normality is essentially the same condition with S^0s replacing the 1-manifolds in this definition.

[53] Ignoring low-dimensional difficulties.

[54] The equivalence relation can be taken to be homeomorphism if we only allow manifolds as strata. However, for our calculation to be correct as stated, it is necessary to allow ANR $\mathbb{Z}$-homology manifolds as strata, and then an s-cobordism theorem is not available. This is related to the discussion of functoriality in §4.7.

So the formal structure set of algebraic surgery theory has an interpretation even for (certain) non-manifolds (without using the artifice of thickening the space to be a manifold). In particular, the Borel conjecture for $\pi_1(X)$ then implies (and clearly is implied by!) the following:

Conjecture 6.32 *If X is an aspherical supernormal space, then any stratified space stratified-homotopy-equivalent to $X \operatorname{rel} \Sigma$ is homeomorphic to it.*

This is equivalent (as surely one would guess) to the statement that, assuming asphericality, $S(X) \to S(\Sigma)$ is an isomorphism. In this view, aspherical manifolds are topologically rigid because they have no singularities!

Remark 6.33 For the conclusions about the $\operatorname{rel} \Sigma$ theory, we only need the simple connectivity condition occurring in the definition for the parts of the top pure stratum X^n near x (i.e. not on the intermediate strata). This follows from "continuously controlled at infinity surgery" (Pedersen, 2000).[55]

The absolute theory is necessarily more complicated. For example, if M is a manifold with boundary ∂M, we can view it as a 2-stratum supernormal stratified space X. In the above notation, $S(X)$ then corresponds to the group $S(M, \partial M)$ (while $S(X\text{rel}, \Sigma)$ is $S(M)$). The strata interact[56].

In Example 6.27, there is a "forgetful" map $S(W, M) \to S(W)$. *This is far from trivial.* (On the other hand, the forgetful – or, better, restriction – map $S(W, M) \to S(W)$ is tautologous.) The reason for this is that the closed stratum of a manifold that is a homotopically stratified space that is stratified homotopy-equivalent to (W, M) is actually a manifold, as we now explain.

It is quite easy to see that W' is an ANR homology manifold and that it has the disjoint disks property (DDP) (see §4.7 for this and the rest of this paragraph).[57] That it is a manifold if the top pure stratum is a manifold requires the theorem of Quinn that gives it a resolution, and then Edwards's theorem that DDP is then sufficient for manifoldness.

Now we can combine the maps $S(W, M) \to S(W) \times S(W)$ and the theorem above directly implies that this is an isomorphism if the codimension of M in W is at least 3.

Wonderful: we've calculated something. But what does it mean?

[55] This is not very hard, and the reader might want to try their hand at verifying this.

[56] Unlike the situation in K-theory, where the stratified object decomposed into pieces. That L-theory works differently could have been predicted by the fact that the involution given by turning h-cobordisms upside down does not preserve this decomposition. But, this is obvious, anyway, as above.

[57] So, if the reader had been content to accept that we could define $S(W)$ using homology manifolds, and that this only differs by some $\mathbb{Z}$s from the one defined using topological manifolds, then the forgetful map was easy to define!

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It certainly includes the statement that if $M \subset W$ is a locally flat submanifold of codimension at least 3, then any manifold M' homotopy-equivalent to M embeds in any manifold W' homotopy-equivalent to W.[58] Moreover, this embedding "has the same homotopy theory" as that of $M \subset W$. For example, $M' \subset W'$ is homotopy-equivalent to $M \subset W$, as a pair, and $W' - M'$ is homotopy-equivalent to $W - M$.

The first observation is a completely natural (but perhaps surprising) statement to someone studying embedding theory, but the second one, while strengthening our conclusion (the embedding we produce of $M \subset W$ has even more properties than we might have asked for), is not particularly a natural one to someone who studies embeddings for a living.

For example, consider the two (homotopic) embeddings of $\mathcal{S}^{k-1} \vee \mathcal{S}^{k-1}$ in $\mathcal{S}^{2k-1}$, according to whether the $\mathcal{S}^{k-1}$ are linked as in the Hopf link, or just embedded in two disjoint disks: the first has complement homotopy-equivalent to $\mathcal{S}^k \times \mathcal{S}^k$; and the other, $\mathcal{S}^k \vee \mathcal{S}^k \vee \mathcal{S}^{2k}$, is completely different.[59]

We also note that there is also a third condition that comes out of isovariant homotopy equivalence regarding the normal bundles[60] of the submanifolds. More precisely, associated to a codimension-c locally flat[61] embedding there is a spherical fibration $S^{c-1} \to E \to M$. These spherical fibrations must match for M and M'. The proof of this goes by comparing the neighborhood systems near M and M' that are mapped to each other by f and g.

These three conditions serve to define the notion of a *Poincaré embedding*. A Poincaré embedding of M in W consists of a triple $\big((X,E),\pi,f\big)$, where (X,E) is a pair, $\pi\colon E \to M$ is a spherical fibration with fiber S^{c-1} and, denoting the mapping cylinder of a map by Cyl, $f\colon X \cup \mathrm{Cyl}(\pi) \to W$ is a homotopy equivalence. The stratified map $(W,M) \to (W',M')$ gives us an isomorphism of the underlying Poincaré embeddings, and the theorem that $S(W,M) \to S(W) \times S(M)$ is an isomorphism says that there is a unique isotopy class of embedding of M in W associated to any Poincaré embedding.

In general, we have to be careful in thinking through what we get out of a

58 Actually, we should use simple homotopy equivalence. However, there are straightforward arguments that allow us to deduce the homotopy equivalence result from the above.

59 The complements do have the same stable homotopy types, but this does not suffice for the application of (stratified) surgery.

60 We are abusing terminology here, since locally flat submanifolds don't necessarily have bundle neighborhoods.

61 With a little effort, this spherical fibration can be associated to non-locally flat embeddings. Combining this observation with the result about the stratified structure set quickly gives a proof (in codimension greater than 2) that topological embeddings can always be approximated by locally flat ones: see Daverman and Venema (2009), for a thorough discussion of such results. (In codimension 2, this is not true according to examples of Matumoto; the codimension-1 result is true, but would involve a little more work to deduce from these methods, since the embedding problem does not reduce to homotopy theory.)

calculation of $S(X)$ – it gives some useful information, but frequently not all the information we want: for instance, it won't classify for us the embeddings in a given homotopy class.

A full discussion of how to calculate $S(X)$ for a stratified X is outside the scope of our current exposition, but, *ignoring algebraic K-theory issues*, we can give a quick summary.

(1) There are spectra $\mathbf{L}(X)$ associated to a stratified space X; one has that $\pi_i\mathbf{L}(X) = \mathbf{L}(X \times D^i$relative to the boundary).[62] If Y is a union of closed strata of X, there is a restriction map $\mathbf{L}(X) \to \mathbf{L}(Y)$, whose fiber is $\mathbf{L}(X \mathrm{rel}\, Y)$. If $X \subset Z$ is an open inclusion then there is an induced map of $\mathbf{L}(X \mathrm{rel}\, \infty) \to \mathbf{L}(Z)$.
(2) $\mathbf{L}(X_n \mathrm{rel}\, X_{n-1}) \cong \mathbf{L}_n\big(\pi_1(X^n)\big)$.
(3) There is an exact sequence $\cdots \to S(X \mathrm{rel}\, Y) \to H(X; \mathbf{L}(\mathrm{loc}(X \mathrm{rel}\, Y)$, where H is the spectral cosheaf homology of the cosheaf that associates to a small open set U in X, the $\mathbf{L}$-space of $(U, U \cap Y \mathrm{rel}\, \infty)$.

Items (1) and (2) together say that $\mathbf{L}(X \text{ rel } Y)$ is built up out of the $\mathbf{L}$-spectra of the fundamental groups of the pure strata of X that are not in Y. They do not say exactly how they fit together – this is the issue of interaction mentioned above – and I will ignore it here, although *for our situation of discrete group actions*,[63] it turns out that there is no interaction after inverting 2 (see Chapter 13 of Weinberger (1994)).

Note the special case where (X, Y) is a supernormal pair, with Y the singularity set of X. In that case, all of the $\mathbf{L}(\mathrm{loc})$ are just $\mathbf{L}(R^x \text{ rel } \infty)$ (induced locally by the inclusion of a neighborhood of any manifold point in the neighborhood). In that case, the cosheaf homology is essentially the ordinary[64] $H_x(X; \mathbf{L})$ that arises in surgery theory. The global $\mathbf{L}$ term $\mathbf{L}(X \mathrm{rel} Y)$ is just the L-group of the top pure stratum, which has fundamental group $\pi_1(X)$ by Van Kampen's theorem. This explains (aside from K-theory[65]) Example 6.30.

If X is a manifold with boundary, then the homology term has the usual spectrum at the interior points, but is contractible (by the π–π theorem for the trivial group π) at the boundary points. If we work relative to the boundary,

[62] The L-groups of stratified spaces are sometimes written $L^{\mathrm{BQ}}(X)$ in recognition of the paper Browder and Quinn (1973) that initiated their study.

[63] Acting preserving orientations.

[64] There is a twist in this group when the top pure stratum of X is nonorientable. The reason that these cosheaf homology groups become more conventional (generalized) homology groups is because of the great rigidity that $\mathbf{L}$ cosheaves have, referred to as "flattening" in Weinberger (1994).

[65] In this case $\mathrm{Wh}^{\mathrm{top}}(X \mathrm{rel} \Sigma) = \mathrm{Wh}(\pi_1 X)$, so the K-theory agrees exactly with the manifold situation.

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then the spectral term will be $\mathbf{L}(\mathbb{R}^x)$ everywhere, and the global term will be the ordinary relative L-group. Both of these calculations are completely in accord with the classical calculations.

Finally, let us consider Example 6.26. We suppose that $X = M^+$. As we are ignoring algebraic K-theory, we can safely view M as interior$(W, \partial W)$. In that case, $S(X) \cong S(W, \partial W)$. As the structures of a point are contractible (i.e. they form a trivial group), for ease of exposition, we will compute $S(X\mathrm{rel}\,*)$ where $*$ is the compactification point. In that case the global spectral term is $L(\pi_1 M)$ (rather than $L(\pi_1 M, \pi_1 \partial W)$. However, the homology term is different – noting that at the cone point the cosheaf is $L(\pi_1 \partial W)$, it fits into the exact sequence

$$H_x(X, L(\mathrm{loc}(X\mathrm{rel}\,*)) \to H_x(X - *, L\big(\mathrm{loc}(X\mathrm{rel}\,*)\big)$$
$$\cong H_x\big(W, \partial W, L(e)\big) \to L_{x-1}(\pi_1 \partial W).$$

So the homology term absorbs the difference between the absolute and relative global L-groups. *In a stratified space, there is little difference between local and global problems*, i.e. what is local from the point of view of a k-stratum space is global from the point of view of $(k-1)$-strata spaces.

The reader can gain some insight into this surgery sequence by thinking about the PL situation where the strata have regular neighborhoods, and the boundaries block fiber over the previous strata, and then use the theory of blocked surgery to give directly a proof of the "Verdier dual" exact sequence.

Let us now return to the situation of Γ acting properly discontinuously on X a contractible manifold, with all fixed sets contractible locally flat submanifolds, of codimension greater than 2. In that case:

(1) $\mathrm{Wh}^{\mathrm{top}}(X/\Gamma, \mathrm{rel\ sing})$ is the fiber of the assembly map $H(X/\Gamma; K(\Gamma_x)) \to K(\Gamma)$.

This group is thus related to the Nil terms and can vanish but can certainly be nontrivial. This would give one set of obstructions to the equivariant Borel conjecture, had we not already made the assumption that our maps are equivariantly-simple-homotopy equivalences.

Realizing elements of this group, one obtains counterexamples if the boundaries of the appropriate h-cobordism are not homeomorphic; if they are, then one can glue them together and get a counterexample for the group $\mathbb{Z} \times \Gamma$ (i.e. for $\mathcal{S}^1 \times X/\Gamma$).[66]

[66] One can actually see that, for $\mathbb{Z}/p^2$ acting on $\mathcal{S}^1 \times \mathbb{T}^{p(p-1)}$ (here the action is the one associated to the action of $\mathbb{Z}/p^2$ on the torus associated to the ring of integers in the cyclotomic field of p^2 roots of unity), there are a number of Nil terms in the topological Whitehead group, and that, when one realizes an h-cobordism with suitable torsion, the "other end" is not topologically simple homotopy-equivalent to the original manifold. This actually gives an infinite number of examples (for each p).

Now, assume that this vanishes:[67]

(2) $S(X/\Gamma \mathrm{rel\,sing})$ is isomorphic to the fiber of the assembly map

$$H(X/\Gamma; L(\Gamma_x)) \to L(\Gamma).$$

We have seen that this can be nontrivial because of UNil. On the other hand, if, instead of $\mathbb{Z}$ we deal with rings, R, in which 2 is inverted, there are no known counterexamples to an isomorphism statement[68] – this is not so useful directly for classification problems, but it is useful for understanding invariants of manifolds (and group actions).

Notation 6.34 We will usually denote the singularity set of a stratified space by Σ, unless otherwise stated.

In §6.7 we will discuss a form of equivariant Novikov conjecture and some evidence for it. This conjecture does not take into account the Nil and UNil phenomena that get in the way of rigidity as we have seen. It is interesting that these always seem to split off structure sets.

We will follow this with a discussion of the Farrell–Jones conjecture, which gives a specific statement about how all of the Nil and UNil contributions to $\mathrm{Wh}^{\mathrm{top}}$ and S can be explicitly computed in terms of the virtually cyclic subgroups of Γ. This will suffice to give an understanding (at least in theory) of what *isovariant* structures should look like (since vanishing is not always true). Finally, we will return to the equivariant Borel conjecture, and discuss the relation between the difference between equivariance and isovariance and embedding theory.

