VII
The differential complex associated with a formally integrable structure

In this chapter we shall introduce the differential complex associated with a formally integrable structure and discuss several aspects of its exactness.

VII.1 The exterior derivative

Let Ω be a differentiable manifold of dimension N. As in Chapter I, we shall denote by $\mathfrak{X}(\Omega)$ the space of all complex vector fields over Ω. We then set $\mathfrak{N}_0(\Omega) = C^\infty(\Omega)$ and if $q \geq 1$ is an integer we shall denote by $\mathfrak{N}_q(\Omega)$ the space of all $C^\infty(\Omega)$-multilinear, alternating forms

$$\omega : \underbrace{\mathfrak{X}(\Omega) \times \ldots \times \mathfrak{X}(\Omega)}_{q} \longrightarrow C^\infty(\Omega).$$

Notice that, according to Section I.4, we have $\mathfrak{N}_1(\Omega) = \mathfrak{N}(\Omega)$; notice also that $\mathfrak{N}_q(\Omega)$ has, for each q, the structure of a $C^\infty(\Omega)$-module. We then generalize the concept of one-forms introduced in Section I.4 and call the elements of the direct sum $\oplus_{q=0}^{\infty}\mathfrak{N}_q(\Omega)$ *differential forms over* Ω. If $\omega \in \mathfrak{N}_q(\Omega)$ we shall say that ω is a *differential form of degree* q (or q*-form* for short). The *exterior product* between $\omega \in \mathfrak{N}_q(\Omega)$ and $\theta \in \mathfrak{N}_r(\Omega)$ is the $(q+r)$-form $\omega \wedge \theta \in \mathfrak{N}_{q+r}(\Omega)$ defined by the formula

$$(\omega \wedge \theta)(X_1, \ldots, X_{q+r}) = \sum_{(A,B)} (\operatorname{sg} \sigma)\, \omega(X_{\sigma(1)}, \ldots, X_{\sigma(q)})\theta(X_{\sigma(q+1)}, \ldots, X_{\sigma(q+r)}), \tag{VII.1}$$

where $X_j \in \mathfrak{X}(\Omega)$ and the summation is over all partitions (A, B) of $\{1, \ldots, q+r\}$ with $|A| = q$, $|B| = r$ and $\sigma \in S^{q+r}$ is such that $\sigma\{1, \ldots, q\} = A$, $\sigma\{q+1, \ldots, q+r\} = B$. It is easy to see that (VII.1) defines indeed a $(q+r)$-form, that the map

$$(\omega, \theta) \mapsto \omega \wedge \theta$$

is $C^\infty(\Omega)$-bilinear, and that the operation so defined is associative. It follows that $\oplus_{q=0}^{\infty}\mathfrak{N}_q(\Omega)$ has a structure of a graded $C^\infty(\Omega)$-algebra. We also remark that

$$\omega \wedge \theta = (-1)^{qr}\theta \wedge \omega, \quad \omega \in \mathfrak{N}_q(\Omega),\ \theta \in \mathfrak{N}_r(\Omega). \tag{VII.2}$$

The *exterior differentiation operator* is a $\mathbb{C}$-linear map

$$\mathrm{d} : \oplus_{q=0}^{\infty}\mathfrak{N}_q(\Omega) \to \oplus_{q=0}^{\infty}\mathfrak{N}_q(\Omega)$$

whose restriction to $\mathfrak{N}_0(\Omega) = C^\infty(\Omega)$ coincides with the operator introduced in Definition I.1.6 [that is, $\mathrm{d}f(X) = X(f)$ if $f \in C^\infty(\Omega)$ and $X \in \mathfrak{X}(\Omega)$] and is characterized by the following additional properties:

(d_1) $\mathrm{d}\mathfrak{N}_q(\Omega) \subset \mathfrak{N}_{q+1}(\Omega)$ for every $q \geq 0$;
(d_2) $\mathrm{d} \circ \mathrm{d} = 0$;
(d_3) if $\omega \in \mathfrak{N}_q(\Omega)$ and $\theta \in \mathfrak{N}_r(\Omega)$ then

$$\mathrm{d}(\omega \wedge \theta) = \mathrm{d}\omega \wedge \theta + (-1)^q \omega \wedge \mathrm{d}\theta. \tag{VII.3}$$

The only operator d which satisfies these properties can be defined by the expression:

$$\begin{aligned} \mathrm{d}\omega(X_1, \ldots, X_{q+1}) &= \sum_{j=1}^{q+1}(-1)^{j+1}X_j\left\{\omega(X_1, \ldots, \hat{X}_j, \ldots, X_{q+1})\right\} \\ &\quad + \sum_{j<k}(-1)^{j+k}\omega([X_j, X_k], X_1, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_{q+1}), \end{aligned} \tag{VII.4}$$

where $\omega \in \mathfrak{N}_q(\Omega)$ and $X_j \in \mathfrak{X}(\Omega)$. (Recall that the sign ^ over a letter means that the letter is missing.)

VII.2 The local representation of the exterior derivative

If $\omega \in \mathfrak{N}_q(\Omega)$ then $\omega(X_1, X_2, \ldots, X_q) = 0$ at p if the vector fields $X_1, \ldots, X_q$ are linearly dependent at p. Indeed if we have, say, $X_1 = \sum_{j=2}^{q} \alpha_j X_j$ at p and if take $g_j \in C^\infty(\Omega)$ with $g_j(p) = \alpha_j$ then

$$\omega(X_1, X_2, \ldots, X_q) = \omega\left(X_1 - \sum_{j=2}^{q} g_j X_j, X_2, \ldots, X_q\right)$$

and our claim follows immediately from Lemma I.4.1 applied to the one-form $X \mapsto \omega(X, X_2, \ldots, X_q)$. In particular, we can restrict a q-form over Ω to an open set $W \subset \Omega$, that is, given $\omega \in \mathfrak{N}_q(\Omega)$ there is $\omega|_W \in \mathfrak{N}_q(W)$ which makes the diagram

$$\begin{array}{ccc}
\mathfrak{X}(\Omega) \times \ldots \times \mathfrak{X}(\Omega) & \xrightarrow{\omega} & C^\infty(\Omega) \\
\downarrow & & \downarrow \\
\mathfrak{X}(W) \times \ldots \times \mathfrak{X}(W) & \xrightarrow{\omega|_W} & C^\infty(W)
\end{array}$$

commutative, where the vertical arrows denote the restriction homomorphisms. Moreover, from (VII.4) it follows easily that the operator d commutes with restrictions.

Let $(U, \mathbf{x})$ be a local chart in Ω. The $C^\infty(U)$-module $\mathfrak{N}_q(U)$ is spanned by the q-forms $\mathrm{d}x_J$, where $J: j_1 < j_2 < \ldots < j_q$ is an ordered multi-index of length q, $j_\ell \in \{1, \ldots, N\}$, and

$$\mathrm{d}x_J = \mathrm{d}x_{j_1} \wedge \ldots \wedge \mathrm{d}x_{j_q}.$$

Every $\omega \in \mathfrak{N}_q(U)$ can be represented as

$$\omega = \sum_{|J|=q} f_J(x)\mathrm{d}x_J, \quad f_J \in C^\infty(U) \tag{VII.5}$$

and the properties that characterize d allow us to write

$$\mathrm{d}\omega = \sum_{|J|=q} \mathrm{d}f_J \wedge \mathrm{d}x_J\,. \tag{VII.6}$$

Remark VII.2.1. The analysis presented at the beginning of this section allows one to extend the notion of pullback for one-forms introduced in Section I.14. If $\mathcal{M}$ is a submanifold of Ω we have well-defined *pullback homomorphisms* $(\iota_{\mathcal{M}})^* : \mathfrak{N}_q(\Omega) \to \mathfrak{N}_q(\mathcal{M})$ defined by

$$(\iota_{\mathcal{M}})^*\omega\,(X_1, \ldots, X_q)(p) = \omega(\tilde{X}_1, \ldots, \tilde{X}_q)(p), \quad \omega \in \mathfrak{N}_q(\Omega), \tag{VII.7}$$

where $p \in \mathcal{M}$, $X_1, \ldots, X_q \in \mathfrak{X}(\mathcal{M})$, and $\tilde{X}_1, \ldots, \tilde{X}_q \in \mathfrak{X}(\Omega)$ are such that $(\iota_{\mathcal{M}})_* X_j|_p = \tilde{X}_j|_p$ for every $j = 1, \ldots, q$. The pullback homomorphisms commute with the exterior derivative, that is

$$(\iota_{\mathcal{M}})^*\mathrm{d}\omega = \mathrm{d}_{\mathcal{M}}(\iota_{\mathcal{M}})^*\omega\,, \quad \omega \in \mathfrak{N}_q(\Omega), \tag{VII.8}$$

where we have denoted by $\mathrm{d}_{\mathcal{M}}$ the exterior derivative operator on the manifold $\mathcal{M}$.

