1

Poisson and Other Discrete Distributions

The Poisson distribution arises as a limit of the binomial distribution. This chapter contains a brief discussion of some of its fundamental properties as well as the Poisson limit theorem for null arrays of integer-valued random variables. The chapter also discusses the binomial and negative binomial distributions.

1.1 The Poisson Distribution

A random variable X is said to have a *binomial distribution* $\mathrm{Bi}(n, p)$ with parameters $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $p \in [0, 1]$ if

$$\mathbb{P}(X = k) = \mathrm{Bi}(n, p; k) := \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, \dots, n, \tag{1.1}$$

where $0^0 := 1$. In the case $n = 1$ this is the *Bernoulli distribution* with parameter p. If $X_1, \dots, X_n$ are independent random variables with such a Bernoulli distribution, then their sum has a binomial distribution, that is

$$X_1 + \cdots + X_n \overset{d}{=} X, \tag{1.2}$$

where X has the distribution $\mathrm{Bi}(n, p)$ and where $\overset{d}{=}$ denotes equality in distribution. It follows that the expectation and variance of X are given by

$$\mathbb{E}[X] = np, \qquad \mathbb{V}\mathrm{ar}[X] = np(1 - p). \tag{1.3}$$

A random variable X is said to have a *Poisson distribution* $\mathrm{Po}(\gamma)$ with parameter $\gamma \geq 0$ if

$$\mathbb{P}(X = k) = \mathrm{Po}(\gamma; k) := \frac{\gamma^k}{k!} e^{-\gamma}, \quad k \in \mathbb{N}_0. \tag{1.4}$$

If $\gamma = 0$, then $\mathbb{P}(X = 0) = 1$, since we take $0^0 = 1$. Also we allow $\gamma = \infty$; in this case we put $\mathbb{P}(X = \infty) = 1$ so $\mathrm{Po}(\infty; k) = 0$ for $k \in \mathbb{N}_0$.

The Poisson distribution arises as a limit of binomial distributions as

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follows. Let $p_n \in [0,1]$, $n \in \mathbb{N}$, be a sequence satisfying $np_n \to \gamma$ as $n \to \infty$, with $\gamma \in (0,\infty)$. Then, for $k \in \{0,\dots,n\}$,

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{(np_n)^k}{k!} \cdot \frac{(n)_k}{n^k} \cdot (1-p_n)^{-k} \cdot \left(1 - \frac{np_n}{n}\right)^n \to \frac{\gamma^k}{k!} e^{-\gamma}, \quad (1.5)$$

as $n \to \infty$, where

$$(n)_k := n(n-1)\cdots(n-k+1) \quad (1.6)$$

is the k-th *descending factorial* (of n) with $(n)_0$ interpreted as 1.

Suppose X is a Poisson random variable with finite parameter γ. Then its expectation is given by

$$\mathbb{E}[X] = e^{-\gamma} \sum_{k=0}^{\infty} k \frac{\gamma^k}{k!} = e^{-\gamma} \gamma \sum_{k=1}^{\infty} \frac{\gamma^{k-1}}{(k-1)!} = \gamma. \quad (1.7)$$

The *probability generating function* of X (or of $\mathrm{Po}(\gamma)$) is given by

$$\mathbb{E}[s^X] = e^{-\gamma} \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} s^k = e^{-\gamma} \sum_{k=0}^{\infty} \frac{(\gamma s)^k}{k!} = e^{\gamma(s-1)}, \quad s \in [0,1]. \quad (1.8)$$

It follows that the *Laplace transform* of X (or of $\mathrm{Po}(\gamma)$) is given by

$$\mathbb{E}[e^{-tX}] = \exp[-\gamma(1-e^{-t})], \quad t \geq 0. \quad (1.9)$$

Formula (1.8) is valid for each $s \in \mathbb{R}$ and (1.9) is valid for each $t \in \mathbb{R}$. A calculation similar to (1.8) shows that the *factorial moments* of X are given by

$$\mathbb{E}[(X)_k] = \gamma^k, \quad k \in \mathbb{N}_0, \quad (1.10)$$

where $(0)_0 := 1$ and $(0)_k := 0$ for $k \geq 1$. Equation (1.10) implies that

$$\mathbb{V}\mathrm{ar}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[(X)_2] + \mathbb{E}[X] - \mathbb{E}[X]^2 = \gamma. \quad (1.11)$$

We continue with a characterisation of the Poisson distribution.

