11

Minimal Models and Their Singularities

We review the definitions and results of the minimal model program that we used repeatedly.

Assumptions The theorems of Sections 11.1–11.3 are currently known in characteristic 0. See Kollár and Mori (1998) or Kollár (2013b) for varieties; Lyu and Murayama (2022) and Fujino (2022) in general.

Most of the older literature works with $\mathbb{Q}$-divisors. We treat $\mathbb{R}$-divisors on arbitrary schemes in Section 11.4.

11.1 Singularities of Pairs

Singularities of pairs are treated thoroughly in Kollár (2013b). Here we aim to be concise, discussing all that is necessary for the main results in this book, but leaving many details untouched.

Definition 11.1 (Pairs) We are primarily interested in pairs (X, Δ) where X is a normal variety over a field and $\Delta = \sum a_i D_i$ a formal linear combination of prime divisors with rational or real coefficients. More generally, X can be a reduced scheme and $\Delta = \sum a_i D_i$ a formal linear combination of prime, Mumford divisors (4.16.4), that is, none of the D_i are contained in $\operatorname{Sing} X$.

For a prime divisor E, we use $\operatorname{coeff}_E(\Delta)$ to denote the *coefficient* of E in Δ. That is, $E \not\subset \operatorname{Supp}(\Delta - \operatorname{coeff}_E(\Delta) \cdot E)$. We use $\operatorname{coeff}(\Delta)$ to denote the set of all nonzero coefficients in Δ.

If Δ is $\mathbb{R}$-Cartier, $\pi\colon X' \to X$ is birational and E' is a prime divisor on X', then $\operatorname{coeff}_{E'}(\Delta) := \operatorname{coeff}_{E'}(\pi^*\Delta)$ defines the coefficient of every prime divisor over X in Δ.

For any $c \in \mathbb{R}$ we set $\Delta^{>c} := \sum_{i:\ a_i>c} a_i D_i$, and similarly for $\Delta^{=c}, \Delta^{<c}$.

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

Definition 11.2 (Canonical or dualizing sheaf) A pure dimensional, projective scheme over a field has a dualizing sheaf as in Hartshorne (1977, III.7), but for arbitrary schemes the existence of a dualizing sheaf is a complicated issue. The following quite general setting is sufficient for our purposes.

Let $g\colon X \to S$ be a finite type morphism. As in Stacks (2022, tag 0E9M), there is a *relative dualizing complex.* If X is pure dimensional, the lowest non-zero cohomology of it is the *relative dualizing sheaf,* or *relative canonical sheaf,* denoted by $\omega_{X/S}$.

We are interested in cases where $\omega_{X/S}$ depends very little on S. This happens when $\mathscr{O}_S$ is a dualizing complex on S Stacks (2022, tag 0AWV). We only need to know that this occurs in four important cases:

- S is the spectrum of a field,
- S is smooth over a field,
- S is regular and of dimension 1, or
- S is the spectrum of a regular, local ring.

We declare $\omega_{X/S}$ to be a *canonical sheaf* of X and denote it by ω_X.

Note that we do not need $X \to S$ to be surjective. So if we want to work over a quasi-projective scheme S, we choose an embedding $S \hookrightarrow \mathbb{P}^N$ and work over $\mathbb{P}^N$. Similarly, if S is the spectrum of a complete local ring, we can embed it into the spectrum of a regular, complete local ring. However, ω_X is well defined only up to tensoring with pull-backs by line bundles from S. Thus one should use it only for properties of X that are local on S.

Definition 11.3 (Canonical class II) Let X be a scheme that has a canonical sheaf ω_X. If ω_X is invertible outside a subset of codimension ≥ 2 – for example, X is normal or demi-normal – then it corresponds to a linear equivalence class of Mumford divisors K_X, called the *canonical class* of X.

Assumptions In Sections 11.1–11.3, we work with pairs that have a canonical class.

Definition 11.4 (Discrepancy of divisors) Let $(X, \Delta = \sum a_i D_i)$ be a pair as in (11.1) that has a canonical class (11.3). We are looking at cases when the pull-back of $K_X + \Delta$ by birational morphisms makes sense. If Δ is a $\mathbb{Q}$-divisor, the natural assumption is that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, that is, $m(K_X + \Delta)$ is Cartier for some $m > 0$.

If Δ is an $\mathbb{R}$-divisor, we need to assume that $K_X + \Delta$ is $\mathbb{R}$-*Cartier*, we discuss this notion in detail in Section 11.4. (See (4.48) for the even more general notion of numerically $\mathbb{R}$-Cartier divisors).

Let $f\colon Y \to X$ be a proper, birational morphism from a demi-normal scheme Y (11.36), $\operatorname{Ex}(f) \subset Y$ the exceptional locus, and $E_i \subset \operatorname{Ex}(f)$ the irreducible exceptional divisors. Assume that $\operatorname{Ex}(f) \cap \operatorname{Sing} Y$ and $f(\operatorname{Ex}(f))$ have codimension ≥ 2 in Y and X; these are automatic if X and Y are normal. Let $f_*^{-1}\Delta := \sum a_i f_*^{-1} D_i$ denote the birational transform of Δ. Fix any canonical divisor K^Y in the linear equivalence class K_Y and set $K^X := f_*(K^Y)$.

Assume next that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then $K^Y + f_*^{-1}\Delta - f^*(K^X + \Delta)$ makes sense and it is exceptional, hence we can write

$$K^Y + f_*^{-1}\Delta = f^*(K^X + \Delta) + \textstyle\sum_i a(E_i, X, \Delta) E_i. \tag{11.4.1}$$

The $a(E_i, X, \Delta) \in \mathbb{R}$ are independent of the choice of K^Y. This defines $a(E, X, \Delta)$ for exceptional divisors. Set $a(E, X, \Delta) := -\operatorname{coeff}_E \Delta$ for non-exceptional divisors $E \subset X$.

The real number $a(E, X, \Delta)$ is called the *discrepancy* of E with respect to (X, Δ); it depends only on the valuation defined by E, not on the choice of f. (See Kollár and Mori (1998, 2.22) for a more canonical definition.)

Warning 11.4.2 For most cases of interest to us, $a(E, X, \Delta) \geq -1$, so some authors use *log discrepancies*, $a_\ell(E, X, \Delta) := 1 + a(E, X, \Delta)$. Unfortunately, some people use $a(E, X, \Delta)$ to denote the log discrepancy, leading to confusion.

The discrepancies have the following obvious monotonicity and linearity properties; see Kollár and Mori (1998, 2.27).

Claim 11.4.3 Let Δ' be an effective, $\mathbb{R}$-Cartier divisor and E a divisor over X. Then $a(E, X, \Delta + \Delta') = a(E, X, \Delta) - \operatorname{coeff}_E \Delta'$. In particular, $a(E, X, \Delta + \Delta') \leq a(E, X, \Delta)$, and $a(E, X, \Delta + \Delta') < a(E, X, \Delta)$ iff $\operatorname{center}_X E \subset \operatorname{Supp} \Delta'$. □

Claim 11.4.4 Assume that $K_X + \Delta_i$ are $\mathbb{R}$-Cartier. Fix $\lambda_i \geq 0$ such that $\sum \lambda_i = 1$ and set $\Delta := \sum \lambda_i \Delta_i$. Then $K_X + \Delta$ is $\mathbb{R}$-Cartier and $a(E, X, \Delta) = \sum \lambda_i a(E, X, \Delta_i)$ for every divisor E over X. In particular, using the next definition, if the (X, Δ_i) are lc (resp. dlt, klt, canonical, terminal) then so is (X, Δ). □

Definition 11.5 Let X be a normal scheme of dimension ≥ 2 and $\Delta = \sum a_i D_i$ an $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that (X, Δ) is

$$\left.\begin{array}{c} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{plt} \\ \textit{dlt} \\ \textit{lc} \end{array}\right\} \quad \text{if } a(E, X, \Delta) \text{ is} \quad \left\{\begin{array}{ll} > 0 & \text{for every exceptional } E, \\ \geq 0 & \text{for every exceptional } E, \\ > -1 & \text{for every } E, \\ > -1 & \text{for every exceptional } E, \\ > -1 & \text{if } \operatorname{center}_X E \subset \text{non-snc}(X, \Delta), \\ \geq -1 & \text{for every } E. \end{array}\right.$$

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

Here klt is short for *Kawamata log terminal*, plt for *purely log terminal,* dlt for *divisorial log terminal,* lc for *log canonical,* and non-snc(X, Δ) denotes the set of points where (X, Δ) is not a simple normal crossing pair (p.xvi).

We define semi-log-canonical or slc pairs in (11.37).

Claim 11.5.1 If (X, Δ) is in any of these 6 classes, $0 \le \Delta' \le \Delta$ and $K_X + \Delta'$ is $\mathbb{R}$-Cartier, then (X, Δ') is also in the same class. □

Claim 11.5.2 Assume that (X, Δ) is terminal (resp. klt) and Θ is an effective $\mathbb{R}$-Cartier divisor. If (X, Δ) has a log resolution (p.xvi), then $(X, \Delta + \varepsilon\Theta)$ is also terminal (resp. klt) for $0 \le \varepsilon \ll 1$. (See (11.10.6) for the other cases.)

We gave some examples in (1.33) and (1.40); see also Section 2.2 for such surfaces, (2.35) for cones, and Kollár (2013b) for a detailed treatment.

For computing discrepancies, the following are useful; see also (Kollár and Mori, 1998, 2.29–30).

Lemma 11.6 *Let $(X, \Delta - \Theta)$ be an snc pair, where $\Delta = \sum(1 - a_i)D_i$ and Θ are effective. Let E be a divisor over X such that $a(E, X, \Delta - \Theta) < 0$. Then $a(E, X, \lceil\Delta\rceil) = -1$ and $a(E, X, \Delta - \Theta) \ge a(E, X, \Delta) = -1 + \sum a_i \cdot \operatorname{coeff}_E D_i < 0$.*

Proof $(X, \lceil\Delta\rceil)$ is lc by Kollár and Mori (1998, 2.31), so $a(E, X, \Delta - \Theta) \ge a(E, X, \lceil\Delta\rceil) = -1$ by (11.4.3). The rest follows from (11.4.3.a). □

Corollary 11.7 *Using the notation of (11.6), for every $\varepsilon > 0$ there is $\eta > 0$ such that the following holds.*

Let $(X, \Delta' - \Theta')$ be a pair, where $\operatorname{Supp}\Theta = \operatorname{Supp}\Theta'$ and $\Delta' = \sum(1 - a_i')D_i$ such that $|a_i - a_i'| < \eta$ for every i and $a_i' = 0$ iff $a_i = 0$. Then, for every E,

$$\left|a(E, X, \Delta - \Theta) - a(E, X, \Delta' - \Theta')\right| < \varepsilon,$$

whenever one of the discrepancies is < 0. □

Definition 11.8 Let (X, Δ) be an lc or slc (11.37) pair and $W \subset X$ an irreducible, closed subset. The *minimal log discrepancy* of W is defined as the infimum of the numbers $1 + a(E, X, \Delta)$ where E runs through all divisors over X such that $\operatorname{center}_X(E) = W$. It is denoted by

$$\operatorname{mld}(W, X, \Delta) \quad \text{or by} \quad \operatorname{mld}(W) \tag{11.8.1}$$

if the choice of (X, Δ) is clear. Note that if W is an irreducible divisor on X and $W \not\subset \operatorname{Sing} X$ then $\operatorname{mld}(W, X, \Delta) = 1 - \operatorname{coeff}_W \Delta$. If $W \subset X$ is a closed subset with irreducible components W_i, then we set $\operatorname{mld}(W, X, \Delta) = \max_i\{\operatorname{mld}(W_i, X, \Delta)\}$.

If (X, Δ) is slc then, by definition, $\operatorname{mld}(W, X, \Delta) \geq 0$ for every W. The subvarieties with $\operatorname{mld}(W, X, \Delta) = 0$ play a key role in understanding (X, Δ).

Definition 11.9 Let (X, Δ) be an slc pair. An irreducible subset $W \subset X$ is a *log canonical center* or *lc center* of (X, Δ) if $\operatorname{mld}(W, X, \Delta) = 0$. If (X, Δ) has a log resolution, then there is a divisor E over X such that $a(E, X, \Delta) = -1$ and $\operatorname{center}_X E = W$.