6.7 The Equivariant Novikov Conjecture

The Novikov conjecture describes the restrictions on the characteristic classes of the tangent bundles of homotopy equivalent manifolds. While it can be phrased as the injectivity of an assembly map, it has other interpretations and implications and analogues, as we saw in Chapter 5. Already in that chapter we discussed some equivariant aspects of the Novikov philosophy, e.g. vanishing of higher A genera for smooth actions of S^1 on the one hand and the higher-signature local formulas for homologically trivial actions on the other.

[67] Note that in the usual Borel conjecture we assume homotopy equivalence, not simple, and we deduce a vanishing of the Whitehead group from the conjecture.

[68] And, indeed, the Farrell–Jones conjecture implies that it is an isomorphism, with $L^{-\infty}$ as the version of L-theory used.

In this section, we will take seriously the issue of how to properly generalize the Novikov conjecture equivariantly. There are several possibilities that interact with each other: wisely did the authors[69] of "An equivariant Novikov conjecture" title their paper.

As in the non-equivariant case, we should be concerned with the issue of tangentiality of (equivariant) homotopy equivalences, and also with the restrictions that can be made on characteristic classes of tangent "bundles" of G-manifolds, as well as assembly maps in L-theory and C^*-algebra theory.

Rather similarly to the classical case, in the situation of equivariant homotopy equivalent compact G-manifolds, the equivariant Novikov conjectures give very similar information away from the prime 2.

We start by noting some of the obstacles to proceeding as we had before:

(1) We needed to put scare quotes around the word "bundles" as we laid out our path two paragraphs ago: in §6.2 we had seen that in the topological setting the tangent theory is not so naturally bundle-theoretic.
(2) Indeed, the local structure of a manifold is simply its dimension; one would want to generalize this to be something like a tangential representation (of the isotropy group) associated to each point. However, we have to confront the phenomenon discovered by Cappell and Shaneson (1981) that for many groups there are (linearly) different representations V and W that are equivariantly homeomorphic. So we don't have the analogue of dimension even if we had bundle theory!
(3) We also have to worry about what characteristic classes we can hope to use: in the topological case, Novikov's theorem on topological invariance of rational Pontrjagin classes led us to use the L-class. What shall we use here?
(4) As a final point to put us in the mood: in §6.6, when we were thinking about the equivariant Borel conjecture, in order to use ordinary manifold surgery techniques, we were led to consider the problem inductively, i.e. assuming that we already had a homeomorphism on the singular set (e.g. on fixed sets of all proper subgroups; abbreviated rel Σ). This led to an assembly map formulation involving

$$H_n(\underline{\mathrm{E}\Gamma}/\Gamma; L(\Gamma_x)) \to L_n(\Gamma).$$

(5) However, we should be interested in formulations that are not rel Σ and also in restrictions on characteristic classes, etc., that are *a priori* not rel Σ, i.e. do not require precise analysis of what is occurring on fixed point

[69] Rosenberg and Weinberger (1990); correspondingly the title of the present section has an unwarranted definite article.

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sets: the many examples we've already seen show that such information is frequently difficult to get.

In achieving all of the above, we will also understand things like the higher-signature localization formula of Chapter 5 as part of this picture. Applications to closed (homology) manifolds will be given in Chapter 7.

Deviating from the way of the wise,[70] we will begin by dealing with the last question first, starting with the simplest situation, the simply connected case. After all, the key class that is relevant for the ordinary Novikov conjecture is the L-class[71] of the manifold M that lies in homology (or $H_m(M; L^*(\mathbb{Z}))$) which itself is a variation of the ordinary signature of the manifold – i.e. it's an encoding of all the signatures of all of the submanifolds of the manifold (taking into account their normal bundles).[72]

The equivariant version is, of course, the G-signature that we first met in §4.10 and again in Chapter 5 when we studied homologically trivial actions. We review some basic aspects of this invariant now[73] as a step towards considering the topological characteristic class theory.

If G acts orientation-preservingly on X^{2k}, an even-dimensional oriented Poincaré complex,[74] the middle cohomology $H^k(X; \mathbb{Q})$ admits a G-invariant $(-)^k$-symmetric inner product pairing.[75] Let's go even further, and consider the situation after extending scalars to $\mathbb{R}$. As such it has some signature-type invariants that occur in the representation theory of G. If k is even, then one chooses a G-invariant positive-definite inner product on H^k and diagonalizes the cup product pairing in terms of this auxiliary pairing. The G-action preserves both, and therefore preserves the positive- and negative-definitive parts, giving an invariant in $\mathrm{RO}(G)$. If k is odd, then one does the same, except that the operator A describing cup product in terms of the auxiliary product is now

[70] See Avot 5:9 (*Ethics of the Fathers*, one of the books of the Talmud) regarding the wise approach to answering a series of questions.

[71] Or, better, the $L^*(\mathbb{Z})$ orientation of a manifold. This is an intrinsic class in $H_m(M; L^*(\mathbb{Z}))$ that defines the Poincaré duality between $H_m(M; L^*(\mathbb{Z}))$ and $[M : \mathbb{Z} \times G/\mathrm{Top}]$ and refines the L-class. Sullivan emphasized that the PL-block bundle away from 2 is a KO[1/2]-oriented spherical fibration, which, away from 2, is this class.

[72] Or all the definable signature-type invariants of all the open subsets of M, in the case of the controlled symmetric signature of M over M.

[73] See Wall (1968) and Atiyah and Singer (1968a,b, 1971) for early references, from different points of view.

[74] Perhaps with boundary; in that case the relevant quadratic form will be singular, and one must mod out by the null vectors, $\ker H^* \to (H^*)^*$, before following the prescription above.

[75] We use rational coefficients, which greatly simplifies our remarks throughout the section, losing information only at the prime 2, thanks to Ranicki's (1979a) localization theory. However, the theory at the prime 2 is indeed much deeper, and much more mysterious, as we shall occasionally indicate.

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skew-adjoint. A positive square root of AA^* gives a canonical complex structure and a representation ρ of G. The G-signature in this case is $\rho - \rho^*$.

One can view this more illuminatingly perhaps by taking the Wedderburn decomposition of the real group ring RG and considering the effect of the anti-involution $g \to g^{-1}$. Up to Morita equivalence one has pieces corresponding to $\mathbb{R}$, $\mathbb{C}$, and **H**. Whereas $\mathbb{C}$ contributes for both k odd and even, $\mathbb{R}$ arises only for k even (every symplectic real vector space, the k-odd situation has a self-annihilating subspace of half the dimension), and the **H** case arises only for k odd (for similar reasons).

It is a nice fact that the maps[76] L(ZG) → L(QG) → L(RG) are also isomorphisms away from the prime 2 (see, for example, Wall, 1968) and computed via these combinations of integer-valued invariants. This means that, for finite groups, *away from* 2, surgery obstructions can be computed as the difference of very simple-minded intrinsic invariants of domain and range (i.e. their G-signatures).

The fact that G-signatures are computed from the action of G on cohomology implies the following strong homotopy-invariance property:

Proposition 6.35 *If $f: M \to N$ is an equivariant map that is a homotopy equivalence, then* $G - \operatorname{sign}(M) = G - \operatorname{sign}(N)$.

A map as in the proposition is called, following Petrie (1978), a *pseudo-equivalence*. It is equivalent to asserting that $f \times \mathrm{id} : M \times \mathrm{EG} \to N \times \mathrm{EG}$ is an equivariant homotopy equivalence. It obviously makes more sense to use the equivalence relation generated by this notion. But in that case, it is exactly equivalent to the homotopy equivalences in the following *pseudo-category*.[77]

Motivation 6.36 Let $G = \mathbb{Z}_2$. A map between G-spaces $f: X \to Y$ might fail to be equivariant, i.e. $f(\mathrm{i}x \ncong f(x)$. If these two maps are not homotopic, then we have no chance of getting (say, up to homotopy) an equivariant map, but if they are homotopic, we are still not done. For example, let F be homotopy between f and $f(\mathrm{i})$. Then $F \circ \mathrm{i}$ is also such a homotopy, and we need F to be homotopic (rel$X \times \{0, 1\}$) to $F \circ \mathrm{i}$. And then that homotopy G must be homotopic to $G \circ \mathrm{i}$, and so on. All of this still won't make you succeed, and you've traded a simple condition of equivariance for an infinite number of homotopies and higher homotopies, and the down-to-earth reader will surely want some justification of this . . . Hopefully the pages that follow will provide some. In any case, this

[76] We can go further and add one more isomorphism to the real K-theory of $\mathbb{C}_{\mathbb{R}} * G$.

[77] The pseudo-category is a category: we use the perjorative "pseudo" to describe the morphisms that are *prima facie* odd. (Of course, mathematics often progresses through non-naive definitions and problems. As Gromov once said, "Naive problems are usually stupid.")

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data is just an equivariant map from $S^\infty \times X \to Y$. The point being that blowing X up in this way makes it slightly easier to build maps (at least in theory).

Definition 6.37 The *pseudo-category* of G consists of G-spaces as objects (just like the usual equivariant category) but it has more morphisms. A morphism from X to Y consists of a G-equivariant map $\mathrm{EG} \times X \to Y$. (This can be thought of as an EG-parameterized family of maps from X to Y that satisfy certain intertwining conditions with respect to the G-action and the parameter.)

A morphism in this category is thus an element of the *homotopy fixed set* $\mathrm{Map}[X : Y]^{\mathrm{hG}}$ (for the reader who remembers this notion from §4.9); this should be compared to the usual equivariant category where morphisms are elements of the *usual fixed set* $\mathrm{Map}[X : Y]^G$.

One reason that this category is important is because pseudo-equivalences arise frequently. For example, any G-action on a contractible space is pseudo-equivalent to the action of G on a point (clearly!), but the fixed sets of such G-actions can be quite different for nontrivial subgroups of G, and thus these actions would not be equivariantly homotopy-equivalent.

If G acts on a space X then there is an equivariant fundamental group associated to the action:

Proposition 6.38 (Definition) *If G acts on a space X then the equivariant fundamental group associated to the action is given by the group of all lifts of the elements of G to the universal cover of X. This group, Π, fits into an exact sequence $1 \to \pi_1 X \to \Pi \to G \to 1$. It is an invariant of the pseudo-equivalence class of the group action on X (if G is discrete, it is the fundamental group of the Borel construction $X \times_G \mathrm{EG} = (X \times \mathrm{EG})/G$).*

Proposition 6.39 *If G is a finite group and acts on an M-manifold, then we can define an invariant $\sigma^*_G(M) \in L^*(\mathbb{Q}\Pi)$. It is a pseudo-equivalence invariant.*

Indeed, the Borel construction $M \times_G \mathrm{EG}$ is a $\mathbb{Q}\Pi$-Poincaré complex,[78,79]

Now, the ideas of controlled topology that we have discussed earlier assert that this invariant disassembles over M/G.

[78] First, note a piece of good news. Since $1/2 \in \mathbb{Q}$, there is no difference between symmetric and quadratic L-theories. I should point out, though, a slight subtlety. The finiteness condition on this chain complex is homological, so one only gets a projective chain complex. (If there is a G-invariant triangulation, this is direct, because all orbits are permutations, and permutation complexes are projective over $\mathbb{Q}G$. This suggests the true statement that for free actions the chain complex is defined in L^h and that one doesn't ever need all projective modules.)

[79] Probably one should point out that one need only invert in the coefficients the orders of the nontrivial isotropy groups. One might also point out that, using intersection homology sheaves, one can define these invariants, for example, for complex varieties with action.

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Proposition 6.40 *There is an assembly map*

$$H_m(M/G; L(\mathrm{QG}_x)) \to L_m(\mathbb{Q}\Pi)$$

and $\sigma^*_G(M)$ *canonically lifts to the domain of this assembly map. We denote this lift by* $\Delta(M)$.

This is completely analogous to the non-equivariant situation (aside from the coefficients being $\mathbb{Q}$ rather than $\mathbb{Z}$). To continue the analogy to the non-equivariant case, we should study the functorialities of this map and factor it through

$$H_m(M/G; L(\mathbb{Q}G_x)) \to H_m(\underline{\mathrm{E}\Pi}/\Pi; L(\mathbb{Q}\Pi_x)) \to L_m(\mathbb{Q}\Pi).$$

This is indeed possible and is part and parcel of the interpretation of the left-hand side as controlled algebraic Poincaré complexes and the arrows as change of control spaces (or the forgetful map).[80]

Now we have a wonderful coincidence. The domain and range of this assembly

map are (away from 2) the same as for the assembly map that arises in the calculation of $S(M/G\,\mathrm{rel}\,\Sigma)$ considered in §6.6.

Corollary 6.41 *Away from* 2, $\Delta(M)$ *is a pseudo-equivalence invariant iff it is an isovariant homotopy-invariant (for maps that are homeomorphisms on the singular set!).*

This is highly significant, because, unfortunately, we do not have a well-understood theory of pseudo-equivalence (especially in the topological category). In addition, this corollary reduces a pseudo-homotopy invariance statement to a *tangentiality type* result in the equivariant Borel conjecture (see Ferry *et al.*, 1988).

Another important point that is almost implicit within the corollary is that $S(X, \mathrm{rel}\Sigma)$ is a summand of $S(X)$ (away from 2) for finite group actions. We record a somewhat more general statement that is proven by induction.

Theorem 6.42 *If G is a finite group tamely and acting orientation-preservingly*[81] *on a manifold M, then, inverting* 2, *the isovariant structure groups decompose:*

$$S(M/G) \otimes \mathbb{Z}[1/2] \cong \bigoplus S(M^H/(\mathrm{N}H/H), \mathrm{rel}\,\Sigma), \otimes\mathbb{Z}[1/2].$$

80 Thus the rel Σ isovariant structure sets have an equivariant functoriality (for finite groups acting orientation-preservingly) like manifold structure sets. The theory of functoriality for the isovariant structures themselves is much more complicated – Cappell, Yan, and I have been thinking about this from a number of points of view for years, with only fragmentary results.