Proposition 1.1 *An $\mathbb{N}_0$-valued random variable X has distribution* $\mathrm{Po}(\gamma)$ *if and only if, for every function $f\colon \mathbb{N}_0 \to \mathbb{R}_+$, we have*

$$\mathbb{E}[Xf(X)] = \gamma\, \mathbb{E}[f(X+1)]. \quad (1.12)$$

Proof By a similar calculation to (1.7) and (1.8) we obtain for any function $f\colon \mathbb{N}_0 \to \mathbb{R}_+$ that (1.12) holds. Conversely, if (1.12) holds for all such functions f, then we can make the particular choice $f := \mathbf{1}_{\{k\}}$ for $k \in \mathbb{N}$, to obtain the recursion

$$k\,\mathbb{P}(X = k) = \gamma\,\mathbb{P}(X = k-1).$$

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This recursion has (1.4) as its only (probability) solution, so the result follows. □

1.2 Relationships Between Poisson and Binomial Distributions

The next result says that if X and Y are independent Poisson random variables, then $X+Y$ is also Poisson and the conditional distribution of X given $X + Y$ is binomial:

Proposition 1.2 *Let X and Y be independent with distributions* $\mathrm{Po}(\gamma)$ *and* $\mathrm{Po}(\delta)$*, respectively, with* $0 < \gamma+\delta < \infty$*. Then $X+Y$ has distribution* $\mathrm{Po}(\gamma+\delta)$ *and*

$$\mathbb{P}(X = k \mid X + Y = n) = \mathrm{Bi}(n, \gamma/(\gamma + \delta); k), \qquad n \in \mathbb{N}_0,\ k = 0, \dots, n.$$

Proof For $n \in \mathbb{N}_0$ and $k \in \{0, \dots, n\}$,

$$\begin{aligned}\mathbb{P}(X = k, X + Y = n) = \mathbb{P}(X = k, Y = n - k) &= \frac{\gamma^k}{k!}e^{-\gamma}\frac{\delta^{n-k}}{(n-k)!}e^{-\delta}\\ &= e^{-(\gamma+\delta)}\Big(\frac{(\gamma+\delta)^n}{n!}\Big)\binom{n}{k}\Big(\frac{\gamma}{\gamma+\delta}\Big)^k\Big(\frac{\delta}{\gamma+\delta}\Big)^{n-k}\\ &= \mathrm{Po}(\gamma+\delta; n)\,\mathrm{Bi}(n, \gamma/(\gamma+\delta); k),\end{aligned}$$

and the assertions follow. □

Let Z be an $\mathbb{N}_0$-valued random variable and let $Z_1, Z_2, \dots$ be a sequence of independent random variables that have a Bernoulli distribution with parameter $p \in [0, 1]$. If Z and $(Z_n)_{n\geq 1}$ are independent, then the random variable

$$X := \sum_{j=1}^{Z} Z_j \tag{1.13}$$

is called a *p-thinning* of Z, where we set $X := 0$ if $Z = 0$. This means that the conditional distribution of X given $Z = n$ is binomial with parameters n and p.

The following partial converse of Proposition 1.2 is a noteworthy property of the Poisson distribution.

Proposition 1.3 *Let* $p \in [0, 1]$*. Let Z have a Poisson distribution with parameter $\gamma \geq 0$ and let X be a p-thinning of Z. Then X and $Z - X$ are independent and Poisson distributed with parameters $p\gamma$ and* $(1 - p)\gamma$*, respectively.*

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Proof We may assume that $\gamma > 0$. The result follows once we have shown that

$$\mathbb{P}(X = m, Z - X = n) = \mathrm{Po}(p\gamma; m)\,\mathrm{Po}((1-p)\gamma; n), \qquad m, n \in \mathbb{N}_0. \tag{1.14}$$

Since the conditional distribution of X given $Z = m + n$ is binomial with parameters $m + n$ and p, we have

$$\begin{aligned}\mathbb{P}(X = m, Z - X = n) &= \mathbb{P}(Z = m + n)\,\mathbb{P}(X = m \mid Z = m + n)\\ &= \left(\frac{e^{-\gamma}\gamma^{m+n}}{(m+n)!}\right)\binom{m+n}{m} p^m (1-p)^n\\ &= \left(\frac{p^m\gamma^m}{m!}\right)e^{-p\gamma}\left(\frac{(1-p)^n\gamma^n}{n!}\right)e^{-(1-p)\gamma},\end{aligned}$$

and (1.14) follows. □

1.3 The Poisson Limit Theorem

The next result generalises (1.5) to sums of Bernoulli variables with unequal parameters, among other things.