11.10 (Properties of log canonical centers) Let (X, Δ) be an slc pair over a field of characteristic 0. (11.10.1) There are only finitely many lc centers.
(11.10.2) Any union of lc centers is seminormal and Du Bois (11.12.1–2).
(11.10.3) Any intersection of lc centers is also a union of lc centers; see Ambro (2003, 2011), Fujino (2017), or (11.12.4).
(11.10.4) If (X, Δ) is snc then the lc centers of (X, Δ) are exactly the *strata* of $\Delta^{=1}$, that is, the irreducible components of the various intersections $D_{i_1} \cap \cdots \cap D_{i_s}$ where the $\operatorname{coeff}_{D_{i_k}} \Delta = 1$; see Kollár (2013b, 2.11). More generally, this also holds if (X, Δ) is dlt; see Fujino (2007, sec.3.9) or Kollár (2013b, 4.16).
(11.10.5) At codimension 2 normal points, the union of lc centers is either smooth or has a node; see Kollár (2013b, 2.31).
(11.10.6) Let (X, Δ) be slc and Θ effective, $\mathbb{R}$-Cartier. Then $(X, \Delta + \varepsilon\Theta)$ is slc for $0 < \varepsilon \ll 1$ iff $\operatorname{Supp}\Theta$ does not contain any lc center of (X, Δ).
(11.10.7) Assume that (X, Δ) is slc and $\varepsilon\Theta \leq \Delta$ is an effective $\mathbb{Q}$-Cartier divisor. Then $\operatorname{Supp}\Theta$ does not contain any lc center of $(X, \Delta - \varepsilon\Theta)$ by (11.4.3).

Definition 11.11 Let (X, Δ) be an slc pair. An irreducible subset $W \subset X$ is a *log center* of (X, Δ) if $\operatorname{mld}(W, X, \Delta) < 1$. (It is frequently convenient to consider every irreducible component of X a log center.)

Building on earlier results of Ambro (2003, 2011), and Fujino (2017), part 1 of the following theorem is proved in Kollár and Kovács (2010). The rest in Kollár (2014); see also Kollár (2013b, chap.7).

Theorem 11.12 *Let (X, Δ) be an slc pair over a field of characteristic 0 and $Z, W \subset X$ closed, reduced subschemes.*
(11.12.1) *If* $\operatorname{mld}(Z, X, \Delta) = 0$*, then Z is Du Bois.*
(11.12.2) *If* $\operatorname{mld}(Z, X, \Delta) < \frac{1}{6}$*, then Z is seminormal (10.74).*
(11.12.3) *If* $\operatorname{mld}(Z, X, \Delta) + \operatorname{mld}(W, X, \Delta) < \frac{1}{2}$*, then $Z \cap W$ is reduced.*
(11.12.4) $\operatorname{mld}(Z \cap W, X, \Delta) \leq \operatorname{mld}(Z, X, \Delta) + \operatorname{mld}(W, X, \Delta)$. □

Adjunction is a classical method that allows induction on the dimension by lifting information from divisors to the ambient scheme.

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

Definition 11.13 (Poincaré residue map) Let X be a (pure dimensional) CM scheme and $S \subset X$ a divisorial subscheme. Then $\omega_S = \mathcal{E}xt^1(\mathcal{O}_S, \omega_X)$ and $\mathcal{E}xt^1(\mathcal{O}_X, \omega_X) = 0$. Thus, applying $\mathcal{H}om(\ ,\omega_X)$ to the exact sequence

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0,$$

we get the short exact sequence

$$0 \to \omega_X \to \omega_X(S) \xrightarrow{\mathcal{R}_S} \omega_S \to 0. \tag{11.13.1}$$

The map $\mathcal{R}_S : \omega_X(S) \to \omega_S$ is called the *Poincaré residue map*. By taking tensor powers, we get maps

$$\mathcal{R}_S^{\otimes m} : (\omega_X(S))^{\otimes m} \to \omega_S^{\otimes m},$$

but, if $m(K_X + S)$ and mK_S are Cartier for some $m > 0$ then we really would like to get a corresponding map between the locally free sheaves

$$\omega_X^{[m]}(mS)|_S \overset{???}{\dashrightarrow} \omega_S^{[m]}. \tag{11.13.2}$$

There is no such map in general; one needs a correction term.

Definition 11.14 (Different) Let X be a demi-normal scheme (11.36), S a reduced divisor (p.xv) on X, and Δ an $\mathbb{R}$-divisor on X. We assume that there are no coincidences, that is, the irreducible components of $\operatorname{Supp} S$, $\operatorname{Supp} \Delta$ and $\operatorname{Sing} X$ are all different from each other.

Let $\pi : \bar{S} \to S$ denote the normalization. There is a closed subscheme $Z \subset S$ of codimension 1 such that $S \setminus Z$ and $X \setminus Z$ are both smooth along $S \setminus Z$, the restriction $\pi : (\bar{S} \setminus \pi^{-1}Z) \to (S \setminus Z)$ is an isomorphism and $\operatorname{Supp} \Delta \cap S \subset Z$.

Assume first that Δ is a $\mathbb{Q}$-divisor and $m(K_X + S + \Delta)$ is Cartier for some $m > 0$. Then the Poincaré residue map (11.13) gives an isomorphism

$$\mathcal{R}^m_{S\setminus Z} : \pi^*\omega_X^{[m]}(mS + m\Delta)|_{(\bar{S}\setminus\pi^{-1}Z)} \simeq \omega_{\bar{S}}^{[m]}|_{(\bar{S}\setminus\pi^{-1}Z)}.$$

Hence there is a unique (not necessarily effective) divisor $\Delta_{\bar{S}}$ on $\bar{S}$ supported on $\pi^{-1}Z$ such that $\mathcal{R}^m_{S\setminus Z}$ extends to an isomorphism

$$\mathcal{R}^m_{\bar{S}} : \pi^*\omega_X^{[m]}(mS + m\Delta)|_{\bar{S}} \simeq \omega_{\bar{S}}^{[m]}(\Delta_{\bar{S}}). \tag{11.14.1}$$

We formally divide by m and define the *different* of Δ on $\bar{S}$ as the $\mathbb{Q}$-divisor

$$\operatorname{Diff}_{\bar{S}}(\Delta) := \tfrac{1}{m}\Delta_{\bar{S}}. \tag{11.14.2}$$

We can write (11.14.1) in terms of $\mathbb{Q}$-divisors as

$$(K_X + S + \Delta)|_{\bar{S}} \sim_{\mathbb{Q}} K_{\bar{S}} + \operatorname{Diff}_{\bar{S}}(\Delta). \tag{11.14.3}$$

Note that (11.14.3) has the disadvantage that it indicates only that the two sides are $\mathbb{Q}$-linearly equivalent, whereas (11.14.1) is a *canonical* isomorphism.

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

If $K_X + S + \Delta$ is $\mathbb{R}$-Cartier, then, by (11.43.4), we can write $\Delta = \Delta' + \Delta''$ where $K_X + S + \Delta'$ is $\mathbb{Q}$-Cartier and Δ'' is $\mathbb{R}$-Cartier. Then we set

$$\mathrm{Diff}_{\bar{S}}(\Delta) := \mathrm{Diff}_{\bar{S}}(\Delta') + \pi^*\Delta''. \tag{11.14.4}$$

If X, S are smooth than $K_S = (K_X + S)|_S$, hence in this case $\mathrm{Diff}_{\bar{S}}(\Delta) = \pi^*\Delta$.

Let $f\colon Y \to X$ be a proper birational morphism, $S_Y := f_*^{-1}S$ and write $K_Y + S_Y + \Delta_Y \sim_{\mathbb{R}} f^*(K_X + S + \Delta)$. Then

$$\mathrm{Diff}_{\bar{S}}(\Delta) = (f|_{\bar{S}_Y})_* \mathrm{Diff}_{\bar{S}_Y}(\Delta_Y). \tag{11.14.5}$$

Proposition 11.15 (Kollár, 2013b, 4.4–8) *Using the notation of (11.14) write* $\mathrm{Diff}_{\bar{S}}(\Delta) = \sum d_i V_i$ *where* $V_i \subset \bar{S}$ *are prime divisors. Then the following hold.*

(11.15.1) *If* $(X, S + \Delta)$ *is lc (or slc) then* $(\bar{S}, \mathrm{Diff}_{\bar{S}}(\Delta))$ *is lc.*

(11.15.2) *If* $\mathrm{coeff}(\Delta) \subset \{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$, *then the same holds for* $\mathrm{Diff}_{\bar{S}}(\Delta)$.

(11.15.3) *If* S *is Cartier, then* $\mathrm{Diff}_{\bar{S}}(\Delta) = \pi^*\Delta$.

(11.15.4) *If* $K_X + S$ *and* D *are both Cartier, then* $\mathrm{Diff}_{\bar{S}} D$ *is a* $\mathbb{Z}$*-divisor and* $(K_X + S + D)|_{\bar{S}} \sim K_{\bar{S}} + \mathrm{Diff}_{\bar{S}} D$. □

The following facts about codimension 1 behavior of the different can be proved by elementary computations; see Kollár (2013b, 2.31, 2.36).

Lemma 11.16 *Let* S *be a normal surface,* $E \subset S$ *a reduced curve and* $\Delta = \sum d_i D_i$ *an effective* $\mathbb{R}$*-divisor. Assume that* $0 \le d_i \le 1$ *and* $D_i \not\subset \mathrm{Supp}\, E$ *for every* i*. Let* $\pi\colon \bar{E} \to E$ *be the normalization and* $x \in \bar{E}$ *a point.*

(11.16.1) *If* E *is singular at* $\pi(x)$, *then* $\mathrm{coeff}_x \mathrm{Diff}_{\bar{E}}(\Delta) \ge 1$, *and equality holds iff* E *has a node at* $\pi(x)$, E *is Cartier at* $\pi(x)$ *and* $\pi(x) \notin \mathrm{Supp}\, \Delta$.

(11.16.2) *If* $\pi(x) \in D_i$, *then* $\mathrm{coeff}_x \mathrm{Diff}_{\bar{E}}(\Delta) \ge d_i$. □

The next theorem – proved in Kollár (1992b, 17.4) and Kawakita (2007) – is frequently referred to as *adjunction* if we assume something about X and obtain conclusions about S, or *inversion of adjunction* if we assume something about S and obtain conclusions about X. See Kollár (2013b, 4.8–9) for a proof of a more precise version.

Theorem 11.17 *Let* X *be a normal scheme over a field of characteristic 0 and* S *a reduced divisor on* X *with normalization* $\pi_S\colon \bar{S} \to S$*. Let* Δ *be an effective* $\mathbb{R}$*-divisor that has no irreducible components in common with* S *and such that* $K_X + S + \Delta$ *is* $\mathbb{R}$*-Cartier. Then*

(11.17.1) $(\bar{S}, \mathrm{Diff}_{\bar{S}}(\Delta))$ *is klt iff* $(X, S + \Delta)$ *is plt in a neighborhood of* S, *and*

(11.17.2) $(\bar{S}, \mathrm{Diff}_{\bar{S}}(\Delta))$ *is lc iff* $(X, S + \Delta)$ *is lc in a neighborhood of* S.

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

(11.17.3) $\operatorname{mld}(Z, \bar{S}, \operatorname{Diff}_{\bar{S}}(\Delta)) = \operatorname{mld}(\pi_S(Z), X, S + \Delta)$ *for any irreducible and closed subset* $Z \subsetneq \bar{S}$*, provided one of them is* ≤ 1.

(11.17.4) *The claims also hold for slc pairs by (11.37).* □

Many divisorial sheaves on an lc pair are Cohen–Macaulay (CM for short). The following variant is due to Kollár and Mori (1998, 5.25) and Fujino (2017, 4.14); see also Kollár (2013b, 2.88).

