1

Maurer–Cartan Methods

The purpose of this chapter is to give a rather exhaustive survey on the Maurer–Cartan equation and its related methods, which lie at the core of the present monograph. We first give a recollection of the Maurer–Cartan equation and its gauge symmetries in differential geometry. This chapter is viewed as a motivation for the rest of the book, which consists of higher algebraic generalisations of the key notions of gauge theory. Reading it is not mandatory to understand what follows, but this might help the reader to get some concrete pictures in mind before passing to a more abstract treatment. Then, we establish the general theory of the Maurer–Cartan equation in differential graded Lie algebras. With that in hand, we discuss the philosophy of deformation theory suggesting that studying Maurer–Cartan elements of differential graded Lie algebras, as well as the symmetries of those elements, is the central question of *any* deformation theory problem in characteristic 0. In Chapter 7, we shall discuss more recent developments making that philosophy precise by means of higher category theory.

Throughout this chapter, various infinite series arise. For simplicity, we work with the strong assumption that the various differential graded Lie algebras are nilpotent, so that all these series are actually finite once evaluated on elements. We refer to the treatment of complete algebras given in Chapter 2 for the correct set-up in which convergence is understood in the rest of the text.

1.1 Maurer–Cartan Equation in Differential Geometry

In this section, we give a short outline of the differential geometric notions of which the subject of this book is a far-reaching algebraic generalisation. We review the fundamental objects and the classical results of gauge theory: vector and principal bundles, connections, and curvatures. Nowadays, these notions play a key role in analysis, geometry, and topology [5, 40, 116, 119]; they also provide physicists with the suitable conceptual language to express modern theories [67, 103, 118].

Throughout this section, we work over the field of real numbers and we denote by M a smooth manifold. We assume the reader is familiar with the basic notions of differential manifolds, as treated in [150], for instance. For more details on this section, we refer the reader to the textbooks [79, 144].

Given a smooth vector bundle $E \to M$, one considers the space

$$\Omega^\bullet(M,E) \coloneqq \Gamma\left(\Lambda^\bullet T^*M \otimes E\right) \cong \Omega^\bullet(M) \otimes_{\Omega^0(M)} \Gamma(E)$$

of *differential forms with values in E*. In order to extend the de Rham differential map to E-valued differential forms, one is led to the following notion.

Definition 1.1 (Connection) A *connection* of a smooth vector bundle $E \to M$ is an $\mathbb{R}$-linear map

$$\nabla\colon \Omega^0(M,E) \cong \Gamma(E) \to \Omega^1(M,E) \cong \Gamma(T^*M \otimes E)$$

satisfying the *Leibniz rule*

$$\nabla(fs) = df \otimes s + f\nabla s,$$

for all $f \in \Omega^0(M)$ and all $s \in \Gamma(E)$.

Lemma 1.2 *For any connection ∇, there is a unique $\mathbb{R}$-linear operator*

$$d^\nabla : \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E)$$

satisfying

(i) $d^\nabla = \nabla$, *for* $\bullet = 0$,
(ii) *and the generalised Leibniz rule*

$$d^\nabla(\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^k \alpha \wedge d^\nabla \omega, \tag{1.1}$$

for any $\alpha \in \Omega^k(M)$ and any $\omega \in \Omega^l(M,E)$.

Proof It is given by the following definition

$$d^\nabla(\alpha \otimes s) \coloneqq d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s,$$

for all $\alpha \in \Omega^k(M)$ and all $s \in \Gamma(E)$. ◻

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Let us now look for a condition implying that d^∇ is a differential, that is squares to 0. In this direction, we first consider the composite

$$d^\nabla \circ \nabla \colon \Omega^0(M, E) \to \Omega^2(M, E),$$

which is actually $\Omega^0(M)$-linear.

Definition 1.3 (Curvature) The *curvature* of a connection ∇ is the $\mathrm{End}(E)$-valued 2-form

$$\theta \in \Omega^2(M, \mathrm{End}(E))$$

obtained from $d^\nabla \circ \nabla$ under the isomorphism

$$\mathrm{Hom}_{\Omega^0(M)}\left(\Omega^0(M, E), \Omega^2(M, E)\right) \cong \Omega^2(M, \mathrm{End}(E)).$$

Definition 1.4 (Flat connection) A connection ∇ is called *flat* when its curvature is trivial, that is $\theta = 0$.