Proposition 1.4 *Suppose for $n \in \mathbb{N}$ that $m_n \in \mathbb{N}$ and $X_{n,1}, \dots, X_{n,m_n}$ are independent random variables taking values in $\mathbb{N}_0$. Let $p_{n,i} := \mathbb{P}(X_{n,i} \geq 1)$ and assume that*

$$\lim_{n\to\infty} \max_{1\leq i\leq m_n} p_{n,i} = 0. \tag{1.15}$$

Assume further that $\lambda_n := \sum_{i=1}^{m_n} p_{n,i} \to \gamma$ as $n \to \infty$, where $\gamma > 0$, and that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \mathbb{P}(X_{n,i} \geq 2) = 0. \tag{1.16}$$

Let $X_n := \sum_{i=1}^{m_n} X_{n,i}$. Then for $k \in \mathbb{N}_0$ we have

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \mathrm{Po}(\gamma; k). \tag{1.17}$$

Proof Let $X'_{n,i} := \mathbf{1}\{X_{n,i} \geq 1\} = \min\{X_{n,i}, 1\}$ and $X'_n := \sum_{i=1}^{m_n} X'_{n,i}$. Since $X'_{n,i} \neq X_{n,i}$ if and only if $X_{n,i} \geq 2$, we have

$$\mathbb{P}(X'_n \neq X_n) \leq \sum_{i=1}^{m_n} \mathbb{P}(X_{n,i} \geq 2).$$

By assumption (1.16) we can assume without restriction of generality that

$X'_{n,i} = X_{n,i}$ for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, m_n\}$. Moreover it is no loss of generality to assume for each (n, i) that $p_{n,i} < 1$. We then have

$$\mathbb{P}(X_n = k) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} \frac{\prod_{j=1}^{m_n}(1 - p_{n,j})}{(1 - p_{n,i_1}) \cdots (1 - p_{n,i_k})}. \tag{1.18}$$

Let $\mu_n := \max_{1 \le i \le m_n} p_{n,i}$. Since $\sum_{j=1}^{m_n} p_{n,j}^2 \le \lambda_n \mu_n \to 0$ as $n \to \infty$, we have

$$\log\left(\prod_{j=1}^{m_n}(1 - p_{n,j})\right) = \sum_{j=1}^{m_n}(-p_{n,j} + O(p_{n,j}^2)) \to -\gamma \text{ as } n \to \infty, \tag{1.19}$$

where the function $O(\cdot)$ satisfies $\limsup_{r \to 0} |r|^{-1}|O(r)| < \infty$. Also,

$$\inf_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} (1 - p_{n,i_1}) \cdots (1 - p_{n,i_k}) \ge (1 - \mu_n)^k \to 1 \text{ as } n \to \infty. \tag{1.20}$$

Finally, with $\sum^{\neq}_{i_1, \ldots, i_k \in \{1,2,\ldots,m_n\}}$ denoting summation over all ordered k-tuples of distinct elements of $\{1, 2, \ldots, m_n\}$, we have

$$k! \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} = \sum^{\neq}_{i_1, \ldots, i_k \in \{1,2,\ldots,m_n\}} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k},$$

and

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{m_n} p_{n,i}\right)^k - \sum^{\neq}_{i_1, \ldots, i_k \in \{1,2,\ldots,m_n\}} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} \\ &\le \binom{k}{2} \sum_{i=1}^{m_n} p_{n,i}^2 \left(\sum_{j=1}^{m_n} p_{n,j}\right)^{k-2}, \end{aligned}$$

which tends to zero as $n \to \infty$. Therefore

$$k! \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} \to \gamma^k \text{ as } n \to \infty. \tag{1.21}$$

The result follows from (1.18) by using (1.19), (1.20) and (1.21). □

1.4 The Negative Binomial Distribution

A random element Z of $\mathbb{N}_0$ is said to have a *negative binomial distribution* with parameters $r > 0$ and $p \in (0, 1]$ if

$$\mathbb{P}(Z = n) = \frac{\Gamma(n + r)}{\Gamma(n + 1)\Gamma(r)}(1 - p)^n p^r, \quad n \in \mathbb{N}_0, \tag{1.22}$$