Theorem 11.18 *Let* (X, Δ) *be a dlt pair over a field of characteristic 0, L a* $\mathbb{Q}$*-Cartier* $\mathbb{Z}$*-divisor, and* $D \leq \lfloor \Delta \rfloor$ *an effective* $\mathbb{Z}$*-divisor. Then the sheaves* $\mathscr{O}_X$*,* $\mathscr{O}_X(-D - L)$ *and* $\omega_X(D + L)$ *are CM.*

If $D + L$ *is effective, then* $\mathscr{O}_{D+L}$ *is also CM.* □

We also need the following; see Kollár (2011a) or Kollár (2013b, 7.31).

Theorem 11.19 *Let* (X, Δ) *be dlt over a field of characteristic 0, D a (not necessarily effective)* $\mathbb{Z}$*-divisor and* $\Delta' \leq \Delta$ *an effective* $\mathbb{R}$*-divisor on X such that* $D \sim_{\mathbb{R}} \Delta'$*. Then* $\mathscr{O}_X(-D)$ *is CM.* □

If (X, Δ) is lc then frequently $\mathscr{O}_X$ is not CM. The following variant of the above theorems, while much weaker, is quite useful. In increasing generality it was proved by Alexeev (2008), Kollár (2011a), and Fujino (2017); see Kollár (2013b, 7.20) for the slc case and Kovács (2011) and Alexeev and Hacon (2012) for other versions. The main applications are in (2.79) and (4.33).

Theorem 11.20 *Let* (X, Δ) *be slc over a field of characteristic 0 and* $x \in X$ *a point that is not an lc center (11.10). Let D be a Mumford* $\mathbb{Z}$*-divisor. Assume that there is an effective* $\mathbb{R}$*-divisor* $\Delta' \leq \Delta$ *such that* $D \sim_{\mathbb{R}} \Delta'$*. Then*

(11.20.1) $\operatorname{depth}_x \mathscr{O}_X(-D) \geq \min\{3, \operatorname{codim}_X x\}$*, and*

(11.20.2) $\operatorname{depth}_x \omega_X(D) \geq \min\{3, \operatorname{codim}_X x\}$.

Proof The first claim is proved in Kollár (2013b, 7.20). To get the second, note that, working locally, $K_X + \Delta \sim_{\mathbb{R}} 0$, thus $-(K_X + D) \sim_{\mathbb{R}} \Delta - \Delta'$ and $\Delta - \Delta' \leq \Delta$ is effective. Thus, by the first part, $\omega_X(D) \simeq \mathscr{O}_X(-(-(K_X + D)))$ has depth $\geq \min\{3, \operatorname{codim}_X x\}$. □

Corollary 11.21 Alexeev (2008) *Let* (X, Δ) *be slc. If x is not an lc center and* $\operatorname{codim}_X x \geq 3$*, then* $\operatorname{depth}_x \mathscr{O}_X \geq 3$ *and* $\operatorname{depth}_x \omega_X \geq 3$. □

11.22 (Hurwitz formula) The main example is when $\pi\colon Y \to X$ is a finite, separable morphism between normal varieties of the same dimension, but we

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

also need the case when $\pi\colon Y \to X$ is a finite, separable morphism between demi-normal schemes such that π is étale over the nodes of X. Working over the closure of the open set where K_X is Cartier, we get that

$$K_Y \sim_{\mathbb{Q}} R + \pi^* K_X, \tag{11.22.1}$$

where R is the *ramification divisor* of π. If none of the ramification indices is divisible by the characteristic, then $R = \sum_D (e(D) - 1)D$ where $e(D)$ denotes the ramification index of π along the divisor $D \subset Y$.

Note that if π is quasi-étale, that is, étale outside a subset of codimension ≥ 2, then $R = 0$, hence $K_Y \sim_{\mathbb{Q}} \pi^* K_X$.

11.23 Let $\pi\colon Y \to X$ be a finite, separable morphism as in (11.22) and Δ_X an $\mathbb{R}$-divisor on X (not necessarily $\mathbb{R}$-Cartier). Set

$$\Delta_Y := -R + \pi^* \Delta_X. \tag{11.23.1}$$

With this choice, (11.22.1) gives that

$$K_Y + \Delta_Y \sim_{\mathbb{R}} \pi^*(K_X + \Delta_X). \tag{11.23.2}$$

Reid's covering lemma compares the discrepancies of divisors over X and Y. For precise forms see Reid (1980), Kollár and Mori (1998, 5.20), or Kollár (2013b, 2.42–43). We need the following special cases.

Claim 11.23.3 Assume also, that Δ_X and Δ_Y are both effective, and, either the characteristic is 0, or π is Galois and $\deg \pi$ is not divisible by the characteristic, or $\deg \pi$ is less than the characteristic. Then (X, Δ_X) is klt (resp. lc or slc) iff (Y, Δ_Y) is klt (resp. lc or slc). □

Special case 11.23.4 If π is quasi-étale, then $\Delta_Y = \pi^* \Delta_X$; thus we compare (X, Δ_X) and $(Y, \pi^* \Delta_X)$.

Special case 11.23.5 Let D_X be a reduced divisor on X such that π is étale over $X \setminus D_X$. Set $D_Y := \operatorname{red} \pi^*(D_X)$. Then $D_Y + R = \pi^*(D_X)$, thus we compare $(X, D_X + \Delta_X)$ and $(Y, D_Y + \pi^* \Delta_X)$.

11.24 (Cyclic covers) See Kollár and Mori (1998, 2.49–52) or Kollár (2013b, sec.2.3) for details.

Let X be an S_2 scheme, L a divisorial sheaf (3.25) and s a section of $L^{[m]}$. These data define a *cyclic cover* or *μ_m-cover* $\pi\colon Y \to X$ such that we have direct sum decompositions into μ_m-eigensheaves

$$\begin{aligned} \pi_* \mathscr{O}_Y &= \oplus_{i=0}^{m-1} L^{[-i]}, \quad \text{and} \\ \pi_* \omega_{Y/C} &\simeq \mathcal{H}om_X(\pi_* \mathscr{O}_Y, \omega_{X/C}) = \oplus_{i=0}^{m-1} L^{[i]} \,{}_{[\otimes]}\, \omega_{X/C}, \end{aligned}$$

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

where $[\otimes]$ denotes the double dual of the tensor product. The morphism π is étale over $x \in X$ iff L is locally free at x, $s(x) \neq 0$ and $\operatorname{char} k(x) \nmid m$. Thus π is quasi-étale iff s is a nowhere zero section and $\operatorname{char} k(x) \nmid m$.

One can reduce many questions about $\mathbb{Q}$-Cartier divisors to Cartier divisors.

Proposition 11.25 *Let (x, X) be a local scheme over a field of characteristic 0 and $\{D_i : i \in I\}$ a finite set of $\mathbb{Q}$-Cartier, Mumford $\mathbb{Z}$- divisors. Then there is a finite, abelian, quasi-étale cover $\pi : \tilde{X} \to X$ such that the $\pi^* D_i$ are Cartier.*

Furthermore, if (X, Δ) is klt (resp. lc or slc) for some $\mathbb{R}$-divisor Δ, then $(\tilde{X}, \tilde{\Delta} := \pi^*\Delta)$ is also klt (resp. lc or slc). □

11.2 Canonical Models and Modifications

We used many times canonical models in the relative setting.

Definition 11.26 Let (Y, Δ_Y) be an lc pair and $p_Y : Y \to S$ a proper morphism. We say that (Y, Δ_Y) is a *canonical model over S*, if $K_Y + \Delta_Y$ is p_Y-ample.

Let (X, Δ) be an lc pair and $p : X \to S$ a proper morphism. We say that (X^c, Δ^c) is a *canonical model* of (X, Δ) *over S* if there is a diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "p"'] & & X^c \arrow[dl, "p^c"] \\ & S & \end{tikzcd} \tag{11.26.1}$$

such that

(11.26.2) (X^c, Δ^c) is a canonical model over S,

(11.26.3) ϕ is a birational contraction (p.xiv),

(11.26.4) $\Delta^c = \phi_*\Delta$, and

(11.26.5) $\phi_*\mathscr{O}_X(mK_X + \lfloor m\Delta \rfloor) = \mathscr{O}_{X^c}(mK_{X^c} + \lfloor m\Delta^c \rfloor)$ for every $m \geq 0$.

Comments 11.26.6 Since ϕ is a birational contraction, there are open sets $U \subset X$ and $U^c \subset X^c$ whose complements have codimension ≥ 2 such that the restriction of ϕ is a morphism $\phi_U : U \to U^c$. Thus (11.26.5) is equivalent to saying that $\phi_*\mathscr{O}_U(mK_U + \lfloor m\Delta|_U \rfloor) = \mathscr{O}_{U^c}(mK_{U^c} + \lfloor m\Delta^c|_{U^c} \rfloor)$ for every $m \geq 0$. (One needs (11.62.2) to see that this is equivalent to Kollár and Mori (1998, 3.50).)

For $\mathbb{Q}$-divisors we have the following direct generalization of (1.38).

Proposition 11.27 *Let (X, Δ) be an lc pair and $p : X \to S$ a proper morphism. Assume that X is irreducible and Δ is a $\mathbb{Q}$-divisor. Then (X, Δ) has a canonical model over S iff the generic fiber is of general type and the canonical algebra*

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

$\bigoplus_{m\geq 0} p_*\mathscr{O}_X(mK_X + \lfloor m\Delta\rfloor)$ is finitely generated. If these hold then the canonical model is $X^c :=$ Proj$_S \bigoplus_{m\geq 0} p_*\mathscr{O}_X(mK_X + \lfloor m\Delta\rfloor)$.

The main conjecture on canonical models says that the relative canonical models always exist if the generic fiber is of general type. The following known cases, due to Birkar et al. (2010) and Hacon and Xu (2013, 2016), and generalized in Lyu and Murayama (2022) are the most important for us.

Theorem 11.28 *Let (X, Δ) be an lc pair over a field of characteristic 0 and $p\colon X \to S$ a proper morphism, S irreducible. The relative canonical model exists in the following cases.*

(11.28.1) *(X, Δ) is klt and the generic fiber is of general type.*

(11.28.2) *(X, Δ) is dlt, the relative canonical model exists over an open $S^\circ \subset S$, and every lc center intersects $p^{-1}(S^\circ)$.* □

Definition 11.29 (Canonical modification) Let Y be a scheme over a field k. (We allow Y to be reducible and nonreduced, but in applications usually pure dimensional.) Its *canonical modification* is the unique proper, birational morphism $p^{\mathrm{cm}}\colon Y^{\mathrm{cm}} \to \operatorname{red} Y$ such that Y^{cm} has canonical singularities and $K_{Y^{\mathrm{cm}}}$ is ample over Y.

Let Δ be an effective divisor on Y. We define the *canonical modification* $p^{\mathrm{cm}}\colon (Y^{\mathrm{cm}}, \Delta^{\mathrm{cm}}) \to (Y, \Delta)$ as the unique proper, birational morphism for which $(Y^{\mathrm{cm}}, \Delta^{\mathrm{cm}})$ has canonical singularities and $K_{Y^{\mathrm{cm}}} + \Delta^{\mathrm{cm}}$ is ample over Y; where Δ^{cm} is the birational transform of $\Delta|_{\operatorname{red} Y}$; see Kollár (2013b, 1.31).

The *log canonical modification* $p^{\mathrm{lcm}}\colon (Y^{\mathrm{lcm}}, \Delta^{\mathrm{lcm}}) \to (Y, \Delta)$ is defined similarly. The change is that $(Y^{\mathrm{lcm}}, \Delta^{\mathrm{lcm}} + E^{\mathrm{lcm}})$ is log canonical and $K_{Y^{\mathrm{lcm}}} + \Delta^{\mathrm{lcm}} + E^{\mathrm{lcm}}$ is ample over Y, where E^{lcm} denotes the reduced exceptional divisor.

The canonical modification of (X, Δ) is unique. It exist in characteristic 0 if coeff $\Delta \subset [0, 1]$ by (11.28). The lc modification is also unique. As for its existence, we clearly need to assume that coeff $\Delta \subset [0, 1]$. Conjecturally, this is the only necessary condition, but this is known only in some cases. C. Xu pointed out that the arguments in Odaka and Xu (2012) give the following.