VII.3 The Poincaré Lemma

Let $D \subset \mathbb{R}^N$ be open and convex and let $\mathcal{U}$ be an open subset of $\mathbb{R}^p$. Denote by $\mathfrak{N}^\bullet_q(D \times \mathcal{U})$ the space of all q-forms

$$f = \sum_{|J|=q} f_J(x, y)\mathrm{d}x_J \tag{VII.9}$$

where $f_J \in C^\infty(D \times \mathcal{U})$.

Fix $x^0 \in D$ and set, for $J = (j_1, \dots, j_q)$,

$$\beta_J(x, x^0) = \sum_{r=1}^{q}(-1)^{r-1}(x_{j_r} - x^0_{j_r})\mathrm{d}x_{j_1} \wedge \dots \wedge \widehat{\mathrm{d}x_{j_r}} \wedge \dots \wedge \mathrm{d}x_{j_q}.$$

Next we introduce the operators, for $q \geq 1$,

$$\mathrm{G} : \mathfrak{N}^\bullet_q(D \times \mathcal{U}) \to \mathfrak{N}^\bullet_{q-1}(D \times \mathcal{U})$$

defined in the following way: if f is as in (VII.9) we set

$$\mathrm{G}(f) = \sum_{|J|=q} \left\{ \int_0^1 f_J\left(x^0 + \tau[x - x^0], y\right) \tau^{q-1}\mathrm{d}\tau \right\} \beta_J(x, x^0). \tag{VII.10}$$

The standard Poincaré Lemma states that

$$\mathrm{d}_x\mathrm{G}(f) + \mathrm{G}(\mathrm{d}_x f) = f \quad \text{if } q \geq 1; \tag{VII.11}$$

$$\mathrm{G}(\mathrm{d}_x f) = f - f(x_0, \cdot) \quad \text{if } q = 0, \tag{VII.12}$$

which are formulae that can be proved by direct computation, using (VII.6). In particular we derive, if $q \geq 1$,

$$\mathrm{d}_x\mathrm{G}(f) = f \quad \text{if} \quad \mathrm{d}_x f = 0.$$

VII.4 The differential complex associated with a formally integrable structure

Let $\mathcal{V} \subset \mathbb{C}T\Omega$ be a formally integrable structure over Ω. For each $q \geq 1$ we denote by $\mathfrak{N}^{\mathcal{V}}_q(\Omega)$ the $C^\infty(\Omega)$-submodule of $\mathfrak{N}_q(\Omega)$ defined by all $\omega \in \mathfrak{N}_q(\Omega)$ for which $\omega(X_1, \dots, X_q) = 0$ if $X_1, \dots, X_q$ are sections of $\mathcal{V}$ over Ω. Observe that $\mathfrak{N}^{\mathcal{V}}_q(\Omega) = \mathfrak{N}_q(\Omega)$ if $q > n$ for the sections of $\mathcal{V}$ form, locally, a free C^∞-module of rank n.

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Since $\mathcal{V}$ satisfies, by definition, the Frobenius condition it follows immediately from (VII.4) that

$$\mathrm{d}\mathfrak{N}_q^{\mathcal{V}}(\Omega) \subset \mathfrak{N}_{q+1}^{\mathcal{V}}(\Omega) \tag{VII.13}$$

for every $q \geq 1$. Finally we set

$$\mathfrak{U}_q^{\mathcal{V}}(\Omega) = \mathfrak{N}_q(\Omega)/\mathfrak{N}_q^{\mathcal{V}}(\Omega), \quad q \geq 1. \tag{VII.14}$$

Thanks to (VII.13) the exterior derivative defines a complex of $\mathbb{C}$-linear mappings

$$C^\infty(\Omega) \xrightarrow{\mathrm{d}'} \mathfrak{U}_1^{\mathcal{V}}(\Omega) \xrightarrow{\mathrm{d}'} \dots \xrightarrow{\mathrm{d}'} \mathfrak{U}_q^{\mathcal{V}}(\Omega) \xrightarrow{\mathrm{d}'} \mathfrak{U}_{q+1}^{\mathcal{V}}(\Omega) \xrightarrow{\mathrm{d}'} \dots, \tag{VII.15}$$

which we shall refer to as the *complex associated with $\mathcal{V}$ over Ω*.

VII.5 Localization

If $W \subset \Omega$ is open there is a well-defined complex homomorphism

$$\big(\mathfrak{U}_q^{\mathcal{V}}(\Omega), \mathrm{d}'\big) \longrightarrow \big(\mathfrak{U}_q^{\mathcal{V}}(W), \mathrm{d}'\big)$$

which is induced by restriction.

Let $p \in \Omega$ and consider an open neighborhood W of p over which there are defined m differential forms $\omega_1, \dots, \omega_m \in \mathfrak{N}(W)$ that span $T'|_W$ at every point. After contracting W around p and a linear change on $\omega_1, \dots, \omega_m$, we can obtain a coordinate system $(x_1, \dots, x_m, t_1, \dots, t_n)$ defined on W and centered at p in such a way that

$$\omega_k = \mathrm{d}x_k - \sum_{j=1}^{n} b_{jk}(x,t)\,\mathrm{d}t_j, \quad k = 1, \dots, m,$$

with $b_{jk} \in C^\infty(W)$. Next we introduce the linearly independent vector fields over W

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} b_{jk}(x,t) \frac{\partial}{\partial x_k}.$$

Since $\omega_k(\mathrm{L}_j) = 0$ for all $j = 1, \dots, n$ and $k = 1, \dots, m$ it follows that $L_1, \dots, L_n$ span $\mathcal{V}|_W$ at each point.

Next the $C^\infty(W)$-module $\mathfrak{N}_q(W)$ is spanned by the q-forms

$$\omega_J \wedge \mathrm{d}t_K, \quad |J| + |K| = q$$

and since

$$|J| > 0 \Longrightarrow \omega_J \wedge \mathrm{d}t_K \in \mathfrak{N}_q^{\mathcal{V}}(W)$$

it follows that $\mathfrak{U}_q^{\mathcal{V}}(W)$ can be identified with the submodule of $\mathfrak{N}_q(W)$ spanned by $\{\mathrm{d}t_K;\ |K|=q\}$.

If $f \in C^\infty(W)$ then it is plain that

$$\mathrm{d}f = \sum_{j=1}^{n}(L_j f)\mathrm{d}t_j \quad \mathrm{mod}\,[\omega_1, \dots, \omega_m]$$

since $\mathrm{d}t_j(L_{j'}) = \delta_{jj'}$. From this we obtain the representation of the operator d' under the preceding identification: if $f = \sum_J f_J\, \mathrm{d}t_J \in \mathfrak{U}_q^{\mathcal{V}}(W)$ then

$$\mathrm{d}'f = \sum_{|J|=q} \sum_{j=1}^{n}(L_j f_J)\mathrm{d}t_j \wedge \mathrm{d}t_J. \tag{VII.16}$$

REMARK VII.5.1. Since $\mathcal{V}$ satisfies the Frobenius condition and since furthermore the vector fields $[L_j, L_{j'}]$ do not involve any differentiation in the t-variables it follows that $[L_j, L_{j'}] = 0$ for every $j, j' = 1, \dots, n$. Now it is easily seen that this condition is equivalent to the fact that formula (VII.16) defines a differential complex, i.e., that $\mathrm{d}' \circ \mathrm{d}' = 0$.

VII.6 Germ solvability

In this section we pause to apply some standard functional analytic methods in order to discuss the notion of *exactness in the sense of germs*. The important conclusion is that such a weak notion indeed implies solvability in fixed neighborhoods, and with a bound on the order of the distribution solutions when we are willing to allow even the existence of weak solutions.

Although this is a preparation for all the discussion that will follow, we allow quite general systems of operators.

Let then Ω now denote an open subset of $\mathbb{R}^N$ and let

$$P(x, D) = \{P_{jk}(x, D)\},\ \ Q(x, D) = \{Q_{\ell j}(x, D)\}$$

be matrices of linear partial differential operators (with smooth coefficients) in Ω. We assume $j = 1, \dots, \beta$, $k = 1, \dots, \alpha$, $\ell = 1, \dots, \gamma$ and that

$$C^\infty(\Omega, \mathbb{C}^\alpha) \xrightarrow{P(x,D)} C^\infty(\Omega, \mathbb{C}^\beta) \xrightarrow{Q(x,D)} C^\infty(\Omega, \mathbb{C}^\gamma) \tag{VII.17}$$

defines a differential complex, that is, $Q(x, D)P(x, D) = 0$.

Let $x_0 \in \Omega$. We shall say that (VII.17) *is exact at x_0 (in the sense of germs)* if for every $f \in C^\infty(\Omega, \mathbb{C}^\beta)$ satisfying $Q(x, D)f = 0$ in a neighborhood of x_0 there is $u \in C^\infty(\Omega, \mathbb{C}^\alpha)$ solving $P(x, D)u = f$ in a neighborhood of x_0.