81 This includes the hypothesis that the fixed sets of all subgroups are orientable.

This decomposition is frequently true integrally, especially for odd-order groups, but it is not true in general. The right-hand sum is over components of strata of the quotient – so we would not count twice a component fixed by a subgroup that is also fixed by a larger subgroup.

The question of integral versions of this splitting is an important one. The difference between "yes" and "no" is often an element of $\mathbb{Z}_2$!

When one has an integral splitting, one knows that a "replacement theorem" holds: any manifold homotopy-equivalent to the fixed set is the fixed set of some action on an equivariant homotopy-equivalent manifold (and similarly for other strata). Sometimes it is even possible to arrange that the new action is on the same manifold, although there are situations (e.g. for some orientation-preserving involutions) for which replacement holds, but this strong form fails.

In any case, the results we have seen in Chapter 5 about higher-signature formulas for S^1-actions show that in codimension $2 \bmod 4$ replacement does not hold for *rational reasons*, and that strong replacement doesn't hold if the fixed set has codimension $0 \bmod 4$. So, our discussion has really required the finiteness of G.

The analysis of the group structure on these structure sets (for the finite case) is also facilitated by the following:

Observation 6.43 $S(M/G \operatorname{rel} \Sigma)$ is a (graded) module over $L^*(\mathbb{Q}G)$.

In particular we can view it as a module over $\mathrm{RO}(G)$ – and therefore apply the ideas of the localization theorem in equivariant K-theory (Atiyah and Segal, 1968).[82] For example:

Corollary 6.44 *If the action of G on M is free and pseudo-trivial, then, assuming the Novikov conjecture, the higher signature of M vanishes.*[83]

Very similar reasoning would give the localization of higher signatures to *twisted higher signatures* of fixed sets (where we twist the L-class of the fixed set by an appropriate characteristic class of the equivariant normal bundle; however, one would not obtain immediately the relevant converse statements from Chapter 5).

The reader should be able to deduce some non-pseudo-trivial results from the equivariant conjecture.

There are several directions in which we can go next, and we will go in many of them!

We should discuss the prime 2 and also generalize the Novikov conjecture

[82] Compare our discussion of ρ-invariants in §4.10.

[83] Indeed, if there is some element whose fixed-point set is empty, one gets the same conclusion (at least after inverting several primes).

from groups to metric spaces – after all, that was the route we had taken to Novikov's theorem on topological invariance of rational Pontrjagin classes, and we shall see that here it rewards us similarly – and we should discuss the index-theoretic version of these (knowing that the prime 2 will be a place of divergence as always), anticipating, at least, applications to other operators.

The characteristic class that we have introduced here, the equivariant controlled symmetric signature in $L^*(\mathbb{Q}\Pi)$, is certainly not the right thing to do. In the non-equivariant setting we would surely have wanted a $\mathbb{Z}$. However, I do not know any one method for defining the most refined "intrinsic" characteristic classes for group actions without taking their fixed sets, isotropy structure, and so on into account. This feels somewhat related to the realization problem: for manifolds all of $L^*(\mathbb{Z})$ arises as a signature, but none of the torsion elements of $L^*(\mathbb{Q})$. However, not every representation of G occurs as the G-signature of a G-manifold. Depending on the category or setting (e.g. smooth, PL locally linear, PL, topological), one gets different subtle phenomena on the interaction between G-signatures of the manifold and the fixed (or, better, the singular) sets. If the singular set is empty, then the G-signature is a multiple of the regular representation, but even if it's not, there is sometimes residual information available at the prime 2 connecting the G-signature to the fixed set – not the germ neighborhood of the fixed set. It is this information that is implicit in the surgery-theoretic formulation of the Novikov conjecture – since the singular sets for equivariant homotopy equivalences are stratified homotopy-equivalent, this information must be encoded – rather like the equality mod 8 of signatures of manifolds when there is a degree-1 normal map (and concomitant implications for characteristic class theory, such as equalities of Stiefel–Whitney classes). However, the intrinsic characteristic class theory has no room for this refinement and it seems that there are different options, and there is no *a priori* reason to imagine that they will capture the essence of Novikov phenomena at 2.

In particular, I see no reason to expect there to be a theory of intrinsic characteristic classes at 2, for which the pseudo-equivalence invariance is equivalent to the equivariant homotopy invariance. But, mathematics is more beautiful than it needs to be and maybe this is one of those opportunities.

Let us now continue our explorations by analogy to the non-equivariant case. The setting, as we described it, already has controlled aspects. It is completely straightforward to formulate bounded equivariant homotopy equivalence conjectures over metric spaces with proper group actions. By considering $M\times[0,\infty)$ as boundedly controlled over the cone cM, the equivariant Novikov conjecture (which is a theorem of cones of G-ANRs, by the non-equivariant proof) then gives:

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Theorem 6.45 *$\Delta(M)$ is a topological invariant of the G-manifold M.*

This is extremely strong. We shall soon see that $\Delta(M)$ is essentially a topological version of the equivariant signature operator. This theorem therefore can be applied to the situation of M being a representation and it implies a celebrated result:

Theorem 6.46 (Based on Cappell and Shaneson, 1982; Hsiang and Pardon, 1982; Madsen and Rothenberg, 1988a,b, 1989) *For G of odd order, linear representations of G are conjugate as topological group actions iff they are conjugate as representations. For all G, the Grothendieck group of representations under topological equivalence has the following partial calculation:*

$$\mathrm{RTop}(G) \otimes \mathbb{Z}[1/2] \cong \mathrm{RK}(G) \otimes \mathbb{Z}[1/2],$$

where RK *denotes the K-representations of G, and K denotes the real subfield of the cyclotomic field of all odd roots of unity.*

(To see why this fact about the equivariant signature operator is enough, one can read the introductions of Cappell and Shaneson, 1982, and of Madsen and Rothenberg, 1988a,b, 1989.)

For G with elements of order 2, one finds that the symbol of the equivariant signature operator is not a unit in the representation ring $R(G)$. This indirectly is related to the existence of nonlinear similarities.[84] It is also responsible for the different behavior that we have seen regarding the stabilization map

$$\mathrm{BPL}_{2k}(\mathbb{Z}_p) \to \mathrm{BPL}_{2k+2}(\mathbb{Z}_p)$$

(mentioned as Example 6.17 among the other trivialities in §6.2) which is highly connected for p odd (and k sufficiently large) but never a rational equivalence on π_2 (even for k large) when $p = 2$.

The beautiful description of the topological representation group[85] given above should not mislead you into thinking that the *size* of the bundle theory relevant to the topological category is smaller (regulated by RK). Not at all. It is the size of KO^G, but the image of (usually real) representation theory into this description is not the most naively expected one – and happily the kernel of this map is succinctly describable. It is a nice problem to analyze equivariant topological equivalence of bundles (sort of like the Adams conjecture does for fiber homotopy equivalence) – even stably.

The class $\Delta(M)$ is essentially a topological analogue of the equivariant signature operator – except that the latter is only defined when one has a Lipschitz

[84] Although nonlinear similarities only exist when G has elements of order 4.

[85] Yes, it's not a ring. Topological equivalence does not play nicely with tensor products. And, the group does indeed contain 2-torsion as Cappell and Shaneson showed.

invariant metric.[86] We shall ignore the details, since the equivariant homotopy equivalence (or pseudo-equivalence) of the higher equivariant symbol class is of interest even in the smooth case.

Theorem 6.47 (Rosenberg and Weinberger, 1990) *Let* Π *be the fundamental group of a* G*-action on a manifold with fundamental group* π*. The injectivity of the assembly map* $\mathrm{KO}^{\Pi}(\underline{E\Pi}) \to K(C^*_R\Pi)$ *implies the pseudo-equivalence invariance of* G*-equivariant higher signature in* $\mathrm{KO}^{\Pi}(\underline{E\Pi})$*. It also implies the vanishing of higher indices of equivariant Dirac operators on manifolds admitting equivariant positive scalar curvature metrics on spin manifolds with isometric* G*-action.*

The left-hand side (here using real C^*-algebras for some slight refinement, as emphasized in Rosenberg's early (e.g. 1991) papers is the domain for the Baum–Connes assembly map, and the injectivity part has all of the implications we would like. I will not bother repeating the details of this type of argument here, but rather will point out two nice advantages of the analytic version over the topological one:

(1) In the analytic situation, there is no trouble dealing directly with G compact, since the Baum–Connes conjecture is a statement about locally compact groups. For the situation where $\underline{E\Pi}$ is finite, one can deduce the relevant injectivity statement in the topological case by using the result of McClure (1986) for finite complexes X that $\mathrm{KO}^G(X) \to \prod \mathrm{KO}^H(X)$ is injective as H runs over the finite subgroups of G. It is clearly necessary to develop a theory of $\Delta(M)$ for G compact, rather than just finite. However, there are considerable technical difficulties to doing this related to the fact that the orbit G-spaces are homogeneous spaces and have interesting topology.
Indeed, this interesting topology also leads to the important point that the equivariant Novikov conjecture does not yield the information one is interested in about the variation of characteristic classes within an equivariant homotopy type for G-manifolds when G is positive dimensional. For instance, we have noted that there are interactions between higher signatures of manifolds and fixed sets that automatically hold and (are accounted for in L-theory but) not accounted for in equivariant K-theory.

(2) For G non-abelian (and connected!) there is a remarkable topological consequence of the above result. Lawson and Yau (1974) have shown that any smooth G-manifold has a G invariant positive scalar curvature metric.[87]

[86] I do not know an example of a C^0-group action that does not preserve a Lipschitz Riemannian metric, but I doubt that they always exist.

[87] Actually they did this for $G = \mathrm{SU}(2)$; I have not checked that their argument works for all G, but presumably it does.

Consequently the manifold M has vanishing higher Dirac class in $\mathrm{KO}(C^*_{\mathbb{R}}\pi)$ and assuming the Novikov conjecture in $\mathrm{KO}^{\pi}(\underline{E\pi})$.[88]
This is nontrivial even in the situation of exotic spheres, because they are known to have a variety of different symmetry properties (as investigated in papers of Reinhardt Schultz).

We close with section by noting that the proof methods discussed so far apply to the situation where $\underline{E\Pi}$ is a non-positively curved locally symmetric manifold. In the analytic case, this is automatic: the machinery handles equivariance with almost no pain.[89] In the topological case, one needs, for example, the equivariant analogue of Ferry's theorem:

Theorem 6.48 (Steinberger and West, 1987[90]) *If $G \times M \to M$ is a homotopy locally-linear action of a compact group on a compact G-manifold,*[91] *then there is an $\varepsilon > 0$, such that if $f : M \to N$ is a G-map to a connected homotopy locally-linear G-manifold of no larger dimension, then f is equivariantly homotopic to a G-homeomorphism.*

(This theorem cannot be yet phrased in the full tame category including homology manifolds, because we do not know enough about homology manifolds to homotop CE maps to homeomorphisms when they should be!)

Appendix: Note on the Formulation of the Equivariant Novikov and Borel Conjectures

I've been blithely arguing by certain analogies with the unequivariant case. Here I would like to point out some dangers with doing this.

One moral is that the Novikov and Borel conjectures are considerably less well founded in the presence of infinite dimensionality (although they are frequently true even in this setting) and that infinite dimensionality is sometimes hidden in group action problems.

Another point that emerges is that, given the pseudo-invariance properties of our signatures, one should perhaps be led to dispense with the notion of the equivariant $K(\pi, 1)$. Better would be to consider the analogous objects in the pseudo-category: these are actually the $\underline{E\Pi}$ that we had been using[92] above without comment.

[88] Assuming the stated generalized Lawson–Yau theorem, one would get the stronger vanishing of the equivariant class in $\mathrm{KO}^{\Pi}(\underline{E\Pi})$.

[89] Provided one sets up K-theory to be equivariant to begin with, and establishes the relevant forms of Bott periodicity and so on, making use of an equivariant Bott element.

[90] In high dimensions, but subsequent developments allow its restatement in the form given.

[91] And one can suitably relax this condition, as we have in Ferry's theorem in §4.6.

[92] Following Kasparov, and Baum and Connes, whose immediate goal had been to give models

A $K(\pi, 1)$ is the terminal object in the homotopy category of spaces with maps that are 1-equivalences: i.e. for maps $f\colon X \to Y$ with the property that given any $g\colon K^2 \to Y$ there is a map $g'\colon K^2 \to X$ with fg' homotopic to g on the 1-skeleton K'. (Actually, the terminal objects are disjoint unions of $K(\pi, 1)$s but this hardly effects any of our conceptual thinking about the topology – everything happens on the components independently).

In the equivariant case, we should actually deal with the equivariant analogue of this notion. This is a G-space (with G compact – but we can easily change our mind and work with locally compact groups by defining an analogous notion in the universal cover) with the same universal property with respect to equivariant 1-equivalences, defined with respect to equivariant 2-complexes.[93] This boils down to the condition that for all $H \subset G$, the fixed set is an equivariant $\mathrm{N}H/H$ aspherical complex. Or, putting it all together, one wants the fixed set of any subgroup to be a disjoint union of aspherical complexes.

We have seen that the integral Novikov conjecture fails for groups with torsion – so it becomes reasonable to assume finite dimensionality. This would boil down to the equivariant Novikov conjecture failing for $G = \mathbb{Z}_p$ with the equivariant aspherical complex chosen to be S^∞. With finite dimensionality, we are led to consider the complex to be a point (which is $\underline{E\mathbb{Z}_p}$).