Theorem 11.30 *Let X be a normal variety and Δ an $\mathbb{R}$-divisor on X with* coeff$(\Delta) \subset [0, 1]$. *If $K_X + \Delta$ is numerically $\mathbb{R}$-Cartier (4.48), then (X, Δ) has a log canonical modification.* □

Proposition 11.31 (Kollár, 2018a, prop.19) *Let (X, Δ) be a potentially lc pair (11.5.1) over a field of characteristic 0. Then*

(11.31.1) *it has a projective, small, lc modification $\pi : (X^{lcm}, \Delta^{lcm}) \to (X, \Delta)$,*

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

(11.31.2) *π is a local isomorphism at every lc center of (X^{lcm}, Δ^{lcm}), and*
(11.31.3) *π is a local isomorphism over $x \in X$ iff $K_X + \Delta$ is $\mathbb{R}$-Cartier at x.* □

The following typical application of (11.31) reduces some questions about Weil divisors to $\mathbb{Q}$-Cartier Weil divisors.

Proposition 11.32 *Let (X, Δ) be an lc pair over a field of characteristic 0 and Θ an effective $\mathbb{R}$-divisor such that* $\operatorname{Supp}\Theta \subset \operatorname{Supp}\Delta$. *Let B be a Weil $\mathbb{Z}$-divisor such that $B \sim_{\mathbb{R}} -\Theta$. Then there is a small, lc modification $\pi\colon X' \to X$ such that the following hold, where we use $'$ to denote the birational transform of a divisor on X'.*
(11.32.1) *B' is $\mathbb{Q}$-Cartier and π-ample,*
(11.32.2) $\operatorname{Ex}(\pi) \subset \operatorname{Supp}\Theta'$,
(11.32.3) *none of the lc centers of $(X', \Delta' - \varepsilon\Theta')$ are contained in* $\operatorname{Ex}(\pi)$,
(11.32.4) $\pi_*\mathscr{O}_{X'}(B') = \mathscr{O}_X(B)$,
(11.32.5) $R^i\pi_*\mathscr{O}_{X'}(B') = 0$ *for $i > 0$, and*
(11.32.6) $H^i(X, \mathscr{O}_X(B)) = H^i(X, \mathscr{O}_{X'}(B'))$.

Proof We construct $\pi\colon (X', \Delta') \to (X, \Delta)$ by applying (11.31) to $(X, \Delta - \varepsilon\Theta)$. Then $-\varepsilon\Theta' \sim_{\mathbb{R}} K_{X'} + \Delta' - \varepsilon\Theta'$ is $\mathbb{R}$-Cartier and π-ample, hence (1) holds by (11.43). This gives (2). Then (3) follows from (11.10.7). Next, (4) holds since π is small. We can write $B' \sim_{\mathbb{R}} K_{X'} + (\Delta' - \varepsilon\Theta') + (1-\varepsilon)(-\Theta')$; then (5) follows from (3) and (11.34). Finally, the Leray spectral sequence shows (6). □

One of the difficulties in dealing with slc pairs is that analogous small modifications need not exist for them; see Kollár (2013b, 1.40).

We use generalizations of Kodaira's vanishing theorem, see Kollár and Mori (1998, secs.2.4–5) for an introductory treatment. The following is proved in Ambro (2003) and (Fujino, 2014, 1.10). See also Fujino (2017, sec.5.7) and Fujino (2017, 6.3.5), where it is called a Reid–Fukuda–type theorem.

Definition 11.33 Let (X, Δ) be an slc pair, $f\colon X \to S$ a proper morphism, and L an $\mathbb{R}$-Cartier, f-nef divisor on X. Then L is called *log f-big* if $L|_W$ is big on the generic fiber of $f|_W\colon W \to f(W)$ for every lc center W of (X, Δ) and also for every irreducible component $W \subset X$.

Theorem 11.34 *Let (X, Δ) be an slc pair over a field of characteristic 0 and D a Mumford $\mathbb{Z}$-divisor on X. Let $f\colon X \to S$ be a proper morphism. Assume that $D \sim_{\mathbb{R}} K_X + L + \Delta$, where L is $\mathbb{R}$-Cartier, f-nef and log f-big. Then*

$$R^i f_*\mathscr{O}_X(D) = 0 \quad \textit{for} \quad i > 0.$$

□

11.3 Semi-log-canonical Pairs

Definition 11.35 Let (R, m) be a local ring such that $\mathrm{char}(R/m) \neq 2$. We say that $\operatorname{Spec} R$ has a *node* if there is a regular local ring (S, m_S) of dimension 2, generators $m_S = (x, y)$, a unit $a \in S \setminus m_S$ and $h \in m_S^3$ such that $R \simeq S/(x^2 - ay^2 + h)$. (See Kollár (2013b, 1.41) for characteristic 2.)

If R is complete, then we can arrange that $h = 0$. If R/m is algebraically closed, then we can take $a = 1$. Over an algebraically closed field we get the more familiar form $k[[x, y]]/(xy)$.

As a very simple special case of (2.27) or of (10.43), over a field all deformations of a node can be obtained, étale locally, by pull-back from

$$(x^2 - ay^2 = 0) \subset (x^2 - ay^2 + t = 0) \subset \mathbb{A}^2_{xy} \times \mathbb{A}^1_t. \tag{11.35.1}$$

Definition 11.36 Recall that, by Serre's criterion, a scheme X is normal iff it is S_2 and regular at all codimension 1 points. As a weakening of normality, a scheme is called *demi-normal* if it is S_2 and its codimension 1 points are either regular points or nodes.

A one-dimensional demi-normal variety is a curve C with nodes. It can be thought of as a smooth curve $\bar{C}$ (the normalization of C) together with pairs of points $p_i, p_i' \in \bar{C}$, obtained as the preimages of the nodes. Equivalently, we have the nodal divisor $\bar{D} = \sum_i (p_i + p_i')$ on $\bar{C}$, plus a fixed point free involution on $\bar{D}$ given by $\tau\colon p_i \leftrightarrow p_i'$.

We aim to get a similar description for any demi-normal scheme X. Let $\pi\colon \bar{X} \to X$ denote the normalization and $D \subset X$ the divisor obtained as the closure of the nodes of X. Set $\bar{D} := \pi^{-1}(D)$ with reduced structure. Then $D, \bar{D}$ are the conductors of π, and the induced map $\bar{D} \to D$ has degree 2 over the generic points. The map between the normalizations $\bar{D}^n \to \bar{D}^n$ has degree 2 over all irreducible components, determining an involution $\tau\colon \bar{D}^n \to \bar{D}^n$, which is not the identity on any irreducible component. We always assume this condition from now on. (Note that τ is only a rational involution on $\bar{D}$.)

It is easy to see (Kollár, 2013b, 5.3) that a demi-normal scheme X is uniquely determined by the triple $(\bar{X}, \bar{D}, \tau)$.

However, it is surprisingly difficult to understand which triples $(\bar{X}, \bar{D}, \tau)$ correspond to demi-normal schemes. The solution of this problem in the log canonical case, given in (11.38), is a key result for us.

Roughly speaking, the concept of semi-log-canonical is obtained by replacing "normal" with "demi-normal" in the definition of log canonical (11.5).

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

Definition 11.37 Let X be a demi-normal scheme with normalization $\pi\colon \bar{X} \to X$ and with conductors $D \subset X$ and $\bar{D} \subset \bar{X}$. Let Δ be an effective $\mathbb{R}$-divisor whose support does not contain any irreducible component of D, and $\bar{\Delta}$ the divisorial part of $\pi^{-1}(\Delta)$. The pair (X, Δ) is called *semi-log-canonical* or *slc* if

(11.37.1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and

(11.37.2) $(\bar{X}, \bar{D} + \bar{\Delta})$ is lc.

Alternatively, one can define $a(E, X, \Delta)$ using semi-resolutions (as in Kollár (2013b, Sec.10.4)) and then replace (2) by

(11.37.3) $a(E, X, \Delta) \geq -1$ for every exceptional divisor E over X.

This is now the exact analog of the definition of log canonical given in (11.5); the equivalence is proved in Kollár (2013b, 5.10).

This formula suggests that if $D_i \subset D$ is an irreducible component, then we should declare that $a(D_i, X, \Delta) = -1$.

Warning 11.37.4 It can happen that (2) holds, hence $K_{\bar{X}} + \bar{D} + \bar{\Delta}$ is $\mathbb{R}$-Cartier, but $K_X + \Delta$ is not; see (2.22.1) for an instructive special case of dimension 2.

By contrast, this cannot happen in codimensions ≥ 3 by (11.42).

The following theorem, proved in Kollár (2016b) and Kollár (2013b, 5.13), describes slc pairs using their normalizations.

Theorem 11.38 *Let S be a scheme over a field of characteristic 0 as in (11.2). Then normalization gives a one-to-one correspondence:*

$$\left\{ \begin{array}{c} \textit{Proper, slc pairs} \\ g\colon (X, \Delta) \to S, \\ K_X + \Delta \textit{ is } g\textit{-ample.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Proper, lc pairs } \bar{g}\colon (\bar{X}, \bar{D} + \bar{\Delta}) \to S \\ \textit{with involution } \tau \curvearrowright (\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}), \\ K_{\bar{X}} + \bar{D} + \bar{\Delta} \textit{ is } \bar{g}\textit{-ample.} \end{array} \right\}.$$

(As in (11.36), τ is not the identity on any irreducible component.) □

In applications, we usually know the codimension 1 points of X and $\bar{X}$. The codimension 1 points of $(\bar{D}^{\mathrm{n}}, \operatorname{Diff}_{\bar{D}^{\mathrm{n}}} \bar{\Delta})$ correspond to codimension 2 points of X and $\bar{X}$. Since we understand two-dimensional slc pairs quite well, we frequently have good control over codimension ≤ 2 points of lc and slc pairs. The next theorems show that one can sometimes ignore the higher codimension points.

The first result of this type, due to Matsusaka and Mumford (1964), shows how to extend isomorphisms across subsets of codimension ≥ 2.

Theorem 11.39 *Let S be a Noetherian scheme, $X_i \to S$ projective morphisms and H_i relatively ample $\mathbb{R}$-divisor classes on X_i. Let $Z_i \subset X_i$ be closed subsets such that* $\operatorname{depth}_{Z_i} X_i \geq 2$. *Let*

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

$$\tau^\circ\colon (X_1 \setminus Z_1, H_1|_{X_1\setminus Z_1}) \simeq (X_2 \setminus Z_2, H_2|_{X_2\setminus Z_2})$$

be an isomorphism. Then τ° extends to an isomorphism $\tau\colon X_1 \simeq X_2$.

Proof Let $\Gamma \subset X_1 \times_S X_2$ be the closure of the graph of τ° with projections $\pi_i\colon \Gamma \to X_i$. Then π_i is an isomorphism over $X_i \setminus Z_i$. By (5.32.4), π_i is an isomorphism iff it is finite. The latter can be checked locally on S after completion. We can now also assume that the X_i are normal and replace Γ with its normalization.

There is a divisor $E \sim_{\mathbb{R}} \pi_1^*H_1 - \pi_2^*H_2$ that is supported on the union of the π_i-exceptional loci. Since $\pi_i(E) \subset Z_i$, we see that $\operatorname{Supp}(E) \subset \operatorname{Ex}(\pi_i)$ for $i = 1, 2$.

Next note that $-E$ is π_1-nef and exceptional, so $E \geq 0$ by (11.60). Also E is π_2-nef and exceptional, so $E \leq 0$. Thus $E = 0$, hence $\pi_1^*H_1 \equiv \pi_2^*H_2$. We now finish by (11.39.1). □

Claim 11.39.1 Let $X_i \to S$ be projective morphisms and H_i relatively ample $\mathbb{R}$-divisor classes on X_i. Let $p\colon Y \to X_1 \times_S X_2$ be a finite morphism such that $p^*\pi_1^*H_1 \sim_{\mathbb{R}} p^*\pi_2^*H_2$. Then $\pi_i \circ p\colon Y \to X_i$ are finite.