THEOREM VII.6.1. *Suppose that (VII.17) is exact at x_0. Then:*

- *for every open neighborhood U_0 of x_0 in Ω there is another such neighborhood $U_1 \subset\subset U_0$ such that given $f \in C^\infty(U_0, \mathbb{C}^\beta)$ satisfying $Q(x, D)f = 0$ in U_0 there is $u \in C^\infty(U_1, \mathbb{C}^\alpha)$ solving $P(x, D)u = f$ in U_1.*

PROOF. The proof is a well-known category argument due to A. Grothendieck. We fix U_0 and select a fundamental system $\{V_\nu\}_{\nu\in\mathbb{N}}$ of open neighborhoods of x_0, each of them with compact closure in U_0. Set

$$E_\nu = \{(f, v) \in C^\infty(U_0, \mathbb{C}^\beta) \times C^\infty(V_\nu, \mathbb{C}^\alpha) : Q(x, D)f = 0,\ Pv = f \text{ in } V_\nu\}.$$

Each E_ν is a Fréchet space and the linear maps

$$\lambda_\nu : E_\nu \to \{f \in C^\infty(U_0, \mathbb{C}^\beta) : Q(x, D)f = 0\}, \quad \lambda_\nu(f, v) = f$$

are continuous. Now the fact that (VII.17) is exact at x_0 means that

$$\{f \in C^\infty(U_0, \mathbb{C}^\beta) : Q(x, D)f = 0\} = \bigcup_{\nu\in\mathbb{N}} \lambda_\nu (E_\nu)\,.$$

By Baire's category theorem there is ν_0 such that $\lambda_{\nu_0}(E_{\nu_0})$ is of second category in $\{f \in C^\infty(U_0, \mathbb{C}^\beta) : Q(x, D)f = 0\}$ and the open mapping theorem implies that λ_{ν_0} is indeed surjective. This proves the theorem. □

The same argument gives a version of Theorem VII.6.1 where the solutions are now allowed to be distributions.

THEOREM VII.6.2. *Assume that for every $f \in C^\infty(\Omega, \mathbb{C}^\beta)$ satisfying $Q(x, D)f =$ 0 in a neighborhood of x_0 there is $u \in \mathcal{D}'(\Omega, \mathbb{C}^\alpha)$ solving $P(x, D)u = f$ in a neighborhood of x_0. Then the following holds:*

- *for every open neighborhood U_0 of x_0 in Ω there are another such neighborhood $U_1 \subset\subset U_0$ and $p \in \mathbb{N}$ such that given $f \in C^\infty(U_0, \mathbb{C}^\beta)$ satisfying $Q(x, D)f = 0$ in U_0 there is $u \in L^2_{-p}(U_1, \mathbb{C}^\alpha)$ solving $P(x, D)u = f$ in U_1.*

PROOF. It suffices to repeat the argument in the proof of Theorem VII.6.1 with

$$E'_\nu = \{(f, v) \in C^\infty(U_0, \mathbb{C}^\beta) \times L^2_{-\nu}(V_\nu, \mathbb{C}^\alpha) : Q(x, D)f = 0,\ Pv = f \text{ in } V_\nu\}$$

in the place of E_ν. □

VII.7 $\mathcal{V}$-cohomology and local solvability

We now return to our original situation where we are given a formally integrable structure $\mathcal{V} \subset \mathbb{C}T\Omega$ over a smooth manifold Ω.

Given $W \subset \Omega$ open we shall denote by $H^q(W; \mathcal{V})$, $q = 0, 1, \ldots, n$, the cohomology spaces of the complex (VII.15). In other words, we have

$$H^0(W; \mathcal{V}) \doteq \mathrm{Ker}\,\{C^\infty(W) \xrightarrow{\mathrm{d}'} \mathfrak{U}_1^{\mathcal{V}}(W)\} \tag{VII.18}$$

$$H^q(W; \mathcal{V}) \doteq \frac{\mathrm{Ker}\,\{\mathfrak{U}_q^{\mathcal{V}}(W) \xrightarrow{\mathrm{d}'} \mathfrak{U}_{q+1}^{\mathcal{V}}(W)\}}{\mathrm{Im}\,\{\mathfrak{U}_{q-1}^{\mathcal{V}}(W) \xrightarrow{\mathrm{d}'} \mathfrak{U}_q^{\mathcal{V}}(W)\}}, \quad q \geq 1. \tag{VII.19}$$

Notice that $H^0(W, \mathcal{V})$ is the space of all smooth functions u on W such that $\mathrm{d}u$ is a section of $T'|_W$.

Given a point $p \in \Omega$ we shall also introduce the direct limits[1]

$$H^q(p; \mathcal{V}) \doteq \lim_{W \to \{p\}} H^q(W; \mathcal{V}), \quad q \geq 0 \tag{VII.20}$$

and the related definition:

Definition VII.7.1. *We shall say that* d' *is solvable in* $W \subset \Omega$ *open in degree* $q \geq 1$ *if* $H^q(W, \mathcal{V}) = 0$. *We shall further say that* d' *is solvable near* $p \in \Omega$ *in degree* $q \geq 1$ *if* $H^q(p; \mathcal{V}) = 0$.

Take an open neighborhood W of p as in Section VII.5. With the identification described there we see that the spaces $\mathfrak{U}_q^{\mathcal{V}}(U)$, $U \subset W$ open, carry natural topologies of Fréchet spaces. As an immediate consequence of Theorem VII.6.1 we derive:

Proposition VII.7.2. *The operator* d' *is solvable near* $p \in \Omega$ *in degree* $q \geq 1$ *if and only if the following holds:*

- *given an open neighborhood* $U \subset W$ *of* p *there is another such neighborhood* $V \subset U$ *such that for every* $f \in \mathfrak{U}_q^{\mathcal{V}}(U)$ *satisfying* $\mathrm{d}' f = 0$ *there is* $u \in \mathfrak{U}_{q-1}^{\mathcal{V}}(V)$ *satisfying* $\mathrm{d}' u = f$ *in* V.

[1] We recall that for a sheaf of $\mathbb{C}$-vector spaces $U \mapsto F(U)$ over a topological space X, the *direct limit*

$$\lim_{W \to \{p\}} F(W)$$

at $p \in X$ is the space of all pairs (W, f), with W an open subset of X that contains p and $f \in F(W)$, modulo the following equivalence relation: $(W_1, f_1) \sim (W_2, f_2)$ if there is an open neighborhood W of p, $W \subset W_1 \cap W_2$, such that $f_1|_W = f_2|_W$.

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VII.8 The Approximate Poincaré Lemma

Now we assume given a locally integrable structure $\mathcal{V}$ over a smooth manifold Ω. Under this stronger hypothesis a richer description of the differential complex associated with $\mathcal{V}$ can be given. Let $p \in \Omega$ and apply Corollary I.10.2. There is a coordinate system

$$(x_1, \ldots, x_m, t_1, \ldots, t_n)$$

centered at p and there are smooth, real-valued functions $\phi_1, \ldots, \phi_m$ defined in a neighborhood of the origin of $\mathbb{R}^{m+n}$ and satisfying

$$\phi_k(0,0) = 0, \quad \mathrm{d}_x\phi_k(0,0) = 0, \quad k = 1, \ldots, m \tag{VII.21}$$

such that the differentials of the functions

$$Z_k(x,t) = x_k + i\phi_k(x,t), \quad k = 1, \ldots, m \tag{VII.22}$$

span T' near $p = (0,0)$. We shall set

$$Z = (Z_1, \ldots, Z_m), \qquad \phi = (\phi_1, \ldots, \phi_m).$$

Thus we can write

$$Z(x,t) = x + i\phi(x,t),$$

which we assume defined in an open neighborhood of the closure of $B_0 \times \Theta_0$, where $B_0 \subset \mathbb{R}^m$ and $\Theta_0 \subset \mathbb{R}^n$ are open balls centered at the corresponding origins. Thanks to (VII.21) we can assume that

$$|\phi(x,t) - \phi(x',t)| \leq \frac{1}{2}|x - x'|, \ x, x' \in B_0, \ t \in \Theta_0. \tag{VII.23}$$

Also recall that $\mathcal{V}$ is spanned, in an open set that contains the closure of $B_0 \times \Theta_0$, by the linearly independent, pairwise commuting vector fields (*cf.* (I.37))

$$L_j = \frac{\partial}{\partial t_j} - i\sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(x,t) M_k, \quad j = 1, \ldots, n, \tag{VII.24}$$

where the vector fields

$$M_k = \sum_{\ell=1}^{m} \mu_{k\ell}(x,t)\,\frac{\partial}{\partial x_\ell}, \quad k = 1, \ldots, m \tag{VII.25}$$

are characterized by the relations $M_k Z_\ell = \delta_{k\ell}$ (*cf.* (I.35) and (I.36)).