When $G = S^1$ and one uses the S^∞ model, even rationally equivariant homotopy equivalence fails. (Unlike the usual Novikov conjecture, which is not known to have any rational counterexamples using $E\pi$ in place of $\underline{E\pi}$; indeed, the rational injectivity statements are equivalent.)

Note that, in our formulation of the equivariant Novikov conjecture in this section, we used $\underline{E\Pi}$ when we had a G-action on a space M with fundamental group with fundamental group π. However, that space need not be the equivariant aspherical space associated to M. It accepts a G-map from M, but it might collapse different components, lose some fundamental group information, etc. In some sense $\underline{E\Pi}$ is the smallest model, and the one most likely to be finite-dimensional, and therefore the one most likely to have "correct" equivariant Novikov and Borel conjectures.

For simplicity let's consider what we can do when $G = \mathbb{Z}_p$ acts on a simply connected manifold. The only finite-dimensional equivariant aspherical spaces can be determined using Smith theory: they are ones that are contractible, have a $\mathbb{Z}_p$-action, with fixed set F that is aspherical and mod p acyclic. The rel Σ assembly map is typically not an isomorphism even in this case, if F

for $K(C^*\Pi)$ and thus could not have been misled by the possibilities suggested by equivariant algebraic topology and surgery.

[93] Note that the dimension of a G-cell is not the same as its non-equivariant dimension when G is positive-dimensional.

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has torsion in its homology away from p – although it will be rationally.[94] *The infinite dimensionality forced, for example, by requiring the modeling of a disconnected fixed set causes even a rational failure of the injectivity of this assembly map. This means that some fundamental groupoid*[95] *situations give rise to wider variation of equivariant characteristic classes than one would have naively expected from the usual analogies between equivariant and ordinary surgery.*

6.8 The Farrell–Jones Conjecture

In §6.6 we were led to consider assembly maps

$$H(X/\Gamma; K(\mathbb{Z}\Gamma_x)) \to K(\mathbb{Z}\Gamma), \tag{6.1}$$

$$H(X/\Gamma; L(\mathbb{Z}\Gamma_x)) \to L(\mathbb{Z}\Gamma), \tag{6.2}$$

for X, e.g. a locally symmetric space on which Γ acts properly discontinuously.

In §6.7, we have seen that these maps are frequently injective by considerations of the relΣ tangentiality part of the equivariant Borel conjecture.[96] If we replace the ring $\mathbb{Z}$ by $\mathbb{Q}$ in the coefficients, then none of the Nil and UNil phenomena we discussed provide a problem, and one can[97] reasonably conjecture isomorphism.

Is there a moral to this?

Let's review our situation. We started by considering the less-refined assembly map:

$$H(\mathrm{B}\Gamma; K(\mathbb{Q})) \to K(\mathbb{Q}\Gamma), \tag{6.3}$$

$$H(\mathrm{B}\Gamma; L(\mathbb{Q})) \to L(\mathbb{Q}\Gamma). \tag{6.4}$$

However, for a finite group, one observes that the right-hand side behaves (completely) differently from the left-hand side; it has a much more number-theoretic nature.

There is a map, though, $\mathrm{B}\Gamma \to X/\Gamma$, where the inverse image of a point $[x]$ in

[94] Using a Davis construction, this can be promoted to an equivariantly aspherical manifold where there is a failure of the tangentiality part of the equivariant Borel conjecture. Moreover, this cannot be attributed to equivariance versus isovariance, like our examples in §6.10, or Nil/UNil problems like our previous ones in §6.5.

This example, though, has no bearing on the form discussed in this section, or on the stratified Borel conjecture, discussed in Chapter 13 of Weinberger (1994); because a key incompressibity (namely π_1 injectivity) condition is violated.

[95] This captures both π_0 and π_1 issues.

[96] Make no mistake: this is an integral result, despite the fact that the version of the pseudo-equivalence version was only sufficiently precise away from the prime 2.

[97] We will not back out of this conjecture, as we have for some others in this book.

X/Γ is $B\Gamma_x$. We have essentially, in the target, "gathered up" the $H(\mathrm{B}\Gamma_x; L(\mathbb{Q}))$ parts of $H(\mathrm{B}\Gamma; L(\mathbb{Q}))$ and replaced them by $L(\mathbb{Q}\Gamma_x)$.

In other words, we can think formally along the following lines. The original Borel conjecture was that K- and L-theory are the simplest possible things (for group rings $R\Gamma$ consistent with $K(R[e])$ and $L(R[e])$. But, when we realized that this was wrong for finite groups, we just punted and said, OK, the correct conjecture should be the one that is correct for $R[G]$ for G finite – i.e. replace any part of the assembly map that maps through a finite group by one where the finite group acts trivially, at the cost of creating a cosheaf whose co-stalk at such a point reflects the correct answer.

Given that we got so much mileage out of doing this for finite groups, it's clear what to do[98] now that we have examples coming out of Nil and UNil. We know about counterexamples to the assembly maps (6.1) and (6.2) being isomorphisms among the class of groups that are virtually cyclic (i.e. have a cyclic subgroup of finite index). So we should "collect" all of these parts of the left-hand side together and make no predictions about their K- and L-theories – just simply predict the simplest possible answer consistent with assembly maps and calculations (left as a problem for the algebraically minded[99]) for these special groups.

The formal way to do this is to introduce a new classifying space $\mathrm{E}_{\mathrm{vc}}\Gamma$ for simplicial actions of Γ whose isotropy is virtually cyclic. There are equivariant maps of classifying spaces

$$\mathrm{E}\Gamma \to \underline{\mathrm{E}\Gamma} \to \mathrm{E}_{\mathrm{vc}}\Gamma.$$

Davis and Lück (1998) have given a nice formulation of the whole theory[100] by adding a final map to this:

$$E\Gamma \to \underline{\mathrm{E}\Gamma} \to E_{\mathrm{vc}}\Gamma \to \mathrm{E}_{\mathrm{groups}}\Gamma = \text{a point},$$

where the last space is the classifying space of actions where any isotropy is allowed – it is a point, since Γ acting on a point is this classifying space. (After all, that space is now allowed, and surely everything has a unique map to it.)

[98] Except that I was shocked when Farrell and Jones took this step. I was certain that this could not be right because of what it implied for free abelian groups. And, indeed they waited until they had *proved* the conjecture that they asserted for many lattices that contain high-rank abelian groups before publicly making this conjecture. The class of virtually cyclic groups arises very naturally in the dynamical method that they had introduced into topological rigidity theory. In short, the genesis of this conjecture was a much less blithe process than I am pretending it to be. Nevertheless, if hindsight cannot be 20/20, what can be?

[99] This feels reasonable (but difficult) because the rings involved are (not necessarily commutative) finite-degree extensions of $\mathbb{Z}[\mathbb{Z}]$. It's the dimension-1 analogue of the dimension-0 issues considered as a major enterprise of the last century: computing $K(\mathbb{Z}G)$ and $L(\mathbb{Z}G)$ for G finite.

[100] Although I am not using their notation.

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However, the domain of the assembly map for any "family" $\mathcal{F}$ is

$$H\big(\mathrm{E}_{\mathcal{F}}/\Gamma; K(R\Gamma_x)\big) \quad \text{and} \quad H\big(\mathrm{E}_{\mathcal{F}}/\Gamma; L(R\Gamma_x)\big)$$

for K- and L-theory, respectively.

Thus the map induced by the last forgetful map (it is forgetful because, when we go from a small family to a larger one, we are forgetting the special property that isotropy lies in the smaller family) is

$$H\big(\mathrm{E}_{\mathrm{vc}}\Gamma/\Gamma; K(R\Gamma_x)\big) \to K(R\Gamma), \tag{6.5}$$

$$H\big(\mathrm{E}_{\mathrm{vc}}\Gamma/\Gamma; L(R\Gamma_x)\big) \to L(R\Gamma). \tag{6.6}$$

These isomorphisms comprise the Farrell–Jones isomorphism conjecture.[101]

As they point out, when their conjecture is disproved by some group (or class of groups) they will be able to immediately generalize their conjecture by including the counterexample into a new one that has a larger family. Of course, it could be that the "final" conjecture would be the one where $\mathcal{F}$ ends up being the family of groups – but such a pessimistic conclusion is surely premature.[102]

An amusing situation arises if one applied this philosophy to the operator algebra context for understanding $K(C^*_{\max}\Gamma)$. We know that for Γ an infinite Property (T) group $K(C^*_{\max}\Gamma)$ is larger than the domain of the Baum–Connes assembly map (the trivial module C is projective in this setting; indeed, all finite-dimensional representations are isolated and give a very large cokernel to the assembly map). We are thus led to study the map

$$H(\mathrm{E}_{\mathrm{T}}\Gamma/\Gamma; K(C^*_{\max}\Gamma_x)) \to K(C^*_{\max}\Gamma).$$

The subscript T should be interpreted as the family of subgroups of Γ that are subgroups of a Property (T) subgroup of Γ. Whether this leads to any insights regarding the right-hand side, I do not know. It gives many new elements of that group.

Let us unravel what the Farrell–Jones conjecture means in some cases. We have to understand what $E_{\mathcal{F}}\Gamma/\Gamma$ looks like as a stratified space. First of all, there are nontrivial strata for subgroups π in $\mathcal{F}$. The fixed set of such a group is contractible, and the group $\mathrm{N}\pi/\pi$ acts on this. At such a stratum we have $K(R\pi)$ or $L(R\pi)$. Strata corresponding to π and π' can intersect only if the groups generated by π and π' lie in the family $\mathcal{F}$.

We have already discussed the situation for torsion-free word hyperbolic groups in §5.5.3. The interesting situation there is K-theory and how the closed

[101] Where the L is decorated with -8.

[102] And, in this case, I would prefer the false conjecture that has been so fruitful over the previous decades to the one that is true yet meaningless.

geodesics, which are the maximal elements of the virtually cyclic family, contribute Nils (i.e. the fiber of the assembly map $H_*(\mathrm{B}\mathbb{Z}, K(R)) \to K(R[\mathbb{Z}])$). In L-theory, it's the usual assembly map (as it is for all torsion-free groups).

If there is torsion then the L-theory situation becomes interesting. For definiteness, assume that Γ comes from a $\mathbb{Z}_2$-action on a closed negatively curved manifold. Then the fiber of the virtually cyclic assembly map comes specifically from closed geodesics that are invariant under $\mathbb{Z}_2$, but, if the action is free, then it will not contribute either. However, each geodesic which is invariant under the involution will go through two fixed points, and give a UNil contribution via the inclusion of $L(\mathbb{Z}_2 * \mathbb{Z}_2)$ in $L(\Gamma)$.

For $\mathbb{Z}^n$, the trivial and finite families both give us $\mathbb{R}^n$ with free $\mathbb{Z}^n$-action. However, when we use the virtually cyclic family, the maximal subgroups correspond to primitive lattice points (up to sign) in $\mathbb{Z}^n$. However, there is a T^{n-1} family of geodesics in each of these free homotopy classes equal to the corresponding stratum in $E_{\mathrm{vc}}\mathbb{Z}^n/\mathbb{Z}^n$. Each of these families produces a $H_*\big(T^{n-1}; \mathrm{Nil}(R)\big)$ contribution.

By the way, note that this description includes an analysis of $\mathrm{Nil}(R[\mathbb{Z}])$ in terms of $\mathrm{Nil}(R)$, but it is not simply the assertion that $\mathrm{Nil}(R[\mathbb{Z}])$ is computed from $\mathrm{Nil}(R)$ via the assembly map isomorphism. The description given by the Farrell–Jones conjecture is strong enough to enable an analysis of the action of $\mathrm{SL}_n(\mathbb{Z})$ on the K-groups.

The story for, for example, $\mathbb{Z}^n \rtimes Z_2$, where the involution acts as multiplication by -1, is similar. In that case there are 2^n fixed points (on the torus), and many such closed geodesics and these tell the whole story (see Connolly *et al.*, 2014). If the involution had a positive eigenspace, then there would be tori of these interesting geodesics. They would then contribute the homology of these tori with coefficients in the Nils corresponding to the geodesics.

Finally, we note that the Farrell–Jones conjecture gives us a description of the isovariant structure set rel Σ: it is the relative homology group describing the difference between $\underline{\mathrm{E}\Gamma}$ and $\mathrm{E}_{\mathrm{vc}}\Gamma$.

6.9 Connection to Embedding Theory

We now return to the equivariant Borel conjecture (or, more generally, to the problem of equivariant surgery classification). Our approach is via a profound connection between group actions and embedding theory that was already hinted at in §6.5.

Surgery theory does quite a good job (with the Farrell–Jones conjecture picking up much of the slack) of analysis of structures within an isovariant

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homotopy type, but does not do a very good job of classification within an equivariant homotopy type. The latter is what the equivariant Borel conjecture (or, surely we should say, question) asks about.

However, analogously, surgery shows that there is a unique isotopy class of embedding realizing a given Poincaré embedding, but it does not directly help with the problem of analyzing the embeddings in a given homotopy class.

Embedding theory has developed a set of geometric tools that enable good classification results in specific settings. We do not know of any way of mimicing these geometric tools (such as general position, multiple disjunction, and so on) directly.

However, *after the fact*, we can make use of the theoretical reduction of both embedding theory and isovariant classification within an equivariant homotopy type to homotopy theory to relate these problems to one another and get concrete and theoretical results. This section is devoted to developing the connections between the subjects, and the next will use this to give some specific analyses. The explicit results which we give there for some crystallographic manifolds are the subject of heretofore unpublished joint work with Sylvain Cappell. I also wish to acknowledge useful conversations with John Klein about the use of categorical techniques and explanations of the calculus of embeddings due to Goodwillie, Weiss, and him.