Proof If a curve $C \subset Y$ is contracted by $\pi_1 \circ p$ then it cannot be contracted by $\pi_2 \circ p$ since p is finite. Thus $(C \cdot p^*\pi_1^*H_1) = 0$, but $(C \cdot p^*\pi_2^*H_2) = (\pi_2 \circ p(C) \cdot H_2) > 0$ since H_2 is ample. □

The $\operatorname{depth}_Z X \geq 2$ assumption in (11.39) holds if X is normal and $Z \subset X$ has codimension ≥ 2; the main case in most applications. If Z has codimension 1, we usually get very little information about X from $X \setminus Z$. Nonetheless, we have the following very useful result about slc pairs.

Theorem 11.40 *Let S be a scheme over a field of characteristic 0 as in (11.2), and let $f_i\colon (X_i, \Delta_i) \to S$ proper morphisms from slc pairs such that $K_{X_i} + \Delta_i$ is f_i-ample. Let $Z_S \subset S$ be a closed subset and set $Z_i := f_i^{-1}(Z_S)$. Let*

$$\tau^\circ\colon (X_1 \setminus Z_1, \Delta_1|_{X_1\setminus Z_1}) \simeq (X_2 \setminus Z_2, \Delta_2|_{X_2\setminus Z_2}) \tag{11.40.1}$$

be an isomorphism. Assume that none of the log centers (11.11) of (X_i, Δ_i) is contained in Z_i for $i = 1, 2$.

Then τ° extends to an isomorphism $\tau\colon X_1 \simeq X_2$.

Proof Since every irreducible component of X is a log center, the Z_i are nowhere dense in X_i.

Using (11.38) we may assume that the X_i are normal. Let $\Gamma \to X^1 \times_S X^2$ be the normalization of the closure of the graph of τ° with projections $\pi_i\colon \Gamma \to X_i$.

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

As in (1.28), we use the log canonical class to compare the X_i. If F is an irreducible component of Δ_i then $a(F, X_i, \Delta_i) = -\operatorname{coeff}_F \Delta_i < 0$, thus $F \not\subset Z_i$. In particular, $(\pi_1)_*^{-1}\Delta^1 = (\pi_2)_*^{-1}\Delta^2$; let us denote this divisor by Δ_Γ. Write

$$K_\Gamma + \Delta_\Gamma \sim_{\mathbb{R}} \pi_i^*(K_{X_i} + \Delta_i) + E_i, \tag{11.40.2}$$

where E_i is π_i-exceptional and $\pi_i(\operatorname{Supp} E_i) \subset Z_i$. Note that E_i is effective by our assumption on the log centers.

Subtracting the $i = 1, 2$ cases of (11.40.2) from each other, we get that

$$E_1 - E_2 \sim_{\mathbb{R}} \pi_2^*(K_{X_2} + \Delta_2) - \pi_1^*(K_{X_1} + \Delta_1). \tag{11.40.3}$$

Thus $E_1 - E_2$ is π_1-nef and $-(\pi_1)_*(E_1 - E_2) = (\pi_1)_*(E_2)$ is effective. Thus $E_2 - E_1$ is effective by (11.60). Using π_2 shows that $E_1 - E_2$ is effective, hence $E_1 = E_2$. Thus $\pi_1^*(K_{X_1} + \Delta_1) \sim_{\mathbb{R}} \pi_2^*(K_{X_2} + \Delta_2)$. We finish by (11.39.1). □

Remark 11.40.4 The assumption on log centers is crucial. To see an example, consider the family of curves

$$X := \left(xyz(x + y + z) + t(x^4 + y^4 + z^4) = 0\right) \subset \mathbb{P}^2_{xyz} \times \mathbb{A}^1_t.$$

It is smooth along the central fiber X_0, which consists of four lines L_i, each with self-intersection -3. We can contract any of them $p_i \colon X \to X_i$, to get $f_i \colon X_i \to \mathbb{A}^1$. Note that $p_j \circ p_i^{-1} \colon X_i \dashrightarrow X_j$ is an isomorphism over $\mathbb{A}^1 \setminus \{0\}$, but not an isomorphism for $i \neq j$. Here X_i has a singularity of type $\mathbb{A}^2/\frac{1}{3}(1, 1)$, which is log terminal and the singularities are log centers of $(X_i, 0)$.

Corollary 11.41 *Let S be a scheme over a field of characteristic 0 as in (11.2) and $S^\circ \subset S$ a dense, open subscheme. Let $g^\circ \colon (X^\circ, \Delta^\circ) \to S^\circ$ be a proper, slc pair with normalization $\pi^\circ \colon (\bar{X}^\circ, \bar{\Delta}^\circ + \bar{D}^\circ) \to (X^\circ, \Delta^\circ)$.*

Assume that there is an slc pair $(\bar{X}, \bar{\Delta} + \bar{D}) \supset (\bar{X}^\circ, \bar{\Delta}^\circ + \bar{D}^\circ)$ that is proper over S such that $K_{\bar{X}} + \bar{\Delta} + \bar{D}$ is ample over S and every codimension ≤ 2 log center of $(\bar{X}, \bar{\Delta} + \bar{D})$ has nonempty intersection with $\bar{X}^\circ$.

Then there is a unique slc pair $(X, \Delta) \supset (X^\circ, \Delta^\circ)$ that is proper over S and whose normalization is $(\bar{X}, \bar{\Delta} + \bar{D})$.

Proof Since every irreducible component of $\bar{X}$ is a log center, $\bar{X}^\circ$ is dense in $\bar{X}$. Let $n \colon \bar{D}^n \to \bar{D}$ denote the normalization. By inversion of adjunction (11.17.2), $(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta})$ is also lc and $K_{\bar{D}^n} + \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}$ is ample over S.

Using (11.17.3), every irreducible component of $\operatorname{Diff}_{\bar{D}^n}$ lies over a codimension 2 log center of $(\bar{X}, \bar{\Delta} + \bar{D})$, hence none of the irreducible components of $\operatorname{Diff}_{\bar{D}^n}$ is disjoint from $\bar{X}^\circ$.

Thus the involution τ° of $(\bar{D}^\circ)^n$ extends to an involution τ on $\bar{D}^n$ by (11.40), and $\operatorname{Diff}_{\bar{D}^n}$ is $\bar{\tau}$-invariant. Hence (11.38) gives the existence of (X, Δ). □

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

Corollary 11.42 *Let* (X, Δ) *be demi-normal with lc normalization* $(\bar{X}, \bar{\Delta} + \bar{D})$. *Assume that there is a closed subset* $W \subset X$ *of codimension* ≥ 3 *such that* $(X \setminus W, \Delta|_{X \setminus W})$ *is slc. Then* (X, Δ) *is slc.*

Proof Apply (11.41) with $S = X$, $X^\circ = X \setminus W$. □

11.4 $\mathbb{R}$-divisors

It is easy to see that, on a $\mathbb{Q}$-factorial scheme, $\mathbb{R}$-divisors behave very much like $\mathbb{Q}$-divisors. The same holds in general, but it needs a little more work. The basics are discussed in Lazarsfeld (2004, sec.1.3), but most other facts are scattered in the literature; see, for example, Kollár (2013b, 2.21), Birkar (2017), or Fujino and Miyamoto (2021).

11.43 ($\mathbb{R}$-divisors) Let X be a reduced, S_2 scheme and $\Delta = \sum b_i B_i$ a Mumford $\mathbb{R}$-divisor. There is a unique such way of writing Δ where the B_i are irreducible, distinct and $b_i \neq 0$ for every i. The $\mathbb{Q}$-vector space spanned by the coefficients is denoted by $\operatorname{CoSp}(\Delta) = \sum_i \mathbb{Q} \cdot b_i \subset \mathbb{R}$.

We say that Δ is $\mathbb{R}$-*Cartier* if it can be written as an $\mathbb{R}$-linear combination of Cartier $\mathbb{Z}$-divisors $\Delta = \sum r_i D_i$. By (11.46), we can choose the D_i to have the same support as Δ, but we do not assume this to start with. Two $\mathbb{R}$-divisors are $\mathbb{R}$-*linearly equivalent,* denoted by $\Delta_1 \sim_{\mathbb{R}} \Delta_2$, if $\Delta_1 - \Delta_2$ is an $\mathbb{R}$-linear combination of principal divisors. Claim (11.43.2.d) shows that for $\mathbb{Q}$-divisors we do not get anything new.

Let $\sigma\colon \mathbb{R} \to \mathbb{Q}$ be a $\mathbb{Q}$-linear map. It extends to a $\mathbb{Q}$-linear map from $\mathbb{R}$-divisors to $\mathbb{Q}$-divisors as $\sigma(\sum d_i D_i) := \sum \sigma(d_i) D_i$.

Claim 11.43.1 Let $\sigma\colon \mathbb{R} \to \mathbb{Q}$ be a $\mathbb{Q}$-linear map. Then

(a) $\operatorname{Supp}(\sigma(D)) \subset \operatorname{Supp}(D)$,
(b) if $D_1 \sim_{\mathbb{R}} D_2$ then $\sigma(D_1) \sim_{\mathbb{Q}} \sigma(D_2)$,
(c) if D is $\mathbb{R}$-Cartier then $\sigma(D)$ is $\mathbb{Q}$-Cartier, and
(d) $D \mapsto \sigma(D)$ commutes with pull-back for $\mathbb{R}$-Cartier divisors.

Proof The first claim is clear. If $D_1 - D_2 = \sum c_i (f_i)$ then $\sigma(D_1) - \sigma(D_2) = \sum \sigma(c_i)(f_i)$, showing (b), which in turn implies (c) and (d) is clear. □

Let D be an $\mathbb{R}$-divisor. Choosing a $\mathbb{Q}$-basis $d_i \in \operatorname{CoSp}(D)$, we can write $D = \sum d_i D_i$ where the D_i are $\mathbb{Q}$-divisors (usually reducible). The D_i depend on the choice of the basis. Nonetheless, they inherit many properties of D.

Claim 11.43.2 Let D_i be $\mathbb{Q}$-divisors and $d_i \in \mathbb{R}$ linearly independent over $\mathbb{Q}$. Then

(a) $\sum d_i D_i$ is $\mathbb{R}$-Cartier iff each D_i is $\mathbb{Q}$-Cartier.
(b) $\sum d_i D_i \sim_{\mathbb{R}} 0$ iff $D_i \sim_{\mathbb{Q}} 0$ for every i.
(c) If X is proper then $\sum d_i D_i \equiv 0$ iff $D_i \equiv 0$ for every i.
(d) A $\mathbb{Q}$-divisor D_i is $\mathbb{R}$-Cartier iff it is $\mathbb{Q}$-Cartier.
(e) $D_1 \sim_{\mathbb{R}} D_2$ iff $D_1 \sim_{\mathbb{Q}} D_2$.
(f) $\operatorname{Supp} D_i \subset \operatorname{Supp} D$.

Proof If the $d_i \in \mathbb{R}$ are linearly independent then we can choose σ_i such that $\sigma_i(d_i) = 1$ and $\sigma_i(d_j) = 0$ for $i \neq j$. Then $\sigma_i(D) = D_i$, thus (11.43.1) shows (a) and (b).

For (c), assume that $\sum d_i D_i \equiv 0$ and let $C \subset X$ be a curve. Then $\sum d_i(D_i \cdot C) = 0$. Since $(D_i \cdot C) \in \mathbb{Q}$ and the d_i are linearly independent, we get that $(D_i \cdot C) = 0$ for every i. Applying (a) to D_i gives (d). Applying (b) to $D_1 - D_2$ gives (e). Finally (f) follows from the linear independence over $\mathbb{Q}$. □

Corollary 11.43.3 Let Θ be a Mumford $\mathbb{R}$-divisor and $\{d_i\}$ a basis of $\operatorname{CoSp}(\Theta)$ over $\mathbb{Q}$. Then we get a unique representation $\Theta = \sum d_i D_i$ where the D_i are $\mathbb{Q}$-divisors. If Θ is $\mathbb{R}$-Cartier, then the D_i are $\mathbb{Q}$-Cartier. □

Corollary 11.43.4 Let Δ be a Mumford $\mathbb{R}$-divisor and $\{d'_i\}$ a $\mathbb{Q}$-basis of $\mathbb{Q} + \operatorname{CoSp}(\Delta)$ such that $\sum d'_i = 1$. Then we get a unique representation $\Delta = \sum d'_i D_i$ where the D_i are $\mathbb{Q}$-divisors. If $K_X + \Delta$ is $\mathbb{R}$-Cartier, then $K_X + D_i$ are $\mathbb{Q}$-Cartier.