Lemma VII.8.1. *Let the x-projection of the support of $u \in C^\infty(B_0 \times \Theta_0)$ be a compact subset of B_0. Then, for each $j = 1, \ldots, n$,*

$$\frac{\partial}{\partial t_j} \int u(y,t) \det Z_y(y,t)\,\mathrm{d}y = \int \left(L_j u\right)(y,t) \det Z_y(y,t)\,\mathrm{d}y. \tag{VII.26}$$

Proof. In order to prove (VII.26) it suffices to show that, for an arbitrary $\varphi \in C_c^\infty(\Theta_0)$,

$$-\int_{\Theta_0}\left\{\int u(y,t)\det Z_y(y,t)\,\mathrm{d}y\right\}\frac{\partial\varphi}{\partial t_j}(t)\,\mathrm{d}t =$$

$$\int_{\Theta_0}\left\{\int \left(L_j u\right)(y,t)\det Z_y(y,t)\,\mathrm{d}y\right\}\varphi(t)\,\mathrm{d}t.$$

We have $\mathrm{d}Z_1(y,t)\wedge\ldots\wedge\mathrm{d}Z_m(y,t)\wedge\mathrm{d}t = \det Z_y(y,t)\mathrm{d}y\wedge\mathrm{d}t$. Hence

$$\int_{\Theta_0}\left\{\int u(y,t)\det Z_y(y,t)\,\mathrm{d}y\right\}\frac{\partial\varphi}{\partial t_j}\,\mathrm{d}t = \int_{B_0\times\Theta_0}\frac{\partial\varphi}{\partial t_j}(t)u(y,t)$$

$$\mathrm{d}Z_1(y,t)\wedge\ldots\wedge\mathrm{d}Z_m(y,t)\wedge\mathrm{d}t.$$

Using now the Leibniz rule

$$\frac{\partial\varphi}{\partial t_j}u = L_j(\varphi u) - \varphi L_j u$$

the lemma will be proved if we observe that

$$\int_{B_0\times\Theta_0}\left[L_j(\varphi u)\right](y,t)\,\mathrm{d}Z_1(y,t)\wedge\ldots\wedge\mathrm{d}Z_m(y,t)\wedge\mathrm{d}t = 0,$$

a fact that follows from Stokes' theorem in conjunction with the identity

$$\mathrm{d}\left\{\varphi(t)u(y,t)\mathrm{d}Z_1(y,t)\wedge\ldots\wedge\mathrm{d}Z_m(y,t)\wedge\mathrm{d}t_1\wedge\ldots\wedge\widehat{\mathrm{d}t_j}\wedge\ldots\wedge\mathrm{d}t_n\right\}$$

$$= (-1)^{m+j-1}\left[L_j(\varphi u)\right](y,t)\,\mathrm{d}Z_1(y,t)\wedge\ldots\wedge\mathrm{d}Z_m(y,t)\wedge\mathrm{d}t. \qquad \square$$

We now let

$$f(x,t) = \sum_{|J|=q} f_J(x,t)\mathrm{d}t_J \in \mathfrak{U}_q^{\mathcal{V}}(B_0\times\Theta_0) \tag{VII.27}$$

satisfy $\mathrm{d}'f = 0$. Take $\Psi(x)\in C_c^\infty(B_0)$, $\Psi = 1$ in an open ball $B \subset\subset B_0$ also centered at the origin of $\mathbb{R}^m$ and form

$$F_\nu(z,t) = \left(\frac{\nu}{\pi}\right)^{\frac{m}{2}}\int \mathrm{e}^{-\nu[z-Z(y,t)]^2}\Psi(y)f(y,t)\det Z_y(y,t)\,\mathrm{d}y. \tag{VII.28}$$

Notice that F_ν is defined in $\mathbb{C}^m\times\Theta_0$ and is holomorphic in the first variable. Applying Lemma VII.8.1 gives

$$\mathrm{d}_t F_\nu(z,t) = \tag{VII.29}$$

$$\left(\frac{\nu}{\pi}\right)^{\frac{m}{2}}\int \mathrm{e}^{-\nu[z-Z(y,t)]^2}(\mathrm{d}_0'\Psi)(y,t)\wedge f(y,t)\det Z_y(y,t)\,\mathrm{d}y.$$

In (VII.29) the integral is over $B_0\backslash B$, since Ψ is identically equal to one over B. On the other hand, the real part of the exponent equals $-\nu\mathcal{Q}(z,y,t)$, where

$$\mathcal{Q}(z,y,t) = |\Re z - y|^2 - |\Im z - \phi(y,t)|^2.$$

Now, thanks to (VII.23) we have

$$|\phi(y,t)| \le |\phi(0,t)| + \frac{1}{2}|y|$$

and then

$$|\phi(y,t)|^2 \le 2|\phi(0,t)|^2 + \frac{1}{2}|y|^2.$$

Denote by $b > 0$ the radius of B and use the fact that $\phi(0,0) = 0$: there is an open ball $\Theta \subset\subset \Theta_1$, centered at the origin in $\mathbb{R}^n$, such that

$$2|\phi(0,t)|^2 \le \frac{1}{4}b^2, \quad t \in \Theta.$$

If $y \in B_0\backslash B$ and $t \in \Theta$ then we obtain

$$\mathcal{Q}(0,y,t) \ge \frac{1}{2}|y|^2 - 2|\phi(0,t)|^2 \ge \frac{1}{4}b^2$$

and consequently, by continuity we conclude that there are $r > 0$ and $\lambda > 0$ such that

$$(y,t) \in (B_0\backslash B) \times \Theta, \quad |z| < r \implies \mathcal{Q}(z,y,t) \ge \lambda.$$

We can state:

Lemma VII.8.2. *Given $\alpha \in \mathbb{Z}_+^m$, $\beta \in \mathbb{Z}_+^n$ there is a constant $C_{\alpha,\beta} > 0$ such that*

$$|\partial_z^\alpha \partial_t^\beta \mathrm{d}_t F_\nu(z,t)| \le C_{\alpha,\beta}\, \mathrm{e}^{-\lambda\nu}, \quad |z| < r, \quad t \in \Theta. \tag{VII.30}$$

Next we apply the Poincaré Lemma, more precisely the homotopy formula (VII.11) in t-space, with base point $t_0 = 0$, considering z as a parameter:

$$F_\nu(z,t) = \mathrm{d}_t \mathrm{G}(F_\nu)(z,t) + \mathrm{G}(\mathrm{d}_t F_\nu)(z,t). \tag{VII.31}$$

If we use Lemma VII.8.2, a close inspection of the formula that defines the operator G (*cf.* (VII.10)) allows us to state:

Lemma VII.8.3. *Let $\mathcal{U} = \{(z,t) : |z| < r,\ t \in \Theta\}$. Then, for every $\alpha \in \mathbb{Z}_+^m$, $\beta \in \mathbb{Z}_+^n$*

$$\sup_{(z,t)\in\mathcal{U}} |\partial_z^\alpha \partial_t^\beta \mathrm{G}(\mathrm{d}_t F_\nu)(z,t)| \longrightarrow 0 \quad \textit{as} \quad \nu \to \infty. \tag{VII.32}$$

□

Taking into account the fact that

$$F_\nu(Z(x,t),t) \longrightarrow \Psi(x)\, f(x,t) \quad \text{as} \quad \nu\to\infty$$

in the topology of $\mathfrak{U}_q^{\mathcal{V}}(B_0\times\Theta_0)$, we obtain from (VII.31) and (VII.32) the following result:

THEOREM VII.8.4. *Given $B_0\times\Theta_0$ as above, there is $B\times\Theta\subset\subset B_0\times\Theta_0$, where $B\subset B_0$ and $\Theta\subset\Theta_0$ are also open balls centered at the origin in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, such that if f is as in (VII.27) and satisfies $\mathrm{d}'f=0$ then*

$$\{\mathrm{d}_t\mathrm{G}(F_\nu)(z,t)\}_{z=Z(x,t)} \longrightarrow f(x,t) \tag{VII.33}$$

in the topology of $\mathfrak{U}_q^{\mathcal{V}}(B\times\Theta)$. $\square$

We observe that we can write

$$Q_\nu(z,t) \doteq \mathrm{G}(F_\nu)(z,t) = \sum_{|J|=q-1} Q_{J,\nu}(z,t)\,\mathrm{d}t_J,$$

where the coefficients are entire holomorphic in $z\in\mathbb{C}^m$ and smooth in $\mathbb{C}^m\times\Theta_0$; moreover (VII.33) gives

$$\mathrm{d}'\{Q_\nu(Z(x,t),t)\} = (\mathrm{d}_tQ_\nu)\,(Z(x,t),t) \longrightarrow f(x,t)$$

in the topology of $\mathfrak{U}_q^{\mathcal{V}}(B\times\Theta)$. This justifies us referring to the result stated in Theorem VII.8.4 as the *Approximate Poincaré Lemma for the differential complex* d'.