Recall the definition of a *Poincaré embedding* (see Wall, 1968). A Poincaré embedding of M in W consists of a triple $((X,E),\pi,f)$, where (X,E) is a Poincaré pair, $\pi\colon E\to M$ is a spherical fibration with fiber $\mathcal{S}^{c-1}$, and $f\colon X\cup \mathrm{Cyl}(\pi)\to W$. In the PL and topological(ly locally flat) category, every Poincaré embedding can be realized by an embedding (that is unique up to concordance – by the relative version of this realization statement).

By analogy we can define a similar notion of an isovariant Poincaré complex. For notational simplicity we shall only discuss the semi-free case.

Definition 6.49 An isovariant G-Poincaré complex consists of a triple

$$\big((X,E),\pi\big)$$

with $\pi\colon E\to F$ an equivariant spherical fibration with fiber $\mathcal{S}^{c-1}$ having a free G-action, (X,E) with free G-action (i.e. covering a Poincaré pair) so that $Y=X\cup \mathrm{Cyl}(\pi)$ is a Poincaré complex.

We are interested in the possible isovariant Poincaré complexes (up to isovariant homotopy equivalence) within a given equivariant homotopy type. In other words:

Definition 6.50 If $G\times M\to M$ is a group action, then $\mathrm{Iso}(M)$ is the set of isovariant Poincaré complexes *with an equivariant homotopy equivalence* to M (up to isovariant homotopy type). This can be made into a Δ-set in the usual way, and we shall denote this by the same symbol. (The set of isovariant homotopy Poincaré complexes within the given equivariant homotopy type can be thought of as π_0 of the space $\mathrm{Iso}(M)$.)

Similarly we will denote the Poincaré embeddings of F in M which are homotopy equivalent as a pair to a given embedding as $\mathrm{PE}(M, \mathrm{F})$. Note that there is a map $\mathrm{Iso}(M) \to \mathrm{PE}(M, \mathrm{F})$ when F is the fixed set of the G-action on M.c Note that G acts on $\mathrm{PE}(M, \mathrm{F})$ by composing the map f with elements of G. Our main results concern the relation of $\mathrm{Iso}(M)$ to $S^{\mathrm{equi}}(M)$ and the relation of $\mathrm{Iso}(M)$ to $\mathrm{PE}(M, \mathrm{F})^{\mathrm{hG}}$.

Theorem 6.51 (Decomposition theorem) *For $G = \mathbb{Z}_p$, with p odd, acting tamely and supposing that the fixed set is of codimension greater than 2, there is an isomorphism*

$$S^{\mathrm{equi}}(M) \cong S^{\mathrm{iso}}(M) \times \mathrm{Iso}(M).$$

For applications to disproving the equivariant Borel conjecture, the above theorem is not even necessary. The point is that there is a total surgery obstruction to realizing elements of $\mathrm{Iso}(M)$ (or even $\mathrm{Iso}(M \operatorname{rel} \Sigma)$) that lies in a group[103] that is (assuming Farrell–Jones) trivial. As a result, even in the absence of the above theorem, $\mathrm{Iso}(M)$ would provide counterexamples to the equivariant Borel conjecture.[104]

It is necessary, though, for understanding the set of all counterexamples, when the dimension of the fixed set is relatively high compared to $\dim M$.

The proof of the theorem is not purely stratified in nature, but relies on connections between the isovariant and equivariant categories that were pioneered by Browder (and continued to be studied by Dovermann and Schultz, 1990, Yan, 1993, and others).

This decomposition theorem is surely true in much greater generality (at least I think so). I hope to return to this in a later paper; below we will give a small extension of it.

We will not discuss the algebraic K-theoretic aspect of the decomposition theorem: essentially this is handled by the way that isovariant finiteness (when all strata are codimension 3 in one another) is equivariant in nature. We focus on the algebraic topology and the surgery.

[103] The delooping of the isovariant structure space.
[104] Ironically, we would be using the isovariant Borel conjecture to disprove the equivariant Borel conjecture in following such a route!

The first interesting ingredient is a variant of the Whitney embedding theorem, due to Browder, that asserts (in the semi-free case):

Theorem 6.52 *In the semi-free case, if* $\dim M > 2\dim \mathrm{F} + 1$*, then every equivariant homotopy equivalence is equivariantly homotopic to an isovariant homotopy equivalence.*

In addition, we need a method for getting into the stable range that doesn't lose surgery obstructions. Again following Browder, we cross with the G-manifold $(\mathbb{CP}^2)^{\mathrm{G}}$, where the superscript G denotes the product of #G copies of the projective plane, with the G-action given by permutation. Browder observed that:

Theorem 6.53 *If G is odd order, then, if taking the product* $\times(\mathbb{CP}^2)^{\mathrm{G}}$ *does not change the number of strata in the quotient spaces, it induces an isomorphism on L-groups,*

$$L^{\mathrm{strat}}(M/G) \to L^{\mathrm{strat}}((M \times (\mathbb{CP}^2)^{\mathrm{G}})/G).$$

As a consequence of this and stratified surgery $\times(\mathbb{CP}^2)^{\mathrm{G}}$ induces an injection of structure sets. Since existence of a structure underlying a given isovariant Poincaré structure is a surgery problem (i.e. lies in the delooping of the structure space) if the realization exists after crossing $\times(\mathbb{CP}^2)^{\mathrm{G}}$ it will exist before crossing. However, by Browder's first theorem above, if the structure exists equivariantly, it exists isovariantly after crossing with $(\mathbb{CP}^2)^{\mathrm{G}}$, explaining the above theorem.

Remark 6.54 The decomposition theorem holds at least in the greater generality of G is of odd order and acting semi-freely, working relative to the fixed point. (This suffices for our applications below.) Of course, the problem is that crossing with $(\mathbb{CP}^2)^{\mathrm{G}}$ has more strata. However, working rel F will enable us to get around this as follows.

Note that the product map sends[105]

$$S(M/G, \mathrm{rel}\,\mathrm{F}) \quad \text{to} \quad S(M \times (\mathbb{CP}^2)^{\mathrm{G}}/G, \mathrm{rel\,singularities})$$

and we can study the existence problem relative to the singular set. The rel sing structure set can be thought of as the fiber of a *conventional* assembly map[106] –

105 It is actually true that $S(M/G)$ can be decomposed as $S(\mathrm{F}) \times S(M/G, \mathrm{relF})$, because the symmetric signature of the space form normal to F vanishes. (This is enough because of the way the symmetric signature of the link enters in the definition of L^{BQ}, the key object in stratified surgery. This vanishing can be deduced from the fact that the symmetric signature can be lifted under an assembly map to an odd torsion group, but the L-group has only 2-torsion.) For some detail, see Cappell and Weinberger (1995).

106 With non-constant coefficient, when interpreted in the quotient.

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interpreted as forgetting control – from the controlled free equivariant Poincaré complexes over the space mapping to the uncontrolled Poincaré complexes (which end up in L (the orbifold fundamental group)).

With this interpretation, there is a projection map

$$S\big(M \times (\mathbb{CP}^2)^{\mathrm{G}}/G, \text{rel singularities}\big) \to \S(M/G, \text{rel F})$$

whose precomposition with the product map

$$S(M/G, \text{rel F}) \to S\big(M \times (\mathbb{CP}^2)^{\mathrm{G}}/G, \text{rel singularities}\big)$$

is an isomorphism – this map is a transfer associated to a fiber bundle with fiber $(\mathbb{CP}^2)^{\mathrm{G}}$ and that it induces an isomorphism on L-groups is part of the proof of Browder's theorem (see Yan, 1993).

Now we turn to the map $\mathrm{Iso}(M) \to \mathrm{PE}(M, \mathrm{F})^{\mathrm{hG}}$ alluded to above. The map is obtained by sending a typical vertex $\big(((X, E), \pi), \Phi\big)$, where $\Phi\colon X \cup \mathrm{Cyl}(\pi) \to M$ is an equivariant homotopy equivalence, to the vertex of the homotopy-fixed-set $\big(X \cup \mathrm{Cyl}(\pi)\big) \to (M \times_G \mathrm{EG} \downarrow \mathrm{BG})$ in the homotopy-fixed-set of G acting on $\mathrm{PE}(M, \mathrm{F})$. Higher simplices are mapped similarly.

Theorem 6.55 *If M has boundary and each component of the fixed set touches the boundary, then*

$$\mathrm{Iso}(M, \text{rel}\,\partial) \to \mathrm{PE}(M, \mathrm{F}, \text{rel}\,\partial)^{\mathrm{hG}}$$

is a homotopy equivalence.

Without the boundary condition, this theorem is hopelessly false: on π_0 one can get uncountably many components on the right-hand side, while the left is clearly always countable. The problem is that one produces in the homotopy-fixed-set group actions on infinite-dimensional spaces that don't have a reasonable geometric interpretation.

On the other hand, the condition is not an unreasonable one, since it can be arranged through strategic puncturing of M at various fixed points.

The reader can wonder whether this theorem is ever of use, in that homotopy-fixed-sets involve maps of infinite-dimensional spaces into other objects. We close the section with some examples of how one can use this machinery, even in the absence of concrete information about the classification of embeddings. In §6.10 we will give some additional illustrations that have some more computational input that I hope are convincing that this approach is not completely worthless.

The proof of the theorem is quite simple and quite analogous to the old result of George Cooke about realizing homotopy actions by actions: a homotopy action in a map of groups $G \to \pi_0\mathrm{Aut}(X)$, and the question is which of these are realized by group actions?

Theorem 6.56 (Cooke, 1978) *A homotopy action is realized by an action iff the induced map on classifying spaces has a lift:*

$$\begin{array}{c} \mathrm{BAut}(X) \\ \downarrow \\ \mathrm{BG} \to \pi_0\mathrm{Aut}(X) \end{array}$$

Proof If there is an action, there's a lift. If one has a lift, then the associated X fibration over BG has as induced G-cover a space homotopy equivalent to X on which the G-action by covering translates is the desired realization. □

A warning, though, is that the space on which G acts could well be infinite and even infinite-dimensional when the cohomological dimension of G is infinite.

Similar is the following:

Proposition 6.57 *If X and Y are free G-spaces, then the map of mapping spaces*

$$[X,Y]^G \to [X,Y]^{\mathrm{hG}}$$

is a homotopy equivalence.

This is a triviality from covering space theory and the homotopy equivalences between X/G and Y/G and their respective Borel constructions.

If we stare at what the right-hand side $\mathrm{PE}(M,\mathrm{F})^{\mathrm{hG}}$ means, one sees an $\mathrm{F}\times\mathrm{BG}$ with a spherical fibration over it, together with some pair that is also given as a fibration over BG. The spherical fibration over $\mathrm{F} \times \mathrm{BG}$ can be thought of as a family of "spherical fibrations over BG" parameterized by F. A spherical fibration over BG is like the output of Cooke's theorem – it corresponds to a free action of G on a space of the homotopy type of $\mathcal{S}^{c-1}$ but it's not necessarily finite, i.e. corresponding to a homotopy lens space (or space form). This is a question that needs answering over each component of F once – which is why we need the boundary conditions.

But, if it is finite, then we have obtained the relevant equivariant spherical fibration over F. The total space of this is now included into a complement, which, if it were finite, would be exactly what we need for an isovariant Poincaré complex. The finiteness follows from:

(1) codimension greater than 2;
(2) the already established finiteness of the boundary of the regular neighborhood;

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(3) the comparison of the relative chain complexes for $(X, \partial\text{Nbd}(\text{F}))$ and (M, F) (this is a chain equivalence by excision);
(4) the fact that isovariant finiteness obstructions are equivalent to equivariant finiteness obstructions. This statement is pretty obvious in the PL case (because the relevant K-groups are sums of the K-groups of various strata) and[107] it is a consequence of Carter's vanishing theorem for negative K-groups, and the calculations of both of these obstruction groups for the topological case.

Some consequences of the above theorems are worth pointing out immediately – although they involve some massaging to get them for general finite groups (since the decomposition theorem wasn't proved in appropriate generality).

I conjecture that for semi-free actions, the decomposition theorem holds for all G. Indeed, I suspect that the phenomenon is extremely broad (and perhaps only requires a very small gap hypothesis).

(1) The pseudo-trivial orientation-preserving[108] G-action situation produces pairs (M, F) where the inclusion is a homology isomorphism at #G. As a result the homotopy-fixed-set analysis is straightforward, and one obtains that $S^{\text{equi}}(M) \cong S^{\text{iso}}(M\text{rel F}) \times S(\text{F}) \times \text{Emb}(\text{F} \subset M)$ and a complete reduction of the equivariant classification problem to embedding theory!
(2) For orientation-reversing involutions on the sphere, Chase showed in unpublished work that a mod 2 homology subsphere Σ of the sphere is the fixed set of an orientation-reversing involution of codimension greater than 1 iff Σ is isotopic to its mirror image – exactly the π_0 part of the homotopy-fixed-set condition. That the remaining part follows automatically follows from ideas of Dwyer (1989).
(3) If M is a G-manifold, then $S^{\text{equi}}(M \times \mathcal{D}^i \text{rel}\, \partial)$ is an abelian group for $i > 1$. The embeddings $(\text{F} \times \mathcal{D}^i \subset M \times \mathcal{D}^i \text{rel}\, \partial)$ also form a group for $i = 1$ and is abelian for $i > 1$. These are the π_i of the spaces $S^{\text{equi}}(M)$ and $\text{PE}(M, \text{F})$. Sometimes I like to refer to the embeddings $(\text{F} \times \mathcal{D}^i \subset M \times \mathcal{D}^i \text{rel}\, \partial) = C_i(\text{F}, M)$ as the ith *concordance embedding group* of F in M. One obtains[109] an isomorphism

$$S^{\text{equi}}(M \times \mathcal{D}^i \text{rel}\, \partial) \otimes \mathbb{Z}[1/\#G] \cong S^{\text{iso}}(M \text{rel F}) \otimes \mathbb{Z}[1/\#G] \\ \times C_i(\text{F}, M) \otimes \mathbb{Z}[1/\#G].$$

107 As we have already remarked in §6.5.
108 Note that the $G = \mathbb{Z}_2$ case can be pseudo-trivial and orientation-reversing. In that case, the restriction to the boundary is not pseudo-trivial, which interferes with inductive arguments.
109 Here, since we are inverting 2, one can rehabilitate the argument for odd-order groups to apply in general.