Proof Note that $K_X + \Delta = \sum d'_i(K_X + D_i)$, so the last assertion follows from (11.43.2.a). □

Next we show that $\mathbb{R}$-divisors can be approximated by $\mathbb{Q}$-divisors in a way that many properties are preserved. We start with some general comments on vector spaces and field extensions. At the end we care only about $\mathbb{R} \supset \mathbb{Q}$.

Definition–Lemma 11.44 Let K/k be a field extension, V a k-vector space and $w \in V \otimes_k K$. The *linear k-envelope* of w, denoted by $\operatorname{LEnv}_k(w) \subset V$, is the smallest vector subspace such that $w \in \operatorname{LEnv}_k(w) \otimes_k K$. Then $\operatorname{LEnv}_k(w)$ is spanned by any of the following three sets, where σ runs through all k-linear maps $K \to k$.

(11.44.1) All $(1_V \otimes \sigma)(w)$.
(11.44.2) All $\sum \sigma(c_i)v_i$, where $v_i \in V$ is a basis and $w = \sum c_i v_i$.
(11.44.3) All $\sum_i a_{ij} v_i$, where $e_j \in K$ is a k-basis and $w = \sum_{ij} a_{ij} e_j v_i$.

The *affine k-envelope* of w, denoted by $\operatorname{AEnv}_k(w) \subset V$, is the smallest affine-linear subspace such that $w \in \operatorname{AEnv}_k(w) \otimes_k K$. Then $\operatorname{AEnv}_k(w)$ is spanned by

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

any of the following three sets, where σ runs through all k-linear maps $K \to k$ such that $\sigma(1) = 1$.

(11.44.4) All $(1_V \otimes \sigma)(w)$.

(11.44.5) All $\sum \sigma(c_i)v_i$, where $v_i \in V$ is a basis and $w = \sum c_i v_i$.

(11.44.6) All $\sum_i a_{ij}v_i$, where $e_j \in K$ is a k-basis such that $e_1 = 1$ and $w = \sum_{ij} a_{ij}e_j v_i$.

11.45 (Approximating by rational simplices) Fix real numbers $d_1, \dots, d_m$ and consider a $\mathbb{Q}$-vector space W with basis $\mathfrak{d}_1, \dots, \mathfrak{d}_m$. Set $\mathfrak{d} := \sum d_i \mathfrak{d}_i \in W_{\mathbb{R}}$ and $V := \mathrm{AEnv}_{\mathbb{Q}}(\mathfrak{d})$. We inductively construct a sequence of simplices

$$V \supset S_1 \supset S_2 \supset \cdots \quad \text{such that} \quad \cap_n S_n = \{\mathfrak{d}\}.$$

Set $S_0 := V$. For each $n \in \mathbb{N}$ the cubes of the lattice $\frac{1}{n}\mathbb{Z}^m$ give a cubical chamber decomposition of $W_{\mathbb{R}}$. There is a smallest chamber C_n that contains $\mathfrak{d}$. Then $\mathfrak{d}$ is an interior point of $C_n \cap S_{n-1}$ (in its affine-linear span). The vertices of $C_n \cap S_{n-1}$ are in $\mathbb{Q}^n$. Thus $\mathfrak{d}$ can be written as a convex linear combination of suitably chosen $\dim V + 1$ vertices of $V_{\mathbb{R}} \cap C_n$; denote them by $\mathfrak{d}^n_j$. These span S_n. By (11.44), there are $\mathbb{Q}$-linear maps $\sigma^n_j \colon \mathbb{R} \to \mathbb{Q}$ such that $\mathfrak{d}^n_j = \sigma^n_j(\mathfrak{d})$. We can thus write

(11.45.1) $\mathfrak{d} = \sum_j \lambda^n_j \mathfrak{d}^n_j$, where

(11.45.2) $\mathfrak{d}^n_j = \sum_i \sigma^n_j(d_i)\mathfrak{d}_i$,

(11.45.3) $\sum_j \lambda^n_j = 1$ and $\sum_j \lambda^n_j \sigma^n_j(d_i) = d_i \quad \forall i$,

(11.45.4) $\lim_{n\to\infty} \mathfrak{d}^n_j = \mathfrak{d} \quad \forall j$, and

(11.45.5) for fixed n, the λ^n_j are linearly independent over $\mathbb{Q}$. (To see this, note that 1 and the d_i are $\mathbb{Q}$-linear combinations of the λ^n_j for fixed n.)

Remark 11.45.6 The choice of the vertices is not unique, but once we choose them, the constants λ^n_j are unique, and so are the restrictions of σ^n_j to $\mathrm{LEnv}_{\mathbb{Q}}(\mathfrak{d})$. Thus, from now on, we view σ^n_j and λ^n_j as depending only on $j, n \in \mathbb{N}$ and $d_1, \dots, d_m \in \mathbb{R}$. Note that these are not continuous functions of the d_i, even the number of the j-indices varies discontinuously with $d_1, \dots, d_m$.

Also, we only care about the restriction of the σ^n_j to $\mathrm{LEnv}_{\mathbb{Q}}(\mathfrak{d})$, so we are really dealing with finite dimensional linear algebra.

Proposition 11.46 (Convex approximation of $\mathbb{R}$-divisors I) *Let X be a reduced, S_2 scheme and $\Theta = \sum_i d_i D_i$ a Mumford $\mathbb{R}$-divisor, where the D_i are $\mathbb{Q}$-divisors. Let σ^n_j and λ^n_j be as in (11.45) and set $\Theta^n_j := \sum \sigma^n_j(d_i)D_i$. Then*

(11.46.1) *$\Theta = \sum_j \lambda^n_j \Theta^n_j$ and the Θ^n_j are $\mathbb{Q}$-divisors.*

(11.46.2) *Let $E \subset X$ be a prime divisor on X. Then* $\lim_{n\to\infty} \mathrm{coeff}_E \Theta^n_j = \mathrm{coeff}_E \Theta$ *and* $\mathrm{coeff}_E \Theta^n_j = \mathrm{coeff}_E \Theta$ *if* $\mathrm{coeff}_E \Theta \in \mathbb{Q}$.

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

(11.46.3) *Θ is effective iff the Θ_j^n are effective for every j for $n \gg 1$ (then they have the same support).*

Assume next that Θ is $\mathbb{R}$-Cartier. Then the following also hold.

(11.46.4) *The Θ_j^n are $\mathbb{Q}$-Cartier.*

(11.46.5) *Let E be prime divisor over X (11.1). Then $\lim_{n\to\infty} \operatorname{coeff}_E \Theta_j^n = \operatorname{coeff}_E \Theta$ and $\operatorname{coeff}_E \Theta_j^n = \operatorname{coeff}_E \Theta$ if $\operatorname{coeff}_E \Theta \in \mathbb{Q}$.*

(11.46.6) *Let C be a proper curve on X. Then $\lim_{n\to\infty}(C \cdot \Theta_j^n) = (C \cdot \Theta)$ and $(C \cdot \Theta_j^n) = (C \cdot \Theta)$ if $(C \cdot \Theta) \in \mathbb{Q}$.*

(11.46.7) *Θ is ample (11.51) iff the Θ_j^n are ample for every j for $n \gg 1$.*

Proof (1) is a formal consequence of (11.45.2), while the limit in (2) follows from (11.45.3). If $\operatorname{coeff}_E \Theta =: c \in \mathbb{Q}$, then $\sum_i x_i \operatorname{coeff}_E D_i = c$ defines a rational hyperplane in W (as in (11.45)). It contains $\mathfrak{d}$, hence also V and the other $\mathfrak{d}_j^n$. The Θ_j^n are the images of the $\mathfrak{d}_j^n$.

By (11.45.4) the λ_j^n are linearly independent over $\mathbb{Q}$. Thus, if Θ is $\mathbb{R}$-Cartier then the Θ_j^n are $\mathbb{Q}$-Cartier by (11.43.2), proving (4). Also, in this case, $\operatorname{coeff}_E \Theta$ makes sense for divisors over X and same for the intersection numbers $(C \cdot \Theta)$. The proofs of (5–7) are now the same as for (2). □

Proposition 11.47 (Convex approximation of $\mathbb{R}$-divisors II) *Let X be a demi-normal scheme and $\Delta = \sum d_i D_i$ a Mumford $\mathbb{R}$-divisor, where the D_i are $\mathbb{Q}$-divisors. Assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let σ_j^n and λ_j^n be as in (11.45) and set $\Delta_j^n := \sum \sigma_j^n(d_i) D_i$. Then*

(11.47.1) *$\Delta = \sum_j \lambda_j^n \Delta_j^n$ and $K_X + \Delta = \sum_j \lambda_j^n (K_X + \Delta_j^n)$.*

(11.47.2) *Δ is effective iff the Δ_j^n are effective for every j for $n \gg 1$ (then they have the same support).*

(11.47.3) *$K_X + \Delta_j^n$ are $\mathbb{Q}$-Cartier.*

(11.47.4) *$K_X + \Delta$ is ample iff the $K_X + \Delta_j^n$ are ample for every j for $n \gg 1$.*

(11.47.5) *Let E be a prime divisor. Then $\lim_{n\to\infty} a(E, X, \Delta_j^n) = a(E, X, \Delta)$ and $a(E, X, \Delta_j^n) = a(E, X, \Delta)$ if $a(E, X, \Delta) \in \mathbb{Q}$.*

(11.47.6) *Let C be a proper curve. Then $\lim_{n\to\infty}(C \cdot (K_X + \Delta_j^n)) = (C \cdot (K_X + \Delta))$ and $(C \cdot (K_X + \Delta_j^n)) = (C \cdot (K_X + \Delta))$ if $(C \cdot (K_X + \Delta)) \in \mathbb{Q}$.*

Assume next that (X, Δ) has a log resolution and fix $\varepsilon > 0$. Then, for every j and every $n \gg 1$, the following hold.

(11.47.7) *$|a(E, X, \Delta) - a(E, X, \Delta_j^n)| < \varepsilon$ for every divisor E over X, whenever one of the discrepancies is < 0.*

(11.47.8) *(X, Δ) is lc (resp. dlt or klt) iff (X, Δ_j^n) is lc (resp. dlt or klt).*

(11.47.9) *(X, Δ) and (X, Δ_j^n) have the same dlt modifications.*

Proof (1–2) follow directly from (11.46) and (3) follows from (11.46.4) and (1). Since ampleness is an open condition, (3) implies (4).

The proofs of (5) and (6) are the same as the proof of (11.46.2). If (X, Δ) has a log resolution then (7) follows from (11.7) and being lc (resp. dlt or klt) can be read off from the discrepancies, hence (7) implies (8) and (9). □

In the slc case, we have the following remarkable sharpening.

Complement 11.48 (Han et al., 2020, 5.6) *In (11.47) assume in addition that $(X, \Delta = \sum d_i D_i)$ is slc. Then we can choose the σ_j^n and λ_j^n to depend only on $(d_1, \dots, d_r)$ and the dimension.* □

We also get some information about pluricanonical sheaves for $\mathbb{R}$-divisors.

Theorem 11.49 *Fix a finite set $C := \{c_1, \dots, c_r\} \subset [0, 1]$. Then there is a subset $M(C, n) \subset \mathbb{Z}$ of positive density such that, if $(X, \Delta = \sum c_i D_i)$ is an slc pair of dimension n, then $(X, \lfloor\Delta\rfloor + \sum\{mc_i\}D_i)$ is slc for $m \in M(C, n)$, and has the same lc centers as (X, Δ).*

Proof Let $A \subset \mathbb{R}^r$ be the affine envelope of $\mathbf{c} := (c_1, \dots, c_r) \in \mathbb{R}^r$ and $H \subset \mathbb{R}^n$ the closed subgroup generated by A. Then A is a connected component of H and $H/(H \cap \mathbb{Z}^r) \subset \mathbb{R}^r/\mathbb{Z}^r$ is a closed subgroup. Furthermore, by a theorem of Weyl, the multiples of $\mathbf{c}$ are equidistributed in $H/(H \cap \mathbb{Z}^r)$; see, for example, Kuipers and Niederreiter (1974, sec.1.1).