VII.9 One-sided solvability

Let $\mathcal{V}$ be a formally integrable structure over an N-dimensional smooth manifold Ω and let $\Sigma\subset\Omega$ be an embedded submanifold of dimension $N-1$. We assume that Σ is *noncharacteristic with respect to* $\mathcal{V}$, that is

$$T_p^0\cap N^*\Sigma_p = 0, \quad \forall p\in\Sigma. \tag{VII.34}$$

Notice that (VII.34) is equivalent, in this particular situation, to

$$T_p'\cap\mathbb{C}N^*\Sigma_p = 0, \quad \forall p\in\Sigma. \tag{VII.35}$$

Indeed it is clear that (VII.35)$\Rightarrow$(VII.34); on the other hand, suppose that for some $p\in\Sigma$ there is $0\neq\zeta\in T_p'\cap\mathbb{C}N^*\Sigma_p$. Since Σ is one-codimensional it follows that $\mathbb{C}N^*\Sigma_p$ is spanned by one of its nonzero real elements. In particular there are $0\neq\xi\in N^*\Sigma_p$ and $z\in\mathbb{C}$ such that $\zeta=z\xi$ and thus $\xi=z^{-1}\zeta\in N^*\Sigma_p\cap\left(T_p'\cap T_p^*\Omega\right)$, which contradicts (VII.34).

Thanks to (VII.35) it follows that $(\iota_p)^*$ restricted to T'_p is injective and consequently we obtain isomorphisms

$$(\iota_p)^*|_{T'_p} : T'_p \longrightarrow T'\Sigma_p.$$

In particular it follows that $\dim \mathcal{V}(\Sigma)_p = n-1$ for every $p \in \Sigma$. By Proposition I.14.2, we conclude that Σ is compatible with $\mathcal{V}$; thus $\mathcal{V}(\Sigma)$ defines a formally integrable structure over Σ of rank $n-1$. One important situation occurs when Σ is the boundary of a *regular* open subset $\Omega_\bullet \subset \Omega$: this means that the topological boundary of $\Omega_\bullet$ equals Σ and that for each $p \in \Sigma$ there is a coordinate system $(U, \mathbf{x})$ centered at p such that $\mathbf{x}(U \cap \Omega_\bullet) = \mathbf{x}(U) \cap \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}$. Notice that *a fortiori* $\Omega \backslash \overline{\Omega_\bullet}$ is also regular with boundary Σ.

Let $U \subset \Omega$ be an open set such that $U \cap \Sigma \neq \emptyset$. For each $q = 0, 1, \ldots, n$ we shall set

$$\mathfrak{U}_q^{\mathcal{V}}(U \cap \overline{\Omega_\bullet}) = \{f \in \mathfrak{U}_q^{\mathcal{V}}(U \cap \Omega_\bullet) : \exists \tilde{f} \in \mathfrak{U}_q^{\mathcal{V}}(U),\ \tilde{f}|_{U \cap \Omega_\bullet} = f\}.$$

The operator d' induces a differential complex

$$C^\infty(U \cap \overline{\Omega_\bullet}) \xrightarrow{\mathrm{d}'} \mathfrak{U}_1^{\mathcal{V}}(U \cap \overline{\Omega_\bullet}) \xrightarrow{\mathrm{d}'} \ldots \xrightarrow{\mathrm{d}'} \mathfrak{U}_q^{\mathcal{V}}(U \cap \overline{\Omega_\bullet}) \tag{VII.36}$$

$$\xrightarrow{\mathrm{d}'} \mathfrak{U}_{q+1}^{\mathcal{V}}(U \cap \overline{\Omega_\bullet}) \xrightarrow{\mathrm{d}'} \ldots$$

whose cohomology will be denoted by $H^q(U \cap \overline{\Omega_\bullet}; \mathcal{V})$, $q = 0, 1, \ldots, n$. If $p \in \Sigma$ we shall set

$$H^q(p, \Omega_\bullet; \mathcal{V}) \doteq \lim_{U \to \{p\}} H^q(U \cap \Omega_\bullet, \mathcal{V}) \tag{VII.37}$$

$$H^q(p, \overline{\Omega_\bullet}; \mathcal{V}) \doteq \lim_{U \to \{p\}} H^q(U \cap \overline{\Omega_\bullet}; \mathcal{V}). \tag{VII.38}$$

Definition VII.9.1. *Let $1 \leq q \leq n$. We say that d' is solvable near $p \in \Sigma$ in degree q with respect to $\Omega_\bullet$ if $H^q(p, \Omega_\bullet; \mathcal{V}) = 0$. We further say that d' is solvable near $p \in \Sigma$ in degree q with respect to $\overline{\Omega_\bullet}$ if $H^q(p, \overline{\Omega_\bullet}; \mathcal{V}) = 0$.*

The following result is an immediate consequence of the arguments in Section VII.6:

Proposition VII.9.2. *Let $1 \leq q \leq n$ and assume that d' is solvable near $p \in \Sigma$ in degree q with respect to $\Omega_\bullet$ (resp. with respect to $\overline{\Omega_\bullet}$). Then to every open neighborhood U of p in Ω there is another such neighborhood $U' \subset U$ such that the natural homomorphism $H^q(U \cap \Omega_\bullet; \mathcal{V}) \to H^q(U' \cap \Omega_\bullet; \mathcal{V})$ (resp. $H^q(U \cap \overline{\Omega_\bullet}; \mathcal{V}) \to H^q(U' \cap \overline{\Omega_\bullet}; \mathcal{V})$) is trivial.* $\square$

VII.10 Localization near a point at the boundary

Let $p \in \Sigma$, the boundary of a regular open set $\Omega_\bullet \subset \Omega$. We assume that Σ is noncharacteristic with respect to a *locally integrable* structure $\mathcal{V}$ over Ω of rank n. There is a coordinate system $(y_1, \ldots, y_N)$ defined on an open neighborhood U of p and centered at p such that

$$\Sigma \cap U = \{(y_1, \ldots, y_N) : y_N = 0\};$$

$$\Omega_\bullet \cap U = \{(y_1, \ldots, y_N) : y_N > 0\}.$$

Next, after a possible contraction of U around p, we can select first integrals $Z_1^\flat, \ldots, Z_m^\flat \in C^\infty(U)$ whose differentials span $T'|_U$. Thanks to (VII.35) the forms $\mathrm{d}Z_1^\flat(0,0), \ \ldots, \mathrm{d}Z_m^\flat(0,0), \mathrm{d}y_N$ are linearly independent and consequently, after relabeling, we can assume that

$$A \doteq \frac{\partial(Z_1^\flat, \ldots, Z_m^\flat)}{\partial(y_1, \ldots, y_m)}(0)$$

is nonsingular. We then set

$$Z_k = \sum_{r=1}^m A^{kr}\left[Z_r^\flat - Z_r^\flat(0)\right], \quad k = 1, \ldots, m,$$

where (A^{kr}) denotes the inverse of A. We define

$$x_k = \Re Z_k(y), \quad t_j = y_{m+j}, \qquad k = 1, \ldots, m, \ j = 1, \ldots, n. \tag{VII.39}$$

Notice that

$$\frac{\partial Z_k}{\partial y_r}(0) = \delta_{kr}; \tag{VII.40}$$

in particular it follows from (VII.40) that (VII.39) defines a local diffeomorphism in a neighborhood of the origin. In the new variables $(x_1, \ldots, x_m, t_1, \ldots, t_n)$ we have

$$Z_k(x,t) = x_k + i\phi_k(x,t),$$

where the functions ϕ_k are smooth, real-valued and vanish at the origin. Furthermore, we have

$$\phi_k(x,t) = \Im Z_k(y(x,t))$$

and consequently

$$\frac{\partial \phi_k}{\partial x_s} = \sum_{\ell=1}^N \frac{\partial(\Im Z_k)}{\partial y_\ell}(y(x,t))\frac{\partial y_\ell}{\partial x_s}(x,t)$$

which, thanks to (VII.40), implies

$$\frac{\partial \phi_k}{\partial x_s}(0,0) = 0 \quad k, s = 1, \ldots, m.$$

We summarize:

PROPOSITION VII.10.1. *Let $\Omega_\bullet \subset \Omega$, $p \in \Sigma$ and $\mathcal{V}$ as in the beginning of this section. Then there is a coordinate system $(x_1, \ldots, x_m, t_1, \ldots, t_n)$ centered at p and defined in $B_0 \times \Theta_0$, where $B_0 \subset \mathbb{R}^m$ (resp. $\Theta_0 \subset \mathbb{R}^n$) is an open ball centered at the origin of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$) such that*

$$\Sigma \cap (B_0 \times \Theta_0) = \{(x, t) \in B_0 \times \Theta_0 : t_n = 0\};$$

$$\Omega_\bullet \cap (B_0 \times \Theta_0) = \{(x, t) \in B_0 \times \Theta_0 : t_n > 0\}$$

and there are smooth, real-valued functions $\phi_1, \ldots, \phi_m$ defined in $B_0 \times \Theta_0$ satisfying (VII.21) in such a way that the differential of the functions (VII.22) span T' over $B_0 \times \Theta_0$.