6.10 Embedding Theory

We begin by ignoring all the stuff about Poincaré embeddings and their connection to group actions discussed in §6.9.

It requires herculean effort to deduce from surgery even the most basic embedding theorem, that of Whitney:

Theorem 6.58 *If $f: M^m \to W^w$ is a continuous map, and $w > 2m$, then f can be approximated by an embedding.*

To do this, we would need to construct a spherical fibration (which can be done by the methods of Spivak, 1967), and a homotopy complement (which is very difficult, but clearly related to Spanier–Whitehead duality: see, for example, Spanier, 1981). All in all, a lot of work.

But embedding theory goes much further than this. Whitney proved a much deeper embedding theorem for when $\dim W = 2\dim M$ using the famous "Whitney trick" that underlies the h-cobordism theorem and the process of surgery, and therefore underpins almost all that we know about high-dimensional topology. However, that embedding theorem is more subtle; the above is sharp as the "8 curve" in the plane cannot be approximated by an embedding.

The embedding of two kissing circles in the plane, then thought of as lying in $\mathbb{R}^3$, can be approximated by infinitely many *non-isotopic* embedded $\mathcal{S}^1 \cup \mathcal{S}^1$s distinguished by their *linking number*. So there is not a uniqueness theorem that goes with the above existence result (unless the dimension of the ambient space is even larger than what is demanded above).

Recall that the linking number of two disjoint oriented (compact) cycles X^x and Y^y in $\mathbb{R}^n$ (or S^n) with $n = x + y + 1$, namely $\mathrm{lk}(X, Y) \in \mathbb{Z}$, can be defined as the intersection number $\mathrm{int}(Z, Y)$, where Z is any chain bounded by X. This definition is not quite symmetric; viewing X and Z as cycles on the boundary of $\mathcal{D}^{n+1}$ we can define the linking number more symmetrically as $\mathrm{int}(Z, Z')$ where Z bounds X and Z' bounds Y. This then shows that

$$\mathrm{lk}(X, Y) = (-1)^{(x+1)(y+1)}\mathrm{lk}(Y, X).$$

Linking invariants and their generalization are fundamental to the theory of embeddings.

We will need variants for non-simply connected situations, and for more general targets. Note that the current definition only really involves knowing the vanishing of certain homology classes and certain (other) homology groups. If the cycles involved are simply connected, such a theory already arises in the proof of the h-cobordism theorem and in surgery theory – intersection (and self-intersection) numbers take values in (a quotient of) $\mathbb{Z}\pi$ – and one can occasionally define an associated linking theory.

If the cycles are non-simply connected there is more indeterminacy in their definition and we have to mod out by the influences of the fundamental groups of X and Y.

Theorem 6.59 *Suppose M is a connected oriented submanifold in W and that $w = 2m + 1$. Then the embeddings of M homotopic to the given one are in a one-to-one correspondence with*

$$\mathbb{Z}[\pi_1 M \backslash \pi_1 W / \pi_1 M]/\{g \not\cong 1, g - (-)^m g^{-1}\}.$$

Addendum 6.60 If M consists of several components, then there are additional invariants that live in $\mathbb{Z}[\pi_1 M_i \backslash \pi_1 W / \pi_1 M_j]$. These have appropriate symmetry associated to interchanging i and j.

We shall only prove the theorem for the topological locally flat case (or PL case) and shall avoid thereby some arguments necessary for the smooth case (which are given in Whitney's well-known paper). We shall use the following basic theorem (concordance implies isotopy) that is an elementary consequence of the h-cobordism theorem:

Theorem 6.61 *If $i\colon V \subset W$ is an embedding with codimension greater than 2, then any proper embedding of $V \times [0,1]$ in $W \times [0,1]$ which restricts to i on $V \times \{0\}$ is equivalent to $i \times [0,1]$.*

This is completely false in codimension 2, and in codimension 1 it is true for "incompressible" (i.e., π_1-injective locally two-sided) embeddings.

By the way, note that this theorem implies the Zeeman unknotting theorem: any locally flat embedding of a sphere in another with codimension greater than 2 is equivalent to the inclusion of an equator. (In other words, there's only one embedding.)

Suppose now we take two homotopic embeddings of M into W. We can homotop the map of $M \times [0,1]$ into an immersion by Whitney's theorem. We are interested in the self-intersection of this immersion and will try to use the Whitney trick to remove them – completely analogously to what occurs in Wall (1968) in the description of the even-dimensional surgery groups – just taking into account the fact that M is not simply connected.

The self-intersection points (for a generic immersion) are all labeled by ± 1 according to orientation conventions. Moreover, choosing a base point and a path to each sheet of the intersection, we can get an element of the fundamental group by going along one sheet to the intersection and back to the other sheet. Note that there is an indeterminacy of which is the first or second sheet, and also of the paths from the base point – this gives only a well-defined double coset. Now, as usual, when two intersection points have the same group element and opposite signs, they can be cancelled.

The coefficient of the identity can be modified by changing the immersion near a point, or by dealing with embeddings of punctured versions of M and using the uniqueness of the embedding of $\mathcal{S}^{m-1}$ in $\mathcal{S}^{2m}$ to complete the discussion.

These kinds of invariants are relevant exactly at the "edge of the gap hypothesis." To go further, all of these linking invariants need to take values in homotopy groups of spheres, rather than $\mathbb{Z}$ (which equals π_0^s). This will suffice for getting through the metastable range. That this should be the case is pretty clear: if one considers embeddings of

$$\mathcal{S}^n \cup \mathcal{S}^n \subset \mathcal{S}^{2n+1-k},$$

for n large, there is a natural invariant in π_k^s that turns out to determine the embedding. Using Zeeman unknotting, the complement of the first sphere is homotopy-equivalent to $\mathcal{S}^{n-k}$ (the simpler observation that the linking $\mathcal{S}^{n-k}$ included in the complement is a homotopy equivalence, by the Whitehead theorem and Alexander duality, or even by a Mayer–Vietoris argument, suffices for this purpose), and therefore the second sphere defines an element of $\pi_n(\mathcal{S}^{n-k}) \cong \pi_k^S$. The relevant symmetry property can be proved similarly to the symmetry of the linking numbers in the stable range.[110]

In a less stable range of embeddings, e.g. for $\mathcal{S}^3 \cup \mathcal{S}^3 \subset \mathcal{S}^6$ so that the corresponding invariant would take values in $\pi_3(\mathcal{S}^2) \cong \mathbb{Z}$ via the Hopf invariant – and the two definable "Hopf linking numbers" can be different[111] (although they must agree mod 2 by the stable result).

Moreover, for embeddings of $\mathcal{S}^n$ in, for example, $\mathbb{T}^{2n+1-k}$, one would get an invariant in the "group"[112] $\pi_k^s[\mathbb{Z}^{2n+1-k}]$ with the coefficient of 0 being 0, and there being a symmetry condition connecting the coefficients of g and $-g$.

To illustrate the key ideas, let's work out some especially nice cases; for convenience, I will concentrate on crystallographic groups with holonomy $\mathbb{Z}_p$ an odd prime, acting with connected fixed set. We assume that p is odd so as not to get caught up in the surgery difficulties; no problems due to Nil or UNil – all of the isovariant structure sets vanish in this case, and this helps both for the existence of actions as well for their classification. It also helps with actually

110 This uses the Pontrjagin interpretation of stable homotopy groups of spheres are framed cobordism.

111 As John Klein pointed out to me.

112 It's actually a tensor product, but if we were in the stable range, this notation would evoke a group ring.

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doing the homotopy theory. As mentioned above, the $\mathbb{Z}$ gets replaced by π_i^s as we move forward, and for $p = 2$ we are not given much slack as $\pi_1^s = \mathbb{Z}_2$.

Theorem 6.62 (Cappell and Weinberger, unpublished) *If p is an odd prime, then (assuming $k > 1$ if $p = 3$)*

$$S^{\text{equi}}\left((\mathbb{T}^p)^k \times \mathbb{T}^{(p-2)k-1}\right) \cong \mathbb{Z}[\mathbb{Z}^{(p-1)k+(p-2)k-1} - \{0\}]^{\mathbb{Z}_{2p}}.$$

Here $\mathbb{Z}_p$ acts on $\mathbb{T}^p$ by permutation, and otherwise trivially. The extra $\mathbb{Z}_2$-action reflects the symmetry that linking numbers satisfy, so it gives a +/− factor depending on some parities relating the coefficients of g and g^{-1}.

If one increases the size of the $\mathbb{T}^{(p-2)k-1}$ factor, then one moves deeper into the metastable range, and the one gets additional factors. One extra circle then gives another factor of $\mathbb{Z}[\mathbb{Z}_2^{(p-1)k+(p-2)k-1} - \{0\}]_{2p}^{\mathbb{Z}}$ where this corresponds to the π_1^s linking, etc. Throughout the metastable range we have that $S^{\text{equi}} \cong \text{Emb}(\text{F} \subset \mathbb{T})^{\mathbb{Z}_p}$. The original method will be explained in §6.11, but *morally* it follows from the fact that the Tate cohomology of $\mathbb{Z}_p$ acting on the embeddings is trivial.[113] Alas, this vanishing of Tate is computational in nature.

Remark 6.63 I believe that there is an example where the equivariant Borel conjecture fails for an isovariant Poincaré complex reason when the Tate cohomology is nontrivial. Indeed, I would not be much surprised if the analogous crystallographic actions for $\mathbb{Z}_2$ already include such examples, but I did not succeed at doing these calculations.

To go beyond the metastable range,[114] first of all, the homotopy theory becomes unstable (it should go without saying).

One also needs versions that are "multiple linking invariants" that arise from triple points and higher. This is because of the phenomenon of the Borromean rings (Figure 6.4): one can have three linked spheres that are pairwise unlinked in this range.And as one goes further into deeper ranges, there are higher and higher-order Borromean phenomena. Examples of this are the μ-invariants of Milnor (related to Massey products in the way that intersection numbers are related to cup products; see Milnor, 1954).

[113] There are convergence issues in the "obvious" spectral sequence argument that would lead to this conclusion. Note, however, that there is a similar issue that arises in trying to compare the equivariant maps from X to Y to $[X, Y]^G$ (the homotopy classes of maps that are homotopic to themselves after composing with elements of G). If the action of G on X is free, then the spectral sequence has better convergence properties, because X/G is finite-dimensional, and one does not really have to go to infinite dimensions, despite the implicit infinite-dimensionality of BG that arises in the definition of homotopy fixed sets.

[114] Embedding theory is essentially the homotopy theory of the map when $w \gg 2m$, the "stable range," because of the Whitney embedding theorem; the metastable range is when $w \gg 3m/2$. The next range is when $w \gg 3m/3$, etc. Each successive range requires yet higher-order information. The "calculus of embeddings" described below is one version of how to do this.

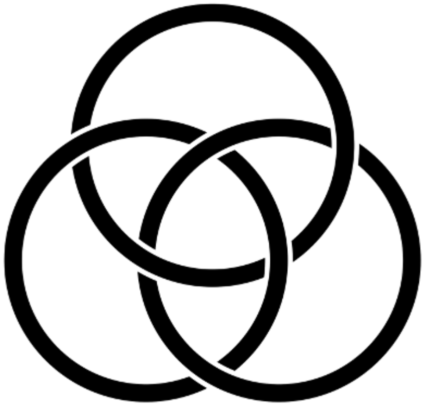

Figure 6.4

Let me explain this in the simplest case, in a somewhat non-classical way, relevant to embeddings of aspherical manifolds in one another (and therefore to the equivariant Borel conjecture).

For simplicity let's think about the classification of k-component linked spheres $\bigcup_k \mathcal{S}^n \subset \mathcal{S}^{n+c}$ in the sphere in codimension $c > 2$. This classification (even when the spheres have different dimensions) was established by Haefliger (1966). Now let's consider this from the Poincaré embedding point of view.

Firstly it is easiest to replace the given problem by the embeddings of $\bigcup_k \mathcal{D}^n \subset \mathcal{D}^{n+c}\text{rel}\partial$ (so-called "disk links"). Note that this is $\pi_n\text{PE}(\mathcal{D}^c, k) = C_n(k \subset \mathcal{D}^c)$, where k denotes any k-point subspace of the disk.

Incidentally, it is interesting to note here how different is the (Δ)-space of embeddings of points in the disk to which we are led from the more naive "genuine embeddings." *That* space is a configuration space, and very well studied. In particular, it is a finite-dimensional space in this case – but PE is actually a function space.

The reason is because concordance implies isotopy. All of the embeddings classified in $C_n(\text{F} \subset M)_{(n\geq 1)}$ are, abstractly, i.e. not relative to the boundary, just product embeddings[115] $\text{F} \times \mathcal{D}^n \subset M \times \mathcal{D}^n$. What makes the element nontrivial is that on the top face $\text{F} \times \mathcal{D}^{n-1} \subset M \times \mathcal{D}^{n-1}$ we have a nontrivial automorphism (relative to the boundary).

Thus, we are interested in the automorphisms of $\mathcal{D}^c$ that are the identity on the boundary and send k points to themselves, and the spheres $\mathcal{S}^{c-1}$ normal to these points to themselves, and finally map the complement to the complement.