Pick now σ_j^n as in (11.48). Then the convex linear combinations of the $\sigma_j^n(\mathbf{c})$ give an open neighborhood $\mathbf{c} \in U \subset H/(H \cap \mathbb{Z}^r)$. If $(\{mc_1\}, \dots, \{mc_r\}) \in U$ then (1–2) hold. □

Applying (11.20) gives the following.

Corollary 11.50 *Using the notation of (11.49), let $(X, \Delta = \sum c_i D_i)$ be an slc pair of dimension n. Then, for every $m \in M(C, n)$,*

$$\operatorname{depth}_x \omega_X^{[m]}\left(\textstyle\sum \lfloor mc_i \rfloor D_i\right) \geq \min\{3, \operatorname{codim}_X x\} \tag{11.50.1}$$

whenever x is not an lc center of (X, Δ). □

Example 11.50.2 Let $X \subset \mathbb{A}^4$ be the quadric cone and $|A|, |B|$ the two families of planes on X. Fix $r \in \mathbb{N}$ and for $0 < c \leq 1/r$ consider the pair

$$(X, \Delta_c := B + cA_1 + \cdots + cA_r + (1 - rc)A_0).$$

Then (X, Δ_c) is canonical and

$$\mathscr{O}_X(\lfloor m\Delta_c \rfloor) \simeq \begin{cases} \mathscr{O}_X(-A) & \text{if } \{mc\} \leq 1/r, \quad \text{and} \\ \mathscr{O}_X(-dA) & \text{for some } d \geq 2 \text{ otherwise.} \end{cases}$$

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

An easy computation, as in Kollár (2013b, 3.15.2), shows that $\mathscr{O}_X(\lfloor m\Delta_c \rfloor)$ is CM iff $\{mc\} \leq 1/r$. If c is irrational, then the set $\{m\colon \{mc\} \leq 1/r\}$ has no periodic subsets.

Definition 11.51 Let $g\colon X \to S$ be a proper morphism. An $\mathbb{R}$-Cartier divisor H is g-ample iff it is linearly equivalent to a positive linear combination $H \sim_{\mathbb{R}} \sum c_i H_i$ of g-ample Cartier divisors.

Ampleness is preserved under perturbations. Indeed, let $D_1, \dots, D_r$ be $\mathbb{Q}$-Cartier divisors. There are $m_j > 0$ such that the $m_j H_1 + D_j$ are g-ample. Then

$$H + \textstyle\sum_j \eta_j D_j \sim_{\mathbb{R}} (c_1 - \sum_j \eta_j m_j) H_1 + \sum_{i \neq 1} c_i H_i + \sum_j \eta_j (m_j H_1 + D_j)$$

shows that $H + \sum_j \eta_j D_j$ is g-ample if $\eta_j \geq 0$ and $\sum_j \eta_j m_j \leq c_1$.

This implies that if H is g-ample, $m \gg 1$ and $\lfloor mH \rfloor$ is Cartier, then $\lfloor mH \rfloor$ is very g-ample. However, frequently $\lfloor mH \rfloor$ is not even $\mathbb{Q}$-Cartier for every $m > 0$, making the proofs of the basic ampleness criteria more complicated.

Theorem 11.52 (Asymptotic Riemann–Roch) *Let X be a normal, proper algebraic space of dimension n and D a nef $\mathbb{R}$-Cartier divisor. Then*

$$\begin{aligned} h^0(X, \mathscr{O}_X(\lfloor mD \rfloor)) &= \tfrac{m^n}{n!}(D^n) + O(m^{n-1}), \quad \textit{and} \\ h^0(X, \mathscr{O}_X(\lceil mD \rceil)) &= \tfrac{m^n}{n!}(D^n) + O(m^{n-1}). \end{aligned} \tag{11.52.1}$$

Proof By Chow's lemma we may assume that X is projective. Write $D = \sum a_i A_i$ where the A_i are effective, ample $\mathbb{Z}$-divisors and $a_i \in \mathbb{R}$. Then

$$\textstyle\sum \lfloor m a_i \rfloor A_i \leq \lfloor mD \rfloor \leq mD \leq \lceil mD \rceil \leq H + \sum \lceil m a_i \rceil A_i \tag{11.52.2}$$

for any H ample and effective. It is thus enough to prove that (11.52.1) holds for the two divisors on the sides of (11.52.2) for suitable H. Note that

$$\textstyle\sum \lceil m a_i \rceil A_i \sim_{\mathbb{R}} \sum (\lceil m a_i \rceil - m a_i) A_i + mD,$$

thus $\sum \lceil m a_i \rceil A_i$ is nef for every $m \geq 0$, even though some of the $\lceil m a_i \rceil$ may be negative. Next choose H such that (11.52.4) holds (with $F = \mathscr{O}_X$) and $H + \sum A_i$ is linearly equivalent to an irreducible divisor B. Then, by Riemann–Roch,

$$h^0(X, \mathscr{O}_X(H + \textstyle\sum \lceil m a_i \rceil A_i)) = \chi(X, \mathscr{O}_X(H + \sum \lceil m a_i \rceil A_i)) = \frac{m^n}{n!}(D^n) + O(m^{n-1}).$$

Restricting $\mathscr{O}_X(H + \sum \lceil m a_i \rceil A_i)$ to B, the kernel is

$$\mathscr{O}_X(\textstyle\sum \lceil m a_i \rceil A_i - \sum A_i) \subset \mathscr{O}_X(\sum \lfloor m a_i \rfloor A_i)$$

(the two are equal iff none of the $m a_i$ are integers). Thus

$$h^0(X, \mathscr{O}_X(H + \textstyle\sum \lceil m a_i \rceil A_i)) - h^0(X, \mathscr{O}_X(\sum \lfloor m a_i \rfloor A_i))$$

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

is at most $h^0(B, \mathscr{O}_B(H|_B + \sum\lceil ma_i\rceil A_i|_B))$. The latter is bounded by $O(m^{n-1})$ using (11.52.3). □

11.52.3 (Matsusaka inequality) Let X be a proper variety of dimension n, L a nef and big $\mathbb{Z}$-divisor and D a Weil $\mathbb{Z}$-divisor giving a dominant map $|D|\colon X \dashrightarrow Z$. Then

$$h^0(X, \mathscr{O}_X(D)) \le \frac{(D \cdot L^{n-1})^{\dim Z}}{(L^n)^{\dim Z - 1}} + \dim Z.$$

See Matsusaka (1972) or Kollár (1996, VI.2.15) for proofs.

11.52.4 (Fujita vanishing) Let X be a projective scheme and F a coherent sheaf on X. Then there is an ample line bundle L such that

$$H^i(X, F \otimes L \otimes M) = 0 \quad \forall i > 0,\ \forall \text{ nef line bundle } M.$$

See Fujita (1983) (or Lazarsfeld (2004, I.4.35) for the characteristic 0 case).

Corollary 11.53 (Kodaira lemma) *Let X be a normal, proper, irreducible algebraic space of dimension n and D a nef $\mathbb{R}$-divisor. Then D is big (p.xvi) $\Leftrightarrow$ $(D^n) > 0$ $\Leftrightarrow$ one can write $D = cB + E$, where B is a big $\mathbb{Z}$-divisor, $c > 0$, and E is an effective $\mathbb{R}$-divisor. If X is projective, then one can choose B to be ample.*

Proof With (11.52) in place, the arguments in Kollár and Mori (1998, 2.61) or Lazarsfeld (2004, 2.2.6) work. See also Shokurov (1996, 6.17) (for characteristic 0) and Birkar (2017, 1.5) for the original proofs, or Fujino and Miyamoto (2021, 2.3). □

The proof of the Nakai–Moishezon criterion for $\mathbb{R}$-divisors uses induction on all proper schemes, so first we need some basic results about them.

11.54 ($\mathbb{R}$-Cartier divisor classes) Fujino and Miyamoto (2021) On an arbitrary scheme one can define $\mathbb{R}$-*line bundles* or $\mathbb{R}$-*Cartier divisor classes* as elements of $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. It is better to think of these as coming from line bundles, but writing divisors keeps the additive notation.

Claim 11.54.1 Let X be a proper algebraic space, $p\colon Y \to X$ its normalization and Θ an $\mathbb{R}$-Cartier divisor class on X. Then Θ is ample iff $p^*\Theta$ is ample.

Proof For Cartier divisors, this is Hartshorne (1977, ex.III.5.7), which implies the $\mathbb{Q}$-Cartier case. Next we reduce the $\mathbb{R}$-Cartier case to it.

By assumption, we can write $\Theta \sim_{\mathbb{R}} \sum_i d_i D_i$ where the D_i are $\mathbb{Q}$-Cartier. By (11.46), there are $c_{ij} \in \mathbb{Q}$ and $0 < \lambda_j \in \mathbb{R}$ such that the $\Theta_j^Y := \sum_i c_{ij} p^* D_i$ are ample and $d_i = \sum_k \lambda_j c_{ij}$ for every i. In particular, $p^*\Theta \sim_{\mathbb{R}} \sum_j \lambda_j \Theta_j^Y$.

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

Set $\Theta_j := \sum_i c_{ij} D_i$. Then $\Theta \sim_{\mathbb{R}} \sum_j \lambda_j \Theta_j$ and $p^*\Theta_j = \Theta_j^Y$. The Θ_j are $\mathbb{Q}$-Cartier, hence ample, hence so is Θ. □

Corollary 11.54.2 Let $g\colon X \to S$ be a proper morphism of algebraic spaces and Θ an $\mathbb{R}$-Cartier divisor class on X. Then

$$S^{\mathrm{amp}} := \{s \in S : \Theta_s \text{ is ample on } X_s\} \subset S \quad \text{is open.}$$

Proof Write $\Theta = \sum d_i D_i$ and apply (11.46) to its restriction to X_s. Thus we get $\mathbb{Q}$-Cartier divisors $\Theta_j := \Theta_j^n$ (for $n \gg 1$) such that $\Theta = \sum_j \lambda_j \Theta_j$ and each $\Theta_j|_{X_s}$ is ample. The Θ_j are ample over some open $s \in S^\circ \subset S$, hence so is Θ. □

Theorem 11.55 Fujino and Miyamoto (2021) *Let X be a proper algebraic space and D an $\mathbb{R}$-Cartier divisor class on X. Then D is ample iff $(D^{\dim Z} \cdot Z) > 0$ for every integral subscheme $Z \subset X$.*

Proof By (11.54.1), we may assume that X is normal. By (11.52), we may assume that D is an effective $\mathbb{R}$-divisor. By (11.46), we can write $D = \sum \lambda_i D_i$ where the D_i are effective, $\mathbb{Q}$-Cartier. $D - D_i$ can be chosen arbitrarily small.

Let $p\colon Y \to \operatorname{Supp} D \hookrightarrow X$ be the normalization of $\operatorname{Supp} D$. By dimension induction, p^*D is ample, and so are the p^*D_i if the $D - D_i$ are small enough.

Thus the $D_i|_{\operatorname{Supp} D}$ are ample, hence the D_i are semiample by (11.55.1). Since $(D \cdot C) > 0$ for every curve, $\operatorname{Supp} D$ is not disjoint from any curve, hence the same holds for $\operatorname{Supp} D_i = \operatorname{Supp} D$. So the D_i are ample, and the converse is clear. □

Claim 11.55.1 (Lazarsfeld, 2004, p.35) Let X be a proper algebraic space and D an effective $\mathbb{Q}$-Cartier divisor such that $D|_{\operatorname{Supp} D}$ is ample. Then D is semiample. Thus if D is not disjoint from any curve, then D is ample. □

The usual proof of the Seshadri criterion (see Lazarsfeld (2004, 1.4.13)) now gives the following.

Corollary 11.56 (Seshadri criterion) *Let X be a proper algebraic space and D an $\mathbb{R}$-Cartier divisor on X. Then D is ample iff there is an $\varepsilon > 0$ such that $(D \cdot C) \geq \varepsilon \operatorname{mult}_p C$ for every pointed, integral curve $p \in C \subset X$.* □

Next we study a way to pull back Weil divisors.