This condition is necessary to get a differential; it is actually enough.

Lemma 1.5 *The composite* $d^\nabla \circ d^\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+2}(M, E)$ *is equal to*

$$\left(d^\nabla \circ d^\nabla\right)(\alpha \otimes s) = \alpha \wedge \left(d^\nabla \circ \nabla\right)(s),$$

for $\alpha \in \Omega^\bullet(M)$ *and* $s \in \Gamma(E)$.

Proof The generalised Leibniz rule (1.1) gives

$$\left(d^\nabla \circ d^\nabla\right)(\alpha \otimes s) = d^\nabla\left(d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s\right) = \alpha \wedge \left(d^\nabla \circ \nabla\right)(s),$$

where k stands for the degree of α. □

Proposition 1.6 *For any flat connection* ∇, *the operator* d^∇ *squares to 0.*

Proof This is a direct corollary of Lemma 1.5. □

Definition 1.7 (Twisted de Rham differential/complex) The differential d^∇ on the space of E-valued differential forms associated to a flat connection ∇ is called the *twisted de Rham differential*. The cochain complex $\left(\Omega^\bullet(M, E), d^\nabla\right)$ is called the *twisted de Rham complex.*

Remark 1.8 One recovers the classical de Rham differential on $\Omega^\bullet(M)$ by considering the trivial line bundle.

Let us now look at the local situation. Using the local trivialisation of the vector bundle $\pi : E \to M$ above an open subset $U \subset M$, any basis $(v_1, \ldots, v_n)$ of the typical (finite dimensional) fibre V induces a collection $e = (e_1, \ldots, e_n)$ of sections, with $e_i \in \Gamma(E|_U)$, such that $(e_1(x), \ldots, e_n(x))$ is a basis of the fibre

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$E_x := \pi^{-1}(x)$. Such a collection is called a *local frame over* U. In such a local frame, the data of the connection ∇ is equivalent to a collection of local 1-forms $\omega_{ij} \in \Omega^1(U)$ such that

$$\nabla e_j = \sum_{i=1}^{n} \omega_{ij} \otimes e_i.$$

Definition 1.9 (Local connection form) The *local connection form* with respect to the frame $e = (e_1, \ldots, e_n)$ is the matrix $\omega_e := \left(\omega_{ij}\right)_{i,j=1,\ldots,n} \in \mathrm{gl}_n(\Omega^1(U))$.

Proposition 1.10 *The curvature is given locally by*

$$\theta_e \cong d\omega_e + \omega_e^2 = d\omega_e + \tfrac{1}{2}[\omega_e, \omega_e]$$

in $\mathrm{gl}_n(\Omega^2(U))$.

Proof It is obtained by the following straightforward computation:

$$\begin{aligned} d^\nabla \circ \nabla(e_i) &= \sum_{i=1}^{n} d^\nabla(\omega_{ij} \otimes e_i) = \sum_{i=1}^{n} d\omega_{ij} \otimes e_i - \sum_{i=1}^{n} \omega_{ij} \wedge \nabla e_i \\ &= \sum_{i=1}^{n} \left(d\omega_{ij} - \sum_{k=1}^{n} \omega_{kj} \wedge \omega_{ik} \right) \otimes e_j = \sum_{i=1}^{n} \left(d\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} \right) \otimes e_i. \end{aligned}$$

□

In other words, the twisted map d^∇ is a differential if and only the local connection forms satisfy the following first kind of *'Maurer–Cartan equation'*

$$d\omega_e + \omega_e \cdot \omega_e = d\omega_e + \tfrac{1}{2}[\omega_e, \omega_e] = 0.$$

Such an equation does not depend on the choice of local frames, as the property for a connection to be flat is global. Explicitly, a *change of local frame* over U from $e = (e_1, \ldots, e_n)$ to $e' = (e'_1, \ldots, e'_n)$ is an invertible matrix $A \in \mathrm{GL}_n(\Omega^0(U))$ such that $e' = eA$.