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where the *Gamma function* $\Gamma\colon (0,\infty) \to (0,\infty)$ is defined by

$$\Gamma(a) := \int_0^\infty t^{a-1} e^{-t}\, dt, \quad a > 0. \tag{1.23}$$

(In particular $\Gamma(a) = (a-1)!$ for $a \in \mathbb{N}$.) This can be seen to be a probability distribution by Taylor expansion of $(1-x)^{-r}$ evaluated at $x = 1-p$. The probability generating function of Z is given by

$$\mathbb{E}[s^Z] = p^r(1-s+sp)^{-r}, \quad s \in [0,1]. \tag{1.24}$$

For $r \in \mathbb{N}$, such a Z may be interpreted as the number of failures before the rth success in a sequence of independent Bernoulli trials. In the special case $r = 1$ we get the *geometric distribution*

$$\mathbb{P}(Z = n) = (1-p)^n p, \quad n \in \mathbb{N}_0. \tag{1.25}$$

Another interesting special case is $r = 1/2$. In this case

$$\mathbb{P}(Z = n) = \frac{(2n-1)!!}{2^n n!}(1-p)^n p^{1/2}, \quad n \in \mathbb{N}_0, \tag{1.26}$$

where we recall the definition (B.6) for $(2n-1)!!$. This follows from the fact that $\Gamma(n+1/2) = (2n-1)!!\, 2^{-n}\sqrt{\pi}$, $n \in \mathbb{N}_0$.

The negative binomial distribution arises as a mixture of Poisson distributions. To explain this, we need to introduce the *Gamma distribution* with *shape parameter* $a > 0$ and *scale parameter* $b > 0$. This is a probability measure on $\mathbb{R}_+$ with Lebesgue density

$$x \mapsto b^a \Gamma(a)^{-1} x^{a-1} e^{-bx} \tag{1.27}$$

on $\mathbb{R}_+$. If a random variable Y has this distribution, then one says that Y is Gamma distributed with shape parameter a and scale parameter b. In this case Y has Laplace transform

$$\mathbb{E}[e^{-tY}] = \left(\frac{b}{b+t}\right)^a, \quad t \geq 0. \tag{1.28}$$

In the case $a = 1$ we obtain the *exponential distribution* with parameter b. Exercise 1.11 asks the reader to prove the following result.

Proposition 1.5 *Suppose that the random variable $Y \geq 0$ is Gamma distributed with shape parameter $a > 0$ and scale parameter $b > 0$. Let Z be an $\mathbb{N}_0$-valued random variable such that the conditional distribution of Z given Y is* $\mathrm{Po}(Y)$. *Then Z has a negative binomial distribution with parameters a and $b/(b+1)$.*

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1.5 Exercises

Exercise 1.1 Prove equation (1.10).

Exercise 1.2 Let X be a random variable taking values in $\mathbb{N}_0$. Assume that there is a $\gamma \geq 0$ such that $\mathbb{E}[(X)_k] = \gamma^k$ for all $k \in \mathbb{N}_0$. Show that X has a Poisson distribution. (Hint: Derive the Taylor series for $g(s) := \mathbb{E}[s^X]$ at $s_0 = 1$.)

Exercise 1.3 Confirm Proposition 1.3 by showing that

$$\mathbb{E}[s^X t^{Z-X}] = e^{p\gamma(s-1)} e^{(1-p)\gamma(t-1)}, \quad s, t \in [0, 1],$$

using a direct computation and Proposition B.4.

Exercise 1.4 (Generalisation of Proposition 1.2) Let $m \in \mathbb{N}$ and suppose that $X_1, \dots, X_m$ are independent random variables with Poisson distributions $\mathrm{Po}(\gamma_1), \dots, \mathrm{Po}(\gamma_m)$, respectively. Show that $X := X_1 + \cdots + X_m$ is Poisson distributed with parameter $\gamma := \gamma_1 + \cdots + \gamma_m$. Assuming $\gamma > 0$, show moreover for any $k \in \mathbb{N}$ that

$$\mathbb{P}(X_1 = k_1, \dots, X_m = k_m \mid X = k) = \frac{k!}{k_1! \cdots k_m!} \left(\frac{\gamma_1}{\gamma}\right)^{k_1} \cdots \left(\frac{\gamma_m}{\gamma}\right)^{k_m} \tag{1.29}$$

for $k_1 + \cdots + k_m = k$. This is a *multinomial distribution* with parameters k and $\gamma_1/\gamma, \dots, \gamma_m/\gamma$.

Exercise 1.5 (Generalisation of Proposition 1.3) Let $m \in \mathbb{N}$ and suppose that Z_n, $n \in \mathbb{N}$, is a sequence of independent random vectors in $\mathbb{R}^m$ with common distribution $\mathbb{P}(Z_1 = e_i) = p_i$, $i \in \{1, \dots, m\}$, where e_i is the i-th unit vector in $\mathbb{R}^m$ and $p_1 + \cdots + p_m = 1$. Let Z have a Poisson distribution with parameter γ, independent of $(Z_1, Z_2, \dots)$. Show that the components of the random vector $X := \sum_{j=1}^{Z} Z_j$ are independent and Poisson distributed with parameters $p_1\gamma, \dots, p_m\gamma$.