11.57 (Intersection theory on normal surfaces) Mumford (1961) Let S be a normal, two-dimensional scheme and $p\colon S' \to S$ a resolution with exceptional curves E_i. The the intersection matrix $(E_i \cdot E_j)$ is negative definite by the Hodge index theorem (see Kollár (2013b, 10.1)). Let D be an $\mathbb{R}$-divisor on S. Then there is a unique p-exceptional $\mathbb{R}$-divisor E_D such that

$$(E_i \cdot (p_*^{-1} D + E_D)) = 0 \quad \text{for every} \quad i. \tag{11.57.1}$$

If D is effective, then E_D is effective by Kollár (2013b, 10.3.3) and $(E_i \cdot E_D) \le 0$ for every i.

We call $p^*D := p_*^{-1}D + E_D$ the *numerical pull-back* of D. If D is $\mathbb{R}$-Cartier then this agrees with the usual pull-back.

More generally, the numerical pull-back is also defined if S' is only normal: we first pull-back to a resolution of S' and then push forward to S.

If D_1, D_2 are $\mathbb{R}$-divisors and one of them has proper support, then one can define their intersection cycle as

$$(D_1 \cdot D_2) = p_*(p_*^{-1} D_1 \cdot p^* D_2) = p_*(p_*^{-1} D_2 \cdot p^* D_1). \tag{11.57.2}$$

If S is proper, we get the usual properties of intersection theory, except that, even if the D_i are $\mathbb{Z}$-divisors, their intersection numbers can be rational.

The following connects the numerical and sheaf-theoretic pull-backs.

Claim 11.57.3 Let $p\colon T \to S$ be a proper, birational morphism between normal surfaces with exceptional curve $E = \cup E_i$. Let B be an $\mathbb{R}$-divisor on T such that $-B$ is p-nef. Then $p_*\mathscr{O}_T(\lfloor B \rfloor) = \mathscr{O}_S(\lfloor p_* B \rfloor)$.

Moreover, if E is connected, then $g_*\mathscr{O}_T(\lfloor B - \varepsilon E \rfloor) = \mathscr{O}_S(\lfloor p_* B \rfloor)$ for $0 \le \varepsilon \ll 1$, save when B is a $\mathbb{Z}$-divisor and $B \sim 0$ in a neighborhood of E.

Proof Write $B = B_v + B_h$ as a sum of its exceptional and nonexceptional parts. We can harmlessly replace B_h with its round down, so we assume that B_h is a $\mathbb{Z}$-divisor. Let ϕ be a local section of $\mathscr{O}_S(p_* B_h)$. Then $\phi \circ p$ is a rational section of $\mathscr{O}_T(B_h)$, with possible poles along the exceptional curves. There is thus a smallest exceptional $\mathbb{Z}$-divisor F such that $\phi \circ p$ is a section of $\mathscr{O}_T(B_h + F)$. In particular, $(E_i \cdot (B_h + F)) \ge 0$ for every i. Thus

$$(E_i \cdot (F - B_v)) = (E_i \cdot (B_h + F - B)) \ge (E_i \cdot (B_h + F)) \ge 0$$

for every i. By the Hodge index theorem (Kollár, 2013b, 10.3.3), this implies that $B_v - F$ is effective, thus $B_h + F \le \lfloor B \rfloor$.

Moreover, $B_v - F - \varepsilon E$ is effective, unless

$$(E_i \cdot (-B)) = 0 \quad \text{and} \quad (E_i \cdot (B_h + F)) = 0$$

for every i. Then $B_h + F \sim 0$ and $B_h + B_v \sim_{\mathbb{Q}} 0$. Thus $F = B_v$, hence $B \sim 0$. □

Corollary 11.57.4 Let $p\colon T \to S$ be a proper, birational morphism between normal surfaces and D an $\mathbb{R}$-divisor on S. Then $p_*\mathscr{O}_T(\lfloor p^* D \rfloor) = \mathscr{O}_S(\lfloor D \rfloor)$. □

Next we propose a higher dimensional version of pull-back, focusing on its numerical properties. A different notion, using sheaf-theoretic properties, is defined in de Fernex and Hacon (2009).

11.58 (Numerical pull-back) Let $g\colon Y \to X$ be a projective, birational morphism of normal schemes and H a g-ample Cartier divisor. We define the *H-numerical pull-back* of $\mathbb{R}$-divisors

$$g_H^{(*)}\colon \ \mathrm{WDiv}_{\mathbb{R}}(X) \to \mathrm{WDiv}_{\mathbb{R}}(Y)$$

as follows. Let $D \subset X$ be an $\mathbb{R}$-divisor. We inductively define

$$g_H^{(*)}(D) = g_*^{-1}D + \textstyle\sum_{i\geq 2} F_i(D), \qquad (11.58.1)$$

where $\operatorname{Supp} F_i(D)$ consists of g-exceptional divisors $E_{i\ell}$ for which $g(E_{i\ell}) \subset X$ has codimension i.

Assume that we already defined the $F_i(D)$ for $i < j$. Let $x \in X$ be a point of codimension j. After localizing at x, we have $g_x\colon Y_x \to X_x$. Let $F_j(D)_x$ be the unique divisor supported on $g_x^{-1}(x)$ such that

$$(E_{j\ell} \cdot (g_*^{-1}D + \textstyle\sum_{i<j} F_i(D) + F_j(D)_x) \cdot H^{j-2}) = 0 \quad \forall \ell. \qquad (11.58.2)$$

To make sense of this, we may assume that H is very ample. Let S be a general complete intersection of $j-2$ members of $|H|$. Then S is a normal surface, so we are working with intersection numbers as in (11.57). Also, if S is general, then the $g_x|_S$-exceptional curves are in one-to-one correspondence with the divisors $E_{j\ell}$, so any linear combination of $g_x|_S$-exceptional curves corresponds to a linear combination of the divisors $E_{j\ell}$.

If we have proper, but non-projective $Y \to X$, we can apply our definition to a projective modification $Y' \to Y \to X$ and then push forward to Y. This defines $g_H^{(*)}$ in general.

Already in simple situations, for example, for cones over cubic surfaces, the divisors $g_H^{(*)}(D)$ do depend on H. However, the notion has several good properties and it is quite convenient in some situations. See, for example, (11.52) or Fulger et al. (2016, 3.3).

Theorem 11.59 *Let $g\colon Y \to X$ be a projective, birational morphism of normal schemes and H a g-ample Cartier divisor. Then*

(11.59.1) $g_H^{(*)}\colon \ \mathrm{WDiv}_{\mathbb{R}}(X) \to \mathrm{WDiv}_{\mathbb{R}}(Y)$ *is $\mathbb{R}$-linear,*

(11.59.2) $g_* \circ g_H^{(*)}$ *is the identity,*

(11.59.3) *if D is $\mathbb{R}$-Cartier, then $g_H^{(*)}(D) = g^*(D)$,*

(11.59.4) *if D is effective, then so is $g_H^{(*)}(D)$,*

(11.59.5) $g_H^{(*)}$ *respects $\mathbb{R}$-linear equivalence,*

(11.59.6) $g_*\mathscr{O}_Y(\lfloor g_H^{(*)}(B)\rfloor) = \mathscr{O}_X(\lfloor B\rfloor)$, *and*

(11.59.7) $g_H^{(*)}$ *maps $\mathbb{Q}$-divisors to $\mathbb{Q}$-divisors.*

Proof Here (1–3) are clear from the definition. (4) follows from its surface case, which we noted after (11.57.1). If $D_1 \sim_{\mathbb{R}} D_2$ then, using first (1) and then (3), we get that

$$g_H^{(*)}(D_1) = g_H^{(*)}(D_2) + g_H^{(*)}(D_1 - D_2) = g_H^{(*)}(D_2) + g^*(D_1 - D_2),$$

giving (5). Finally (6) is a local question. We may thus assume that (6) holds outside a closed point $x \in X$. Assume to the contrary that $\mathscr{O}_Y(\lfloor g_H^{(*)}(B)\rfloor)$ has a rational section that has poles along $g^{-1}(x)$. After restricting to a general complete intersection surface $S \subset Y$ as in (11.58), we would get a contradiction to (11.57.3). □

The following negativity lemmas are quite useful.

Lemma 11.60 (Kollár and Mori, 1998, 3.39) *Let $h\colon Z \to Y$ be a proper birational morphism between normal schemes. Let $-B$ be an h-nef $\mathbb{R}$-Cartier divisor on Z. Then*

(11.60.1) *B is effective iff h_*B is.*

(11.60.2) *Assume that B is effective. Then for every $y \in Y$, either $h^{-1}(y) \subset$ Supp B or $h^{-1}(y) \cap \operatorname{Supp} B = \emptyset$.* □

Lemma 11.61 Kollár (2018a) *Let $\pi\colon Y \to X$ be a proper, birational contraction of demi-normal schemes such that none of the π-exceptional divisors is contained in* Sing *Y. Let N, B be Mumford $\mathbb{R}$-divisors such that N is π-nef and B is effective and non-exceptional. Then*

$$\pi_*\mathscr{O}_Y(\lfloor -N - B\rfloor) = \mathscr{O}_X(\lfloor \pi_*(-N - B)\rfloor). \tag{11.61.1}$$

Moreover, fix $x \in X$ and let E_x be the divisorial part of $\pi^{-1}(x)$. Then

$$\pi_*\mathscr{O}_Y(\lfloor -N - B - \varepsilon E_x\rfloor) = \mathscr{O}_X(\lfloor \pi_*(-N - B)\rfloor) \tag{11.61.2}$$

for $0 \le \varepsilon \ll 1$, save when $N+B$ is a $\mathbb{Z}$-divisor and $N+B \sim 0$ in a neighborhood of $\pi^{-1}(x)$.

Proof If dim $Y = 2$, then B is also π-nef, so the claim follows from (11.57.3). In general, we may assume that π is projective, take the normalization, and reduce to the surface case as in the proof of (11.59.6). □

11.62 (Divisorial base locus) Let X be a normal scheme and D a $\mathbb{Z}$-divisor. The divisorial part of the base locus of $|D|$ is denoted by $\mathrm{Bs}^{\mathrm{div}}(D)$. Define the *divisorial base locus* of an $\mathbb{R}$-divisor Δ as $\mathrm{Bs}^{\mathrm{div}}(\Delta) := \mathrm{Bs}^{\mathrm{div}}(\lfloor\Delta\rfloor) + \{\Delta\}$. In particular, $H^0(X, \mathscr{O}_X(\lfloor D - Bs^{\mathrm{div}}\rfloor(D))) = H^0(X, \mathscr{O}_X(\lfloor D\rfloor))$.

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press

Assume now that we can write $\Delta = \sum_j a_j A_j$ where the A_j are Cartier divisors such that $\operatorname{Bs}(A_j) = \emptyset$ and $a_j > 0$ (This is always possible if X is quasi-affine.) Then $\sum_j \lfloor m a_j \rfloor A_j \le \lfloor m\Delta \rfloor$ for any $m > 0$, which shows that

$$\operatorname{Bs}^{\mathrm{div}}(m\Delta) \le \textstyle\sum_j A_j. \tag{11.62.1}$$

Claim 11.62.2 Let $g\colon Y \to X$ be a proper, birational morphism of normal schemes and Δ an $\mathbb{R}$-Cartier, $\mathbb{R}$-divisor on X. Let E be a g-exceptional divisor. Then $g_*\mathscr{O}_Y(\lfloor mg^*\Delta + mE \rfloor) = \mathscr{O}_X(\lfloor m\Delta \rfloor)$ for infinitely many $m \ge 1$ iff E is effective.

Proof Use (11.61) with $B = 0$ and $N = -g^*\Delta$ for the if part. For the converse, we may assume that X is affine. Write $\Delta = \sum_j a_j A_j$ as above.

If $g_*\mathscr{O}_Y(\lfloor mg^*\Delta + mE \rfloor) = \mathscr{O}_X(\lfloor m\Delta \rfloor)$ then $-mE \le \operatorname{Bs}^{\mathrm{div}}(mg^*\Delta) \le \sum_j g^* A_j$ by (11.62.1). If this holds for infinitely many $m \ge 1$, then E is effective. □

https://doi.org/10.1017/9781009346115.013 Published online by Cambridge University Press