VII.11 One-sided approximation

We continue the analysis within the set-up of the last section; in particular we apply the conclusions obtained in Proposition VII.10.1. As usual we shall set $Z = (Z_1, \ldots, Z_m)$, $\phi = (\phi_1, \ldots, \phi_m)$ and thus we can write

$$Z(x, t) = x + i\phi(x, t).$$

After contracting $B_0 \times \Theta_0$ we can assume that ϕ is smooth in an open neighborhood of the closure of $B_0 \times \Theta_0$ and also that (VII.23) holds. The vector fields (VII.24) span $\mathcal{V}|_{B_0 \times \Theta_0}$ and $L_1, \ldots, L_{n-1}$ are tangent to $\Sigma \cap (B_0 \times \Theta_0)$ whereas L_n is transversal to it. Clearly $\mathcal{V}(\Sigma)|_{\Sigma \cap (B_0 \times \Theta_0)}$ is spanned by the restriction of the vector fields $L_1, \ldots, L_{n-1}$ to $\Sigma \cap (B_0 \times \Theta_0)$. We now write $\Theta_0^+ = \{t \in \Theta_0 : t_n > 0\}$ and assume given

$$f(x, t) = \sum_{|J|=q} f_J(x, t)\, \mathrm{d}t_J \in \mathfrak{U}_q^{\mathcal{V}}(B_0 \times \Theta_0^+), \tag{VII.41}$$

where $q \in \{0, 1, \ldots, n\}$ and $\mathrm{d}' f = 0$. We repeat the analysis presented in Section VII.10. We choose Ψ in the same way and define $F_\nu(z, t)$ by formula (VII.28). Notice that now F_ν is defined in $\mathbb{C}^m \times \Theta_0^+$ and holomorphic in the first variable. If we follow with absolutely no changes the argument that precedes Lemma VII.8.2, we reach the following conclusion:

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LEMMA VII.11.1. *Given $\alpha \in \mathbb{Z}_+^m$, $\beta \in \mathbb{Z}_+^n$ and $\varepsilon > 0$ there is a constant $C_{\alpha,\beta,\varepsilon} > 0$ such that*

$$|\partial_z^\alpha \partial_t^\beta \mathrm{d}_t F_\nu(z,t)| \le C_{\alpha,\beta,\varepsilon}\, \mathrm{e}^{-\lambda\nu}, \quad |z| < r, \quad t \in \Theta \cap \Theta_0^+, \quad \varepsilon \le t_n. \tag{VII.42}$$

We now fix $t_0 \in \Theta \cap \Theta_0^+$ and consider the homotopy formulae (VII.11), (VII.12) in t-space, with base point t_0, considering z as a parameter:

$$F_\nu(z,t) = \mathrm{d}_t \mathrm{G}(F_\nu)(z,t) + \mathrm{G}(\mathrm{d}_t F_\nu)(z,t); \tag{VII.43}$$

$$F_\nu(z,t) - F_\nu(z,t_0) = \mathrm{G}(\mathrm{d}_t F_\nu)(z,t). \tag{VII.44}$$

From Lemma VII.11.1 we derive

$$|\partial_z^\alpha \partial_t^\beta \mathrm{G}(\mathrm{d}_t F_\nu)(z,t)| \le C_{\alpha,\beta,\varepsilon}\, \mathrm{e}^{-\lambda\nu}, \quad |z| < r, \quad t \in \Theta \cap \Theta_0^+, \quad \varepsilon \le t_n.$$

Since moreover

$$F_\nu(Z(x,t),t) \longrightarrow \Psi(x)\, f(x,t) \quad \text{as} \quad \nu \to \infty$$

in the topology of $\mathfrak{U}_q^{\mathcal{V}}(B_0 \times \Theta_0^+)$ we obtain, as before:

THEOREM VII.11.2. *Given $B_0 \times \Theta_0$ as above there is $B \times \Theta \subset\subset B_0 \times \Theta_0$, where $B \subset B_0$ and $\Theta \subset \Theta_0$ are also open balls centered at the origin in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, such that if f is as in (VII.41) and satisfies $\mathrm{d}' f = 0$ then*

$$\{\mathrm{d}_t \mathrm{G}(F_\nu)(z,t)\}_{z=Z(x,t)} \longrightarrow f(x,t) \quad \textit{if } q \ge 1; \tag{VII.45}$$

$$F_\nu(Z(x,t),t_0) \longrightarrow f(x,t) \quad \textit{if } q = 0; \tag{VII.46}$$

in the topology of $\mathfrak{U}_q^{\mathcal{V}}(B \times (\Theta \cap \Theta_0^+))$. Moreover, if $f \in \mathfrak{U}_q^{\mathcal{V}}(\overline{\Omega_\bullet} \cap (B_0 \times \Theta_0))$ then the convergence in (VII.45) and (VII.46) occurs in $\mathfrak{U}_q^{\mathcal{V}}(\overline{\Omega_\bullet} \cap (B_0 \times \Theta_0))$.

The only point that remains to verify in the statement of Theorem VII.11.2 is the very last one, and this follows again from an inspection of the argument, observing that the estimates can be obtained uniformly up to $t_n = 0$. Notice also that in this case the base point t_0 can be chosen to be the origin in t-space.

VII.12 A Mayer–Vietoris argument

We continue to work under the following set-up: $\mathcal{V}$ is a locally integrable structure over the smooth manifold Ω, $\Omega_\bullet \subset \Omega$ is a regular open subset of Ω, and the boundary Σ of $\Omega_\bullet$ is noncharacteristic with respect to $\mathcal{V}$. The differential complex on Σ associated with $\mathcal{V}(\Sigma)$ will be denoted by d'_Σ. The next result is one of the main reasons why we introduce such a scheme:

THEOREM VII.12.1. *Let $p \in \Sigma$ and $1 \leq q \leq n-1$.*

(a) *Assume that* d' *is solvable near p in degree $q+1$. If* d' *is solvable near p in degree q with respect to $\overline{\Omega_\bullet}$ and with respect to $\Omega \backslash \Omega_\bullet$, then* d'_Σ *is solvable near p in degree q.*

(b) *Assume that* d' *is solvable near p in degree q. If* d'_Σ *is solvable near p in degree q then* d' *is solvable near p in degree q with respect to $\overline{\Omega_\bullet}$ and with respect to $\Omega \backslash \Omega_\bullet$.*

For the proof we shall first establish some lemmas. We return to the local coordinates and conclusions provided by Proposition VII.10.1. We call attention, in particular, to the properties of the vector fields $L_1, \ldots, L_n$ as described at the beginning of Section VII.11. Recall that $L_1, \ldots, L_{n-1}$ are tangent to Σ and so they have well-defined restrictions to $\Sigma \cap (B_0 \times \Theta_0)$:

$$L_j^0 \doteq L_j|_{t_n=0} .$$

We shall work in an open set of the form $W_0 = B_0 \times (\Theta_0' \times J_0) \subset B_0 \times \Theta_0$, where Θ_0' (resp. J_0) is an open ball (resp. open interval) centered at the origin in $\mathbb{R}^{n-1}$ (resp. $\mathbb{R}$).

Given a smooth function (or even a differential form) g on W_0 the notation $g \sim_\Sigma 0$ will indicate that g vanishes to infinite order on $\Sigma \cap W_0$.

LEMMA VII.12.2. *Given $f \in C^\infty(W_0)$ and $u_0 \in C^\infty(\Sigma \cap W_0)$ there is a solution $u \in C^\infty(W_0)$ to the approximate Cauchy problem*

$$\begin{cases} \mathrm{L}_n u - f & \sim_\Sigma \ 0 \\ \quad u|_{t_n=0} & = \ u_0 . \end{cases} \tag{VII.47}$$

If moreover v is another solution to (VII.47) then $u - v \sim_\Sigma 0$.

PROOF. By the formal Cauchy–Kowalevsky theorem it is possible to solve the Cauchy problem $L_n u = f$, $u|_{t_n=0} = u_0$ uniquely in the ring of formal power series in t_n with coefficients in $C^\infty(B_0 \times \Theta_0')$. If

$$\sum_{j=0}^{\infty} u_j(x, t') t_n^j$$

is such a formal solution we can obtain a solution to (VII.47) by taking

$$u(x, t) = \sum_{j=0}^{\infty} \zeta(\theta_j t_n) u_j(x, t') t_n^j,$$

where $\zeta \in C_c^\infty(\mathbb{R})$, $\zeta(s) = 1$ for $|s| < 1$, $\zeta(s) = 0$ for $|s| > 2$, and (θ_j) is a suitably chosen sequence of real numbers satisfying $\theta_j < \theta_{j+1}$, $\theta_j \to \infty$.