Proposition 1.11 *The local connection form with respect to the frame* e' *is given by the matrix*

$$\omega_{e'} = A^{-1}dA + A^{-1}\omega_e A$$

in $\mathrm{gl}_n(\Omega^1(U))$.

Proof This follows from the straightforward computation:

$$\nabla(e'_j) = \sum_{k=1}^{n} \nabla\left(A_{kj}\, e_k\right) = \sum_{k=1}^{n}\left(dA_{kj} \otimes e_k + A_{kj}\nabla e_k\right)$$
$$= \sum_{i=1}^{n}\left(\sum_{k=1}^{n} A^{-1}_{ik}\left(dA_{kj}\right) + \sum_{k,l=1}^{n} A^{-1}_{il}\,\omega_{lk}\,A_{kj}\right)e'_i.$$

□

This is the first instance of *'gauge group action'* on solutions to the Maurer–Cartan equation.

Any vector bundle E induces a linear dual bundle E^* and an endomorphism bundle $\mathrm{End}(E)$. In turn, any connection ∇ on E gives rise to canonical connections on E^* and $\mathrm{End}(E)$ as follows. Let us first recall the nondegenerate pairing

$$(\ ,\ ) : \Omega^i(M,E) \otimes \Omega^j(M,E^*) \xrightarrow{\wedge} \Omega^{i+j}(M) \otimes_{\Omega^0(M)} \Gamma(E \otimes E^*) \xrightarrow{\langle\,,\rangle} \Omega^{i+j}(M),$$

where $\langle\,,\rangle$ stands for the linear paring, that is

$$(\alpha \otimes s, \beta \otimes t) \coloneqq (\alpha \wedge \beta) \otimes \langle s, t\rangle,$$

for $\alpha \in \Omega^i(M)$, $\beta \in \Omega^j(M)$, $s \in \Gamma(E)$, and $t \in \Gamma(E^*)$. To any connection ∇ on E, one associates a connection ∇^* on E^* characterised by

$$(s, \nabla^* t) = d(s,t) - (\nabla s, t)$$

and then a connection $\widehat{\nabla}$ on $\mathrm{End}(E) \cong E \otimes E^*$ given by

$$\widehat{\nabla}(s \otimes t) \coloneqq \nabla s \otimes t + s \otimes \nabla^* t.$$

Proposition 1.12 *The twisted de Rham differential on the endomorphism bundle* $\mathrm{End}(E)$ *is locally given by*

$$d^{\widehat{\nabla}} f = df + \omega_e f - (-1)^k f \omega_e = df + [\omega_e, f]$$

in $\mathrm{gl}_n(\Omega^{k+1}(U))$, *for any* $f \in \mathrm{gl}_n(\Omega^k(U))$.

Proof Using the local frame over U, one can write

$$f = \sum_{i,j=1}^{n} f_{ij} \otimes \delta_{ij},$$

where $\delta_{ij} = e_i \otimes e^*_j \in \Gamma(\mathrm{End}(E)|_U)$ sends e_j to e_i and where $f_{ij} \in \Omega^k(U)$. By definition, we have

$$\widehat{\nabla}(\delta_{ij}) = \widehat{\nabla}(e_i \otimes e_j^*) = \nabla e_i \otimes e_j^* + e_i \otimes \nabla^* e_j^*$$
$$= \left(\sum_{l=1}^{n} \omega_{li} \otimes e_l\right) \otimes e_j^* - e_i \otimes \left(\sum_{l=1}^{n} \omega_{jl} \otimes e_l^*\right) = \sum_{l=1}^{n} \omega_{li} \otimes \delta_{lj} - \sum_{l=1}^{n} \omega_{jl} \otimes \delta_{il}.$$