115 By the h-cobordism theorem.

Let us call this space $\mathrm{Iso}(\mathcal{D}^c, k)$:

$$C_n(k \subset \mathcal{D}^c \cong \pi_{n-1}\mathrm{Iso}(\mathcal{D}^c, k).$$

Note that there is a restriction map

$$\mathrm{Iso}(D^c, k) \to \prod \text{homotopy equivalences } (\mathcal{S}^{c-1} : \mathcal{S}^{c-1}).$$

This map is null-homotopic because of the condition that the automorphisms restrict to the identity on the outer boundary. (We can study any factor on its own by filling in the other $(k-1)$ holes, and the geometry then gives an explicit null-homotopy.)

The homotopy fiber of this restriction is easily studied by obstruction theory. The complement we are discussing has the homotopy type of the wedge $\bigwedge_k \mathcal{S}^{c-1}$ and we have restricted these maps on a disjoint union of $(k+1)$ copies of $\mathcal{S}^{c-1}$ in this complement. The associated[116] spectral sequence for this situation has $E_2^{p,q} = H^p(\mathcal{S}^c, k+1; \pi_q(\vee_k \mathcal{S}^{c-1}))$. (It abuts to π_{q-p} (iso-invariant relative to the neighborhoods); one has to remove the

$$\Omega \prod \text{homotopy equivalences } (ph^{c-1} : \mathcal{S}^{c-1})$$

that comes from the injection of the Ω-base in the fibration.)

There are just two lines in this sequence: $p = 1$ and $p = c$. In particular, there is just one nontrivial differential. The homotopy groups that occur as coefficients are of $\bigwedge_k S^{c-1}$. These are given by the Hilton–Milnor theorem (see Hilton, 1955). They are homotopy groups of $\mathcal{S}^{r(c-2)+1}$ where there is one sphere for each generator on the free Lie algebra in degree r on the generators of a vector space of dimension k. The Lie algebra operation is Whitehead product, and the nontrivial differential can easily be written down using Whitehead products (using the obstruction theory interpretation). For example, if c is even, working rationally the rank of this group is $kL_r - L_{r+1}$, where $L_r = (1/r)\sum \mu(d)k^{r/d}$ (the sum over divisors of r) is the number of generators of the free Lie algebra of degree r on a vector space of dimension k.

The above reworking of Haefliger's classical work can be modified for the Borel conjecture setting,[117] and, remarkably enough, many of the same features hold. For example, for $C_n(k \subset \mathrm{B}\pi)$ the spectral sequence still has two lines, and one is taking cohomology of π with coefficients in various free Lie algebras on, for example, $(\mathbb{Q}\pi)^k$. This can be interpreted as multivariable "polynomial" invariants of the embeddings, which (together with symmetry properties) will rationally calculate S^{equi} (even outside the metastable range).

The whole story is very complicated, and while the ingredients now seem

[116] See Federer (1956).

[117] As Cappell and I did in our original approach to these calculations.

clear, computationally it currently looks a mess, and surely key aspects of structure elude us. More precisely, there is a calculus of embeddings due to Goodwillie, Klein, and Weiss (see Weiss, 1999; Goodwillie *et al.*, 2001) that puts these types of ingredients together, but it is via a sequence of complicated diagrams, and the analysis of the terms and how they are assembled has only been done in a few cases.[118]

Their theory is a descendant of the work of Whitney and Haefliger, and deals with genuine embeddings,[119] but can be adapted to deal with Poincareé embeddings. The approach goes like this:

One's first approximation to Emb(F $\subset M$) might be the result of "gluing" together the spaces Emb($\mathbb{R}^f \subset M$) over all the submanifolds of M isomorphic to $\mathbb{R}^f$. (Note that when one such submanifold is included in another, the restriction map is a homotopy equivalence.) More explicitly, consider the category F of open subsets of F diffeomorphic to balls, and all smaller in diameter than some ε, say the injectivity radius of F, with morphisms being inclusions. One then can take the limit over F of Emb($\mathbb{R}^f \subset M$) as a guess for Emb(F $\subset M$).

This doesn't quite work: What that actually gives, after doing the bookkeeping, is essentially the Smale–Hirsch description of immersion theory (Hirsch and Smale, 1959). Of course, this means that there is a global effect that immersion theory doesn't solve: the maps are only locally one-to-one, not globally one-to-one.

This might suggest taking a limit over the category of submanifolds of F isomorphic to two (in addition to the one) copies of $\mathbb{R}^f$ to prevent pairwise intersections. In this category there are morphisms where the two components "collide," i.e. are included in a single component of a larger set.

If we work modulo immersions (i.e. in the fiber of Emb $\to$ Imm) then we can elide differences between unions of two points versus unions of two submanifolds each isomorphic to $\mathbb{R}^f$ and get some sort of description involving the configuration of pairs of points in F mapping into M. This is essentially the Haefliger theory in the metastable range (Haefliger, 1964).

But, we know that at the end of the metastable range the Borromean phenomenon begins! There are triple linkings not detected by pairs, so we need to go to the category of triples and further. This is the theory described beautifully in Weiss (1996) and developed in the papers surveyed in Goodwillie *et al.* (2001). The upshot is that we know that the k-tuple theory is not determined

[118] Although it has excellent theoretical implications, e.g. the theorem of Goodwillie and Weiss (1999) that many spaces of embeddings have finitely generated homotopy groups, or the calculations of spaces embeddings of knotted strings in high-dimensional spheres by Volic (2006).

[119] In other words, for points, it is configuration spaces that arise, rather than the concordance embeddings that arose in our analysis.

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by the $(k-1)$-tuple theory beyond a range, which requires serial elements of the categories we use for formulating "the simplest possible answer" to the problem of calculating embeddings. The main theoretical result of the calculus of embeddings is that in codimension greater than 2 this guess is correct.

However, the reduction of (π_0 of) embedding theory to Poincaré embeddings, i.e. the fact that isotopy classes of embeddings up to M in W (in codimension 3) are equivalent to those of M' in W' when M' is homotopy-equivalent to M and W', seems a mystery to these embedding-theoretical methods.[120] As we saw, it is the Poincaré embedding approach that links nicely with the categorical idea of homotopy fixed points. I am optimistic that the coming years will see progress on a useful synthesis of these points of view and their combination with the Farrell–Jones conjecture.

6.11 Notes

In §§6.1 to 6.3, the non-uniqueness of the isometry in a homotopy class is always a torus. This follows from the theorems of Borel explained in §6.1. The fact that the space of homeomorphisms homotopic to the identity is not even connected in high dimensions (except for the case of contractible manifolds relative to the boundary) is due to Hatcher (1978).

Although we make the choice here of "going cubist," i.e. dealing with blocked, rather than parameterized, structures, one need not do so. The way to go would then be obstructed by two issues. The most serious is that we do not understand Homeo(M), the space of homeomorphisms of a compact manifold, in high dimensions *except in a stable range* that is linear in the dimension of the manifold. This story is largely the story of pseudo-isotopy theory and Waldhausen's "algebraic K-theory of spaces." We refer the reader to Weiss and Williams (2001) and Rognes and Waldhausen (2013). The second obstacle is that even in the stable range we do not get contractibility because of non-trivial pseudo-isotopy spaces.[121] However, this is considerably illuminated by the Farrell–Jones conjecture that "blames" the whole difference on the (stable) pseudo-isotopy space of the circle.

[120] Indeed, these methods work more or less the same way in all categories: the essential difference is in their "base case" which is immersion theory, a subject governed by an h-principle, but with different homotopy theory in the different categories. However, the reduction of embeddings to homotopy is only true in PL and Top , not Diff. Presumably, one has to add the immersion of M to the embedding of M' to relate the two. In any case, I do not know how to do this.

[121] Pseudo-isotopy spaces are precisely the difference between blocked and parameterized structures.

The equivariant problems studied in this chapter have a long history – indeed pre-dating Borel. Originally, the philosophy of group actions was to relate all actions to "linear ones." For example, Smith showed that (in the terminology of this chapter) a pseudo-equivalence induces an isomorphism on the F_p homology of the fixed sets of all p-subgroups. (Borel introduced the Borel construction in his famous seminar on transformation groups (Borel, 1960) to give a more conceptual proof of Smith's theorems and extend their scope.) The theme then became to try to compare group actions on "standard" spaces like the disk, sphere, Euclidean space, projective space, etc., to their "linear models" if there were any (see e.g. Bredon, 1972). In low dimensions, there was the goal of geometricization (achieved through the work of Perelman, 2002, Thurston, 2002, and Boileau *et al.*, 2005). In higher dimensions, more and more of the early conjectures of this sort were disproved – first via isolated examples and subsequently systematically.

Among the early results in this "contrary" direction were $\mathbb{Z}_n$-actions on Euclidean space with no fixed points for all n that are not prime powers by Conner and Floyd (1959), an example of Floyd and Richardson (1959) of a group action on the disk with empty fixed set – subsequently developed into the theory of Oliver numbers (Oliver, 1975), and the theory of L. Jones (1971) of converses to Smith theory. Also of great importance was the spherical spaceform problem of determining which finite groups act freely (and to a lesser extent, the classification of these actions) on spheres (which was settled by Madsen *et al.* (1976): all subgroups of order p^2 and $2p$ must be cyclic, the first a fact from Smith theory, and the second a geometric result of Milnor (1957)) – which is different than the situation for free linear actions (where all subgroups of order pq must be cyclic: no metacyclic groups can act linearly).

We were left with a theory of enormous complexity, where all conjectures were false; the positive principles were the conclusions of Smith theory for p-groups, and converses to the combination of Smith theory (due to Jones, 1971) with the Lefshetz fixed-point theorem for non-p-groups (Oliver, 1976b), and a few standout classification results. The differences between the differentiable, PL, and topological categories became abundantly clear from the late 1970s through the 1980s (some of which are explained in §6.2 on trivialities, and some of which depend on the isovariant surgery and equivariant Novikov conjecture results that come later).

There are still a number of standout problems from the early days. My favorites: (Petrie's conjecture) if a homotopy $\mathbb{CP}^n$ has a smooth circle action, must it have the same Pontrjagin classes as $\mathbb{CP}^n$? Does every finite group act freely on a product of spheres? (Or more ambitiously, which groups act on

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which products?) And what are the possible fixed sets of PL $\mathbb{Z}_n$-actions on disks?

I think that the place for new progress in the theory is the world of aspherical manifolds, where rigidity suggests interesting problems. This chapter and the next give some initial results on the equivariant Borel conjecture, on the Nielsen realization problem, and so on. The following is another problem of the same sort.[122]

Conjecture 6.64 *If M is an aspherical manifold whose fundamental group has no center, then only finitely many groups can act effectively on it. If it has center of rank k, then it has a product of at most k-cyclic groups as a subgroup of bounded index.*

Turning now to more specific things mentioned in the body of the text. The construction of counterexamples to the Smith conjecture given here is surely folklore. In dimension 4, the PL Poincaré conjecture is not known, and in any case, the method we used here requires knots whose complement has fundamental group $\mathbb{Z}$. In dimension 4, thanks to the work of Freedman, any such knot is topologically trivial. However, Giffen's construction is very explicit, and is based on "twist-spinning" so one has no need for the Poincaré conjecture.

The theory of Cohen and Sullivan was the PL predecessor to the theory of resolution of homology manifolds. It also foreshadowed the work of Matumoto (1978) and Galewski and Stern (1980) on non-PL triangulations of topological manifolds, leading to the final result of Manolescu (2016) that there are topological manifolds of arbitrarily high dimensions that are not homeomorphic to polyhedra.[123]

The theory by Cappell and me of Rothenberg classes (Cappell and Weinberger, 1991a) measures the lack of homogeneity that might be present in a semi-free PL action whose fixed set is a manifold. It was established in the context of trying to understand the possible neighborhoods of fixed sets of semi-free group actions. It is very similar in spirit to the characteristic class theory $BSRN_2$ of abstract regular neighborhoods in codimension 2 invented by Cappell and Shaneson (1976, 1978).

Examples 6.11 and 6.12 are inspired by Milnor's (1961) counterexamples to the *hauptvermutung* for polyhedra: there are homeomorphic non-PL homeomorphic polyhedra. Milnor relied instead on the "stable classification theory" of manifolds by Mazur. This example has a beautiful irony: Whitehead torsion

[122] As far as I know, this is a conjecture of my own. I don't know whether I really believe it.

[123] For example, the topological manifold which is $S \times E_8$ where E_8 is the unique simply connected closed 4-manifold with quadratic form E_8 can be easily shown to be homeomorphic to a polyhedron, but not to a PL-manifold.

is trivial for homeomorphisms (a theorem of Chapman that follows easily from controlled topology, and also from the work of Kirby and Siebenmann (1977) showing that topological manifolds have handlebody structures).

Rothenberg (1978) developed the PL analogue of torsion for the equivariant setting. The torsions lie in a (group isomorphic to the) sum of Whitehead groups of the equivariant fundamental groups of the various strata (see §6.7). The upshot of this (see Examples 6.12 and 6.16) is that the equivariant torsion is *not* a topological invariant, and that locally linear G-manifolds are *not* equivariantly finite (and equivariant handlebody structures do not exist).

The result about fixed sets of $\mathbb{Q}_8$-actions alluded to (and elaborated on as Example 6.18) is the following. A submanifold of S^n is the fixed set of a PL locally linear $\mathbb{Q}_8$-action iff it is a mod 2 homology sphere of codimension a multiple of 4 and the product of the order of its integral homology groups is $\pm 1 \bmod 8$. It is the fixed set of a topological locally linear action irrespective of the orders of these groups. The necessity of this condition is due to Assadi (1982) who gave a thorough development of finiteness theory for fixed sets and the connections to numerical invariants and how restrictions on isotropy subgroups influence this problem. His work simultaneously extends aspects of the work of Jones and Oliver mentioned above (see also Oliver and Petrie, 1982).