For the uniqueness it suffices to observe that u and v must *a fortiori* have identical formal power series expansions in t_n, whence the assertion. $\square$

LEMMA VII.12.3. *Given $q \geq 0$ and $g \in \mathfrak{U}_q^{\mathcal{V}(\Sigma)}(\Sigma \cap W_0)$ satisfying $\mathrm{d}'_\Sigma g = 0$ there is $G \in \mathfrak{U}_q^{\mathcal{V}}(W_0)$ satisfying $(\iota_\Sigma)^*(G) = g$ and $\mathrm{d}'G \sim_\Sigma 0$.*

PROOF. We write

$$g = \sum_{|I|=q,\, I \subset \{1,\dots,n-1\}} g_I(x, t')\mathrm{d}t_I$$

and apply Lemma VII.12.2 in order to solve, for each I, the approximate Cauchy problems

$$\begin{cases} L_n G_I & \sim_\Sigma & 0 \\ G_I|_{t_n=0} & = & g_I\,. \end{cases} \tag{VII.48}$$

If we set

$$G = \sum_{|I|=q} G_I(x, t)\, \mathrm{d}t_I \in \mathfrak{U}_q^{\mathcal{V}}(W_0)$$

it is clear that, thanks to (VII.48), $(\iota_\Sigma)^* G = g$. We also obtain

$$\mathrm{d}'G = \sum_{|I|=q} \sum_{j=1}^{n-1} L_j G_I(x, t)\mathrm{d}t_j \wedge \mathrm{d}t_I + \sum_{|I|=q} L_n G_I(x, t)\mathrm{d}t_n \wedge \mathrm{d}t_I. \tag{VII.49}$$

The second term on the right in (VII.49) vanishes to infinite order at $\Sigma \cap W_0$ thanks again to (VII.48). On the other hand, since the vector fields L_j are pairwise commuting, we obtain

$$\begin{cases} L_n L_j G_I & \sim_\Sigma & 0 \\ L_j^0 \left(G_I|_{t_n=0}\right) & = & L_j^0 g_I \end{cases}$$

for each $j = 1, \dots, n-1$ and each I. From the uniqueness part in Lemma VII.12.2 together with the fact that $\mathrm{d}'_\Sigma g = 0$ it follows that the first term on the right of (VII.49) also vanishes to infinite order on $\Sigma \cap W_0$. This completes the proof. $\square$

LEMMA VII.12.4. *Let $G \in \mathfrak{U}_q^{\mathcal{V}}(W_0)$ satisfy $(\iota_\Sigma)^*(G) = 0$ and $\mathrm{d}'G \sim_\Sigma 0$. Then:*

(a) If $q = 0$ then $G \sim_\Sigma 0$.
(b) If $q \geq 1$ then there exists $u \in \mathfrak{U}_{q-1}^{\mathcal{V}}(W_0)$ such that $G - \mathrm{d}'u \sim_\Sigma 0$.

PROOF. If $G \in \mathfrak{U}_0^{\mathcal{V}}(W_0)$ is such that $G|_{t_n=0} = 0$ and $\mathrm{d}'G \sim_\Sigma 0$ then in particular we have $L_n G \sim_\Sigma 0$. Consequently we obtain $G \sim_\Sigma 0$ thanks to Lemma VII.12.2.

We now prove (b), whose proof is more involved. We assume $q \geq 1$ and write

$$G = \sum_{|I|=q} G_I(x,t)\,\mathrm{d}t_I = \mathrm{d}t_n \wedge u_1 + \beta_1 = \mathrm{d}'(t_n u_1) + \beta_1 - t_n \mathrm{d}' u_1,$$

where $u_1 \in \mathfrak{U}^{\mathcal{V}}_{q-1}(W_0)$ and $\beta_1 \in \mathfrak{U}^{\mathcal{V}}_q(W_0)$ do not involve $\mathrm{d}t_n$. Since $(\iota_\Sigma)^* G = 0$ we have $G_I|_{t_n=0} = 0$ if $n \notin I$. Consequently, all the coefficients of β_1 vanish when $t_n = 0$ and then we can further write

$$G = \mathrm{d}'(t_n u_1) + t_n h_1, \tag{VII.50}$$

where $h_1 \in \mathfrak{U}^{\mathcal{V}}_q(W_0)$.

We shall construct inductively two sequences $(u_\nu) \subset \mathfrak{U}^{\mathcal{V}}_{q-1}(W_0)$, $(h_\nu) \subset \mathfrak{U}^{\mathcal{V}}_q(W_0)$, where u_ν do not involve $\mathrm{d}t_n$, such that

$$G = \mathrm{d}' \left[\sum_{j=1}^{\nu} \frac{t_n^j}{j} u_j \right] + t_n^\nu h_\nu . \tag{VII.51}$$

Indeed we first observe that (VII.50) gives (VII.51) for $\nu = 1$. We assume then that $u_0, \ldots, u_\nu, h_0, \ldots, h_\nu$ have already been constructed with the required properties and we apply the operator d' to both sides of (VII.51). We obtain

$$\mathrm{d}' G = \nu t_n^{\nu-1}\,\mathrm{d}t_n \wedge h_\nu + t_n^\nu\,\mathrm{d}' h_\nu \tag{VII.52}$$

and then, since $\mathrm{d}'G$ vanishes to infinite order at $t_n = 0$, we conclude that all the coefficients of $\mathrm{d}t_n \wedge h_n u$ vanish at $t_n = 0$. Hence we can write

$$h_\nu = \mathrm{d}t_n \wedge u_{\nu+1} + t_n g_\nu, \tag{VII.53}$$

where $u_{\nu+1} \in \mathfrak{U}^{\mathcal{V}}_{q-1}(W_0)$ and $g_\nu \in \mathfrak{U}^{\mathcal{V}}_q(W_0)$ do not involve $\mathrm{d}t_n$.

Then

$$\begin{aligned}
G - \left[\sum_{j=1}^{\nu+1} \frac{t_n^j}{j} u_\ell \right] &= t_n^\nu h_\nu - \mathrm{d}' \left(\frac{t_n^{\nu+1}}{\nu+1}\, u_{\nu+1} \right) \\
&= t_n^\nu h_\nu - t_n^\nu \mathrm{d}t_n \wedge u_{\nu+1} - \frac{t_n^{\nu+1}}{\nu+1}\,\mathrm{d}' u_{\nu+1} \\
&= t_n^\nu h_\nu + t_n^\nu\,(t_n g_\nu - h_\nu) - \frac{t_n^{\nu+1}}{\nu+1}\,\mathrm{d}' u_{\nu+1} \\
&= t_n^{\nu+1} \left(g_\nu - \frac{1}{\nu+1}\,\mathrm{d}' u_{\nu+1} \right) .
\end{aligned}$$

Defining $h_{\nu+1} = g_\nu - \mathrm{d}' u_{\nu+1}/(\nu+1)$ completes the proof of the inductive argument.

Next we observe that any element $v \in \mathfrak{U}^{\nu}_{q-1}(W_0)$ which does not involve $\mathrm{d}t_n$ can be written as

$$v = v|_{t_n=0} + t_n v_1 ,$$

where $v_1 \in \mathfrak{U}^{\nu}_{q-1}(W_0)$ also does not involve $\mathrm{d}t_n$. Reasoning again by induction we then obtain, from (VII.51), a sequence $(v_\nu) \subset \mathfrak{U}^{\nu(\Sigma)}(W_0)$ such that, for every ν,

$$G = \mathrm{d}'\left[\sum_{j=1}^{\nu} v_j\, t_n^j\right] + \mathrm{O}(t_n^{\nu}). \tag{VII.54}$$

Finally we select $\zeta(s)$ and (θ_j) as in the proof of Lemma VII.12.2 in such a way that

$$\sum_{j=0}^{\infty} \zeta(\theta_j\, t_n) v_j\, t_n^j$$

converges in $\mathfrak{U}^{\nu}_{q-1}(W_0)$. Call $u \in \mathfrak{U}^{\nu}_{q-1}(W_0)$ this sum: for every ν, (VII.54) gives

$$G - \mathrm{d}' u = \mathrm{d}'\left[\sum_{j=1}^{\nu} v_j t_n^j - u\right] + \mathrm{O}(t_n^{\nu}) = \mathrm{O}(t_n^{\nu}) ,$$

which completes the proof. □

Proof of Theorem VII.12.1. In order to shorten the notation it is convenient to work with germs of forms at p. Thus we shall introduce the spaces

$$\mathfrak{U}^{\nu}_q(p) = \lim_{U\to\{p\}} \mathfrak{U}^{\nu}_q(U) ;$$

$$\mathfrak{U}^{\nu(\Sigma)}_q(p) = \lim_{U\to\{p\}} \mathfrak{U}^{\nu}_q(U\cap\Sigma) ;$$

$$\mathfrak{U}^{\nu}_q(p, \overline{\Omega^\bullet}) = \lim_{U\to\{p\}} \mathfrak{U}^{\nu}_q(U\cap\overline{\Omega^\bullet}) ;$$

$$\mathfrak{U}^{\nu}_q(p, \Omega\backslash\Omega^\bullet) = \lim_{U\to\{p\}} \mathfrak{U}^{\nu}_q(U\cap(\Omega\backslash\Omega^\bullet)) .$$

We start by proving (a). Let $\underline{g} \in \mathfrak{U}^{\nu(\Sigma)}_q(p)$ satisfy $\mathrm{d}'_\Sigma \underline{g} = 0$. By Lemma VII.12.3 there is $\underline{f} \in \mathfrak{U}^{\nu}_q(p)$ satisfying $(\iota_\Sigma)^*\underline{f} = \underline{g}$, $\mathrm{d}'\underline{f} \sim_\Sigma 0$. We then define $\mathbf{F} \in \mathfrak{U}^{\nu}_{q+1}(p)$ by the rule

$$\mathbf{F} = \begin{cases} \mathrm{d}'\underline{f} & \text{in } (\overline{\Omega^\bullet}, p) \\ -\mathrm{d}'\underline{f} & \text{in } (\Omega\backslash\Omega^\bullet, p). \end{cases}$$