The generalised Leibniz rule (1.1) gives

$$d^{\widehat{\nabla}} f = d^{\widehat{\nabla}} \left(\sum_{i,j=1}^{n} f_{ij} \otimes \delta_{ij}\right) = \sum_{i,j=1}^{n} df_{ij} \otimes \delta_{ij} + (-1)^k \sum_{i,j=1}^{n} f_{ij} \otimes \widehat{\nabla}(\delta_{ij})$$
$$= \sum_{i,j=1}^{n} df_{ij} \otimes \delta_{ij} + (-1)^k \sum_{i,j,l=1}^{n} f_{ij} \wedge \omega_{li} \otimes \delta_{lj} - (-1)^k \sum_{i,j,l=1}^{n} f_{ij} \wedge \omega_{jl} \otimes \delta_{il}$$
$$= \sum_{i,j=1}^{n} \left(df_{ij} + \sum_{l=1}^{n} \omega_{il} \wedge f_{lj} - (-1)^k \sum_{l=1}^{n} f_{il} \wedge \omega_{lj}\right) \otimes \delta_{ij}.$$

□

For the first time, we encounter a *'differential twisted by a solution to the Maurer–Cartan equation'*.

Let us now pass from the local picture to the global one. To this extend, one needs an extra action of a Lie group of the fibre bundle, leading to the notion of a principal bundle. Developing the notions of connection and curvature at this level will make even more noticeable the role played by the methods from Lie theory. The two theories of vector and principal bundles are essential equivalent: any vector bundle induces a canonical principal bundle, called the frame bundle, and one can associate vector bundles, like the adjoint bundle, to any principal bundle. Let G be a real Lie group with Lie algebra $\mathfrak{g}$.

Definition 1.13 (Principal bundle) A *principal G-bundle* is a fibre bundle $P \to M$ equipped with a smooth (right) action of G, which is free, transitive, and fibre preserving.

The definition implies the identifications $P_x \cong G$, for the fibres, and $P/G \cong M$, for the orbit space.

Example 1.14 The toy model of principal bundle is the *frame bundle* $\mathrm{Fr}(E) \to M$ associated to any vector bundle $E \to M$: elements of its fibres are ordered based on the fibres of E. In this case, the structure Lie group $G = \mathrm{GL}_n$ is the general linear group, where n is the dimension of the fibres of E.

In differential geometry, a *distribution* is a subbundle of the tangent bundle.

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Definition 1.15 (Vertical distribution) The *vertical distribution* $T^vP \subset TP$ of a principal bundle P is defined by

$$T^v_pP := \ker \mathrm{D}_p\,\pi, \quad \text{for any } p \in P,$$

where $\mathrm{D}_p\,\pi \colon T_pP \to T_{\pi(p)}M$ stands for the derivative of the structural projection $\pi \colon P \to M$.

One can see that each fibre of the vertical distribution is isomorphic to the tangent Lie algebra $T^v_pP \cong \mathfrak{g}$.

Definition 1.16 (Horizontal distribution) A *horizontal distribution* $T^hP \subset TP$ of a principal bundle P is a distribution complementary to the vertical distribution:

$$T^vP \oplus T^hP = TP.$$

Notice the discrepancy between these two notions: the vertical distribution is uniquely and canonically defined while there exists possibly many horizontal distributions. Any horizontal distribution gives rise to a $\mathfrak{g}$-valued 1-form ω on P defined by

$$\omega_p \colon T_pP = T^v_pP \oplus T^h_pP \xrightarrow{\text{proj}} T^v_pP \cong \mathfrak{g},$$

for any $p \in P$, where the first map is the projection on T^v_pP along T^h_pP. In order to make explicit its properties, we need first to recall the following notions.

Let us denote by $\mathrm{R} \colon P \times G \to P$ the right action of G on the principal bundle P. (We will use the simpler notation $\mathrm{R}_g(-) := \mathrm{R}(-, g)$ for the right action of an element $g \in G$.) The *fundamental vector field X^ξ associated to $\xi \in \mathfrak{g}$* is defined by

$$X^\xi_p := \mathrm{D}_{(p,e)}\,\mathrm{R}(0, \xi) \in T^v_pP,$$

for any $p \in P$. The *adjoint representation*

$$\mathrm{Ad}_g := \mathrm{D}_e\mathrm{C}_g \colon \mathfrak{g} \to \mathfrak{g}$$

is given by the derivation at the unit e of the Lie group G of the conjugation map $\mathrm{C}_g(x) := gxg^{-1}$, for any $g, x \in G$.