The remaining result is due to Weinberger (1989) – see also Weinberger (1985a) – based on earlier joint work of Cappell and Weinberger (1991a) in order to even build actions on neighborhoods. (The extension of the action from the neighborhood to the whole sphere uses "extension across homology collars"– a result of Assadi and Browder, 1985, and Weinberger, 1985a.) The actual theory is more general – e.g. one can easily replace $\mathbb{Q}_8$ by other groups, but the specific criteria will be different. I picked an example that was easy to state.

Quinn (1988) is a landmark in the application of controlled methods to stratified spaces. The paper is foundational: besides excellent results on concrete problems (e.g. to orbifolds) it puts everything in the right general context. In particular, it contains two very important results: the homogeneity of strata in a homotopically stratified space (which then implies many local linearity results for various constructions of group actions, which typically look locally linear aside from some limit set and can then be deduced to be locally linear everywhere[124]) and the topologically invariant h-cobordism theorem. (In the case of orbifolds, Steinberger (1988) proved essentially the same h-cobordism theorem, expressed very differently, based substantially on extending the ear-

[124] It also is among the motivations for the conjecture in Bryant *et al.* (1993) about the homogeneity of DDP ANR homology manifolds.

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lier ideas of Chapman from the unequivariant case.) This theory justifies the comments in §6.3.

A precise version of the statement about the difference between Top and PL being algebraic K-theory can be found in Anderson and Hsiang (1976, 1977, 1980), which preceded the theory of Quinn (1979, 1982b,c, 1986). This theory can also be used for the purposes of the footnote in Example 6.17.

The problem of the stability of equivariant classifying spaces for neighborhoods is essentially equivalent to the issue of equivariant transversality in the topological setting. This is the point of view of Madsen and Rothenberg (1988a,b, 1989), and the work that they did on nonlinear similarity was part of a deep analysis of the category of locally linear G-actions when G is of odd order. Their approach to geometric topology required transversality and only worked for odd G – because of Example 6.17! The stratified surgery approach works more generally, but it lacks some of the depth of the Madsen–Rothenberg approach. In Chapter 7 we will see some phenomena where equivariant $K(\pi, 1)$ manifolds are really different for $\mathbb{Z}_2$ than for odd-order groups, essentially for reasons that boil down to this transversality issue (although perhaps translated significantly into other more algebraic language).[125]

The fact that, for groups of Oliver number $n_G = 1$, every finite polyhedron[126] occurs as the fixed set of a PL G-action is a modification of a trick of Assadi. It is a completely geometric argument, and I will give it here.

Proposition 6.65 *If G acts piecewise linearly on a disk with empty fixed set, then for any finite complex* F*, G acts on some disk with fixed set* F*.*

Proof Let $\mathcal{D}$ be the G-disk with empty fixed set. Consider $(\mathrm{F} \cup x) * \mathcal{D}$, where x is a disjoint base point and $*\mathcal{D}$ denotes taking the join with $\mathcal{D}$. This has a G-action with fixed set $\mathrm{F} \cup x$. Unfortunately, this join is not a PL disk – it is contractible. One can therefore take an equivariant thickening of this G-space (one essentially replaces each simplex with an equivariant handle (see, for example, Assadi, 1982). This produces a G-disk, denoted Δ, whose fixed set is an abstract regular neighborhood (i.e. a thickening) of $\mathrm{F} \cup x$. Let's denote this by $\mathrm{Nbd}(\mathrm{F}) \cup \mathrm{Nbd}(x)$. *Now* take another join $\Delta * \mathcal{D}$. This produces another G-disk, whose fixed set is again $\mathrm{Nbd}(\mathrm{F}) \cup \mathrm{Nbd}(x)$ with a key difference: the fixed set is now entirely located on the boundary sphere.

Restrict the action to $\partial(\Delta * \mathcal{D})$ and remove an equivariant regular neighborhood of $\mathrm{Nbd}(x)$. This gives an action on a disk with fixed set $\mathrm{Nbd}(\mathrm{F})$ entirely

[125] An equivariant signature class for these manifolds turns out not to be an orientation, or, more fundamentally, its restriction to small balls is not a unit in a suitable ring.

[126] And any finite-dimensional compact ANR occurs for "tame topological actions." For instance, the end-point compactification of any locally finite tree (which will frequently have a Cantor set at infinity) is a fixed set.

included in the interior of this disk! Now recall that regular neighborhoods are mapping cylinders – so collapse the mapping cylinder lines down to F. This is still a disk! (Since these cylinder lines are all interior to the disk.) The G-action has fixed set F. □

Moving on to §6.4, the UNil theory of Cappell applies to all amalgamated free products of groups $A *_B C$, where B injects into A and C. It is equivalent to an appropriate codimension-1 splitting theorem; see Cappell (1976a,b).

Cappell showed that his UNil groups are 2-primary in three senses. First of all, UNil has exponent a power of 2. (It follows from Ranicki's (1979a) localization theorem and the vanishing theorem we will soon assert that it has exponent 8, but Farrell showed that it's actually of exponent 4 in general.[127]) Second, if one studies $L(R\pi)$, and $1/2 \in R$, then UNil vanishes. It is for this reason that when we discuss the L-theory of groups with torsion, we can reasonably conjecture that

$$H_*(\underline{\mathrm{E}\Gamma}/\Gamma; L(R\Gamma_x)) \to L(R\Gamma)$$

is an isomorphism for all Γ, if $1/2 \in R$, while for $R = \mathbb{Z}$, we need to replace $\underline{\mathrm{E}\Gamma}$ by $\mathrm{E}_{\mathrm{vc}}\Gamma$ in the Farrell–Jones conjecture. The third and final vanishing theorem of Cappell is that, if B is square-root-closed in both A and C, then UNil vanishes. This condition means that if, for example, $a^2 = b \in B$, then $a \in B$. So, for the case of connected sums, the square-root-closed condition applies iff the fundamental group has no 2-torsion.

UNil, as mentioned in the text, depends relatively little on the groups A and C, but rather significantly on the group B. The work of Connolly and Davis referred to in the text gives complete information for B trivial. It is clear that the case of B finite should be studied next, especially in light of the Farrell–Jones conjecture.

As we turn more seriously towards the equivariant Borel conjecture, it is important to refer to the early work of Connolly and Kosniewski (1990, 1991) on this problem. Their work (based on the ideas of Farrell and Hsiang that will be explained in Chapter 8) gave a number of cases of odd-order group actions on tori where the equivariant Borel conjecture is true and some counterexamples based on Nil (if one did not assume topological simplicity). They raised the issue of whether the gap hypothesis[128] could be another source counterexample. I had pointed out to them in a letter that UNil was another source counterexample. Thanks to the important work of Connolly and Davis (2004),

[127] And for the infinite dihedral group there are elements of order 4 as Banagl and Ranicki (2006) and Connolly and Davis (2004) show.

[128] That is, when the fixed set of some subgroup was more than around half the dimension of the manifold (or some other stratum it is included in).

Banagl and Ranicki (2006) and Connolly *et al.* (2014), involutions satisfying the gap hypothesis can have their equivariant structure sets analyzed (at least when the fixed set is discrete[129]). In a subsequent paper Connolly *et al.* (2015) deals with the analysis of equivariant structure sets when the singular set is discrete, assuming the Farrell–Jones conjecture holds. This covers many cases in light of the verification of that conjecture by Bartels and Lück (2012a) for all CAT(0) situations.

That there are failures of the equivariant Borel conjecture because of the gap hypothesis as well was first shown in Weinberger (1986). That a tighter connection to embedding theory should exist was explained in Weinberger (1999a). That paper defined concordance embedding groups, proved some of the theorems in §§6.9 and 6.10, and suggested that there might be a "Sullivan conjecture for equivariant structure sets." Shirokova's unpublished University of Chicago thesis showed that the counterexamples given for actions on the torus could be generalized to all finite group actions where the singular set was of the right dimension. In particular, she realized the role of double cosets in the relevant linking theory.

The first precise classification results (where the result was not just that the structure set vanishes) were those arising from joint work with Cappell in the situation of $\mathbb{Z}_p$ odd acting affinely on the torus. The method used an equivariant analogue of Farrell's fibering theorem (see Chapter 7) to reduce it to understanding monodromies, and then calculating with the Federer spectral sequence (and an isovariant analogue). After the event, it seemed that the results could be explained very well by the fact that the Tate cohomology of $\mathbb{Z}_p$ acting on the embeddings vanished. The realization that this would follow from the "Sullivan conjecture" mentioned above and some conversations with John Klein led to the treatment given here.

The conjecture that assembly maps:

$$H_*(\underline{\mathrm{E}\Gamma}/\Gamma; K(R\Gamma_x)) \to K(R\Gamma),$$
$$H_*(\underline{\mathrm{E}\Gamma}/\Gamma; L(R\Gamma_x)) \to L(R\Gamma),$$

could be isomorphisms was made by Quinn (1985b) (see also Quinn, 1987a), recognizing that they were false because of Nils and UNils. The h-cobordism theorem in Quinn (1970) and the stratified surgery theorem in Weinberger (1986) relate these to rigidity, as Quinn points out (at least regarding K-theory). Of course, the issues regarding Nil and UNil were only confronted by the Farrell–Jones conjecture.

129 The case they deal with explicitly. However, most of their paper directly generalizes to the case asserted.

In §§6.5 and 6.6, besides equivariant rigidity, other motivations for stratified surgery were the nonlinear similarity problem and the development of intersection homology.

The main positive results about nonlinear similarity were the case of odd p-groups, proved by Schultz (1977) and Sullivan (unpublished) and then the general odd-order case by Hsiang and Pardon (1982) and Madsen and Rothenberg (1988a,b, 1989). The Hsiang and Pardon approach can be compared to a daring commando raid, while Madsen and Rothenberg's was like a major strategic effort aimed at much broader objectives. Meanwhile for even-order groups, Cappell and Shaneson showed that nonlinear similarities exist (and gave a type of stable classification, as we had mentioned). Hambleton and Pedersen (2005) gave a solution of the problem for all cyclic groups.

Intersection homology (Cheeger, 1980; Goresky and MacPherson, 1980, 1983) gave a perspective from which many stratified spaces (e.g. complex varieties) could be viewed as being like manifolds (e.g. satisfying Poincaré duality). It became natural from that point of view to wonder whether surgery theory could be extended to that setting. In a piece of work that briefly preceded stratified surgery, Cappell and I showed how to extend surgery to the "supernormal even-codimensional stratified spaces" (Cappell and Weinberger, 1991b). In that setting all the usual theorems about manifolds naturally extend (such as the Novikov conjecture for stratified homotopy equivalences).

For §§6.7 and 6.9, the equivariant Novikov conjecture was first studied in Rosenberg and Weinberger (1990). We realized how closely it fit into the framework used by Kasparov – at least in many cases. Our interest was for both its topological and its differential geometric implications. Gong (1998) and Hanke (2008) amplify each of these directions, respectively. One point that we did not appreciate at the time is the one made in the appendix to §6.7 – i.e. that the equivariant Novikov conjecture has systematic failure when one does not have the relevant injectivity of fundamental groupoids of fixed sets.

The connection between the equivariant Novikov conjecture and equivariant surgery was a fortuitous conclusion. That the L-groups break up (for finite group actions) away from 2 is a general phenomenon (Lück and Madsen, 1990a,b; Cappell *et al.*, 1991). At 2, Lück and Madsen (1990a,b) give a general result for locally linear G-manifolds. Cappell *et al.* (2013) describe some integral splitting results of a "replacement theorem" sort. The relΣ theory has good equivariant functoriality, which leads to a good formulation of such isovariant surgery in terms of assembly maps. The not relΣ theory has some functoriality as well – hopefully Cappell, Yan, and I will write a paper about this in due course – but currently it is a difficult and complicated set of examples.

This work on functoriality is based, as is the functoriality relevant to the

Atiyah–Singer theorem (Atiyah and Singer, 1968a,b, 1971), on periodicity theorems. The first periodicity theorem for isovariant structure sets was due to Yan (1993) for odd-order groups, and was based on the method of Browder explained for the decomposition theorem in §6.9. Weinberger and Yan (2005) proved a similar theorem for general compact groups. Unlike the Browder method, this requires stratified spaces rather than G-manifolds. We therefore have not yet been able to prove a decomposition theorem in general.

The pseudo-category is introduced here to foreshadow other uses of homotopy fixed sets in §§6.9 and 6.10. On the other hand, the pseudo-category is very powerful in the theory of group action. The celebrated Sullivan conjecture (now a theorem of H. Miller, 1987; see also Carlsson, 1991; Lannes and Schwartz, 1986) says that the space of pseudo-maps from a point to X is p-adically equivalent to fixed set of $\mathbb{Z}_p$ acting on X (see also Dwyer and Wilkerson, 1988).

The approach we have chosen to give in §§6.8–6.10 for the Farrell–Jones conjecture is the one they gave "after the fact." As I emphasized in Footnote 98, Farrell and Jones were motivated by the role that geodesics played in their proofs of Borel conjecture statements. It was only when they analyzed pseudo-isotopy spaces for non-positively curved closed manifolds (or at least some locally symmetric spaces of that sort) that they were willing to make this bold conjecture.

One point that I think is significant is that the Goodwillie–Klein–Weiss calculus of embedding idea that occurs in §6.10 can be described similarly. Recognizing that an h-principle fails for two points (or larger finite sets), one again finds the simplest functorial expression compatible with true calculations and discovers (following Weiss, 1996) – in their case – a theorem.

Thus, the Farrell–Jones conjecture, the Goodwillie–Klein–Weiss calculus of embeddings, the Sullivan conjecture, and its variant for structure sets are all of one spirit. Given the ubiquity of h-principles (see Gromov, 1986), this somewhat more sophisticated variant might be of help in other circumstances where h-principles fail.

The explicit calculations are influenced by ideas of Kearton, Hacon, Mio, al Rubaee, and Habeggar. I refer to Goodwillie *et al.* (2001) for a survey.