Exercise 1.6 (Bivariate extension of Proposition 1.4) Let $\gamma > 0, \delta \geq 0$. Suppose for $n \in \mathbb{N}$ that $m_n \in \mathbb{N}$ and for $1 \leq i \leq m_n$ that $p_{n,i}, q_{n,i} \in [0, 1)$ with $\sum_{i=1}^{m_n} p_{n,i} \to \gamma$ and $\sum_{i=1}^{m_n} q_{n,i} \to \delta$, and $\max_{1 \leq i \leq m_n} \max\{p_{n,i}, q_{n,i}\} \to 0$ as $n \to \infty$. Suppose for $n \in \mathbb{N}$ that $(X_n, Y_n) = \sum_{i=1}^{m_n} (X_{n,i}, Y_{n,i})$, where each $(X_{n,i}, Y_{n,i})$ is a random 2-vector whose components are Bernoulli distributed with parameters $p_{n,i}, q_{n,i}$, respectively, and satisfy $X_{n,i} Y_{n,i} = 0$ almost surely. Assume the random vectors $(X_{n,i}, Y_{n,i})$, $1 \leq i \leq m_n$, are independent. Prove that X_n, Y_n are asymptotically (as $n \to \infty$) distributed as a pair of independent Poisson variables with parameters γ, δ, i.e. for $k, \ell \in \mathbb{N}_0$,

$$\lim_{n\to\infty} \mathbb{P}(X_n = k, Y_n = \ell) = e^{-(\gamma+\delta)} \frac{\gamma^k \delta^\ell}{k!\ell!}.$$

Exercise 1.7 (Probability of a Poisson variable being even) Suppose X is Poisson distributed with parameter $\gamma > 0$. Using the fact that the probability generating function (1.8) extends to $s = -1$, verify the identity $\mathbb{P}(X/2 \in \mathbb{Z}) = (1 + e^{-2\gamma})/2$. For $k \in \mathbb{N}$ with $k \geq 3$, using the fact that the probability generating function (1.8) extends to a k-th complex root of unity, find a closed-form formula for $\mathbb{P}(X/k \in \mathbb{Z})$.

Exercise 1.8 Let $\gamma > 0$, and suppose X is Poisson distributed with parameter γ. Suppose $f \colon \mathbb{N} \to \mathbb{R}_+$ is such that $\mathbb{E}[f(X)^{1+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Show that $\mathbb{E}[f(X + k)] < \infty$ for any $k \in \mathbb{N}$.

Exercise 1.9 Let $0 < \gamma < \gamma'$. Give an example of a random vector (X, Y) with X Poisson distributed with parameter γ and Y Poisson distributed with parameter γ', such that $Y-X$ *is not* Poisson distributed. (Hint: First consider a pair X', Y' such that $Y'-X'$ *is* Poisson distributed, and then modify finitely many of the values of their joint probability mass function.)

Exercise 1.10 Suppose $n \in \mathbb{N}$ and set $[n] := \{1, \dots, n\}$. Suppose that Z is a uniform random permutation of $[n]$, that is a random element of the space Σ_n of all bijective mappings from $[n]$ to $[n]$ such that $\mathbb{P}(Z = \pi) = 1/n!$ for each $\pi \in \Sigma_n$. For $a \in \mathbb{R}$ let $\lceil a \rceil := \min\{k \in \mathbb{Z} : k \geq a\}$. Let $\gamma \in [0, 1]$ and let $X_n := \mathrm{card}\{i \in [\lceil \gamma n \rceil] : Z(i) = i\}$ be the number of fixed points of Z among the first $\lceil \gamma n \rceil$ integers. Show that the distribution of X_n converges to $\mathrm{Po}(\gamma)$, that is

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \frac{\gamma^k}{k!} e^{-\gamma}, \quad k \in \mathbb{N}_0.$$

(Hint: Establish an explicit formula for $\mathbb{P}(X_n = k)$, starting with the case $k = 0$.)

Exercise 1.11 Prove Proposition 1.5.

Exercise 1.12 Let $\gamma > 0$ and $\delta > 0$. Find a random vector (X, Y) such that X, Y and $X + Y$ are Poisson distributed with parameter γ, δ and $\gamma + \delta$, respectively, but X and Y are not independent.

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