Then $\mathbf{F} \in \mathfrak{U}^{\nu}_{q+1}(p)$ and $\mathrm{d}'\mathbf{F} = 0$. We now apply our hypothesis: we can find $\underline{f}^\star \in \mathfrak{U}^{\nu}_q(p)$ solving $\mathrm{d}'\underline{f}^\star = \mathbf{F}$ and also $\mathbf{u}^\bullet \in \mathfrak{U}^{\nu}_{q-1}(p, \overline{\Omega^\bullet})$, $\mathbf{u}^{\bullet\bullet} \in \mathfrak{U}^{\nu}_{q-1}(p, \Omega\backslash\Omega^\bullet)$ solving $\mathrm{d}'\mathbf{u}^\bullet = \underline{f} - \underline{f}^\star$ in $(\overline{\Omega^\bullet}, p)$, $\mathrm{d}'\mathbf{u}^{\bullet\bullet} = \underline{f} + \underline{f}^\star$ in $(\Omega\backslash\Omega^\bullet, p)$. We then set

$$\mathbf{u} \doteq \frac{1}{2}\left[(\iota_\Sigma)^*\mathbf{u}^\bullet + (\iota_\Sigma)^*\mathbf{u}^{\bullet\bullet}\right].$$

We obtain

$$\begin{aligned} \mathrm{d}'_\Sigma \mathbf{u} &= \frac{1}{2}\left[(\iota_\Sigma)^*\mathrm{d}'\mathbf{u}^\bullet + (\iota_\Sigma)^*\mathrm{d}'\mathbf{u}^{\bullet\bullet}\right] \\ &= \frac{1}{2}\left[(\iota_\Sigma)^*(\underline{f} - \underline{f}^\star) + (\iota_\Sigma)^*(\underline{f} + \underline{f}^\star)\right] \\ &= (\iota_\Sigma)^*\underline{f} \\ &= \underline{g}\,. \end{aligned}$$

Next we prove (b). We shall prove that d' is solvable near p in degree q with respect to $\Omega_\bullet$, the other case being analogous. Let then $\underline{f} \in \mathfrak{U}^{\mathcal{V}}_q(p, \overline{\Omega^\bullet})$ satisfy $\mathrm{d}'\underline{f} = 0$. We can of course assume that $\underline{f}$ has been extended to a germ in $\mathfrak{U}^{\mathcal{V}}_q(p)$ (which in general will no longer be d'-closed). Let $\mathbf{v} \in \mathfrak{U}^{\mathcal{V}(\Sigma)}_{q-1}(p)$ solve $\mathrm{d}'_\Sigma \mathbf{v} = (\iota_\Sigma)^*\underline{f}$. If $\mathbf{V} \in \mathfrak{U}^{\mathcal{V}}_{q-1}(p)$ is such that $(\iota_\Sigma)^*\mathbf{V} = \mathbf{v}$ then $\mathbf{F} \doteq \underline{f} - \mathrm{d}'\mathbf{V}$ satisfies $(\iota_\Sigma)^*\mathbf{F} = 0$ and $\mathrm{d}'\mathbf{F} \sim_\Sigma 0$. By Lemma VII.12.4 there is $\mathbf{u} \in \mathfrak{U}^{\mathcal{V}}_{q-1}(p)$ such that $\mathbf{F} - \mathrm{d}'\mathbf{u} \sim_\Sigma 0$, that is

$$\underline{f} - \mathrm{d}'(\mathbf{u} + \mathbf{V}) \sim_\Sigma 0\,. \tag{VII.55}$$

Define

$$\mathbf{G} = \begin{cases} \underline{f} - \mathrm{d}'(\mathbf{u} + \mathbf{V}) & \text{in } (\overline{\Omega^\bullet}, p) \\ 0 & \text{in } (\Omega \backslash \Omega^\bullet, p). \end{cases}$$

Then $\mathbf{G} \in \mathfrak{U}^{\mathcal{V}}_q(p)$ thanks to (VII.55) and $\mathrm{d}'\mathbf{G} = 0$. By hypothesis we can solve $\mathrm{d}'\mathbf{h} = \mathbf{G}$ for some $\mathbf{h} \in \mathfrak{U}^{\mathcal{V}}_{q-1}(p)$. It follows finally that

$$\mathrm{d}'[\mathbf{u} + \mathbf{V} + \mathbf{h}] = \underline{f} \quad \text{in} \quad (\overline{\Omega^\bullet}, p).$$

The proof of Theorem VII.12.1 is complete. □

VII.13 Local solvability versus local integrability

We conclude the chapter by presenting a natural generalization of Proposition I.13.6 for locally integrable structures of corank one. Thus we assume that $\mathcal{V}$ is a formally integrable structure of rank n over a smooth manifold of dimension $n+1$. Fix $p \in \Omega$ and take an open neighborhood W of p and $\omega \in \mathfrak{N}(W)$ which spans $T'|_W$. As in Section VII.4 we can assume that W is the domain of a coordinate system $(x, t_1, \ldots, t_n)$ centered at p and that ω can be written as

$$\omega = \mathrm{d}x - \sum_{j=1}^{n} b_j(x,t)\,\mathrm{d}t_j\,,$$

where $b_j \in C^\infty(W)$. The linearly independent vector fields

$$L_j = \frac{\partial}{\partial t_j} + b_j(x,t)\frac{\partial}{\partial x}$$

span $\mathcal{V}|_W$ and are pairwise commuting. Since furthermore

$$[L_j, L_{j'}] = \left\{L_j b_{j'} - L_{j'} b_j\right\} \frac{\partial}{\partial x}$$

it follows that

$$0 = \frac{\partial}{\partial x}\left\{L_j b_{j'} - L_{j'} b_j\right\} = L_j\left\{\frac{\partial b_{j'}}{\partial x}\right\} - L_{j'}\left\{\frac{\partial b_j}{\partial x}\right\}$$

and consequently

$$f_0 \doteq -\sum_{j=1}^{n} \frac{\partial b_j}{\partial x}(x,t)\,\mathrm{d}t_j \in \mathfrak{U}_1^{\mathcal{V}}(W) \tag{VII.56}$$

is d'-closed.

THEOREM VII.13.1. *The following properties are equivalent:*

(i) There is $Z \in C^\infty$ in some neighborhood of the origin solving $\mathrm{d}'Z = 0$ and satisfying $Z_x \neq 0$.
(ii) There is $u \in C^\infty$ in some neighborhood of the origin solving $\mathrm{d}'u = f_0$.

In other words, the structure will be locally integrable near p if the class of f_0 in $H^1(p, \mathcal{V})$ vanishes.

PROOF. Assume that (i) holds. If we differentiate the identity

$$Z_{t_j} + b_j Z_x = 0$$

with respect to x we obtain

$$(Z_x)_{t_j} + b_j (Z_x)_x = -(b_j)_x Z_x$$

which gives

$$L_j\{\log Z_x\} = -(b_j)_x\,.$$

Thus

$$\mathrm{d}'\{-\log Z_x\} = f_0,$$

which proves (ii).

Conversely, given u as in (ii) we take

$$G(x,t) = \int_0^x e^{u(y,t)}\,dy.$$

Then

$$\begin{aligned}
L_j G(x,t) &= b_j(x,t)e^{u(x,t)} + \int_0^x u_{t_j}(y,t)e^{u(y,t)}dy \\
&= b_j(x,t)e^{u(x,t)} - \int_0^x [b_j(y,t)u_y(y,t) + (b_j)_y(y,t)]e^{u(y,t)}dy \\
&= b_j(x,t)e^{u(x,t)} - \int_0^x [b_j(y,t)e^{u(y,t)}]_y dy \\
&= b_j(0,t)e^{u(0,t)}.
\end{aligned}$$

If we set

$$B(t) = \sum_{j=1}^n b_j(0,t)e^{u(0,t)}dt_j$$

then $d'G = B$ and consequently, in particular, $d'B = d_t B = 0$. By the Poincaré Lemma we can find $F(t)$ smooth near the origin such that $d_t F = B$ and then if we set $Z(x,t) = G(x,t) - F(t)$ we obtain $Z_x = \exp\{u\} \neq 0$ and $d'Z = 0$. $\square$

Notes

The differential complex associated with a formally integrable structure, first presented in [**T4**], is the natural generalization of the de Rham, Dolbeault, and tangential Cauchy–Riemann complexes, associated respectively with real, complex, and CR structures.

Much of the material of this chapter is preparatory for Chapter VIII, and we should just point out that the Approximate Poincaré Lemma is due to Treves ([**T4**]), whereas Theorem VII.12.1 is a consequence of the existence of a natural Mayer–Vietoris sequence, whose existence for hypersurfaces in the complex space was first proved in [**AH1**].