Definition 1.17 (Connection on a principal bundle) A *connection* on a principal bundle P is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g}) = \Omega^1(P) \otimes \mathfrak{g}$ satisfying the following properties:

VERTICAL VECTOR FIELD: $\omega_p\left(X^\xi_p\right) = \xi$, for any $p \in P$ and $\xi \in \mathfrak{g}$,

EQUIVARIANCE: $\mathrm{Ad}_g\left(\mathrm{R}^*_g\,\omega\right) = \omega$, for any $g \in G$.

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The first conditions says that a connection restricts to the identity map of $\mathfrak{g}$ under the identification $T^v_p P \cong \mathfrak{g}$.

Proposition 1.18 *The data of a horizontal distribution on a principal bundle is equivalent to the data of a connection.*

Proof The left-to-right assignment is defined above. In the other way round, given a connection ω, one defines a horizontal distribution as its kernel $T^h_p P \coloneqq \ker \omega_p$. We refer the reader to [144, Section 28] for complete details about this proof. □

Example 1.19

(i) The *Maurer–Cartan connection* $\omega_G \in \Omega^1(G, \mathfrak{g})$ on the trivial principal bundle $G \to *$ is defined by

$$\omega_G \coloneqq \mathrm{D}_g \mathrm{L}_{g^{-1}} : T_g G \to T_e G \cong \mathfrak{g},$$

where $\mathrm{L}_{g^{-1}} : G \to G$ is the left multiplication by g^{-1}, for $g \in G$.

(ii) Given a connection ∇ on a vector bundle $E \to M$, there is a canonical way [144, Section 29] to define a connection ω on the associated frame bundle $\mathrm{Fr}(E) \to M$ such that the pullback along a local frame $e \colon U \to \mathrm{Fr}(E)|_U$ gives back the local connection form: $e^*\omega = \omega_e$.

The graded vector space $\Omega^\bullet(P, \mathfrak{g}) \coloneqq \Omega^\bullet(P) \otimes \mathfrak{g}$ of $\mathfrak{g}$-valued differential forms on P acquires a canonical differential graded Lie algebra structure (Definition 1.40) as the tensor product of a differential graded commutative algebra with a Lie algebra. This is the relevant algebraic context to express the properties of connections on principal bundle.

Definition 1.20 (Curvature) The *curvature* of a connection ω on a principal bundle is the $\mathfrak{g}$-valued 2-form defined by

$$\Omega \coloneqq d\omega + \tfrac{1}{2}[\omega, \omega].$$

Proposition 1.21 *For any connection ∇ on a vector bundle $E \to M$, the pullback along a local frame $e \colon U \to \mathrm{Fr}(E)|_U$ of the induced curvature of the frame bundle $\mathrm{Fr}(E) \to M$ is equal to the local curvature form:*

$$e^*\Omega = \theta_e.$$

Proof It follows from the direct computation

$$e^*\Omega = e^*(d\omega) + \tfrac{1}{2} e^*[\omega, \omega] = d(e^*\omega) + \tfrac{1}{2}[e^*\omega, e^*\omega] = d\omega_e + \tfrac{1}{2}[\omega_e, \omega_e] = \theta_e,$$

since the pullback preserves the differential and the Lie bracket (second equality), since $e^*\omega = \omega_e$ (third equality), and by Proposition 1.10 (forth equality). □

Any connection on a principal bundle induces a decomposition $TP = T^vP \oplus T^hP$ of the tangent bundle by Proposition 1.18. The associated vertical and horizontal components of a vector field $X \in \Gamma(TP)$ are, respectively, denoted by $X = X^v + X^h$.

Theorem 1.22 *The curvature associated to any connection on a principal bundle satisfies the following properties:*

HORIZONTALITY: $\Omega(X,Y) = d\omega\left(X^h, Y^h\right)$, *for any* $X, Y \in \Gamma(TP)$,
EQUIVARIANCE: $\mathrm{Ad}_g\left(\mathrm{R}_g^*\,\Omega\right) = \Omega$, *for any* $g \in G$,
BIANCHI IDENTITY: $d\Omega = [\Omega, \omega]$.

Proof The proof of the first point is a direct consequence of the definition of the curvature and the fact that the connection ω vanishes on horizontal vectors. The second point is showed by the same type of computation as in the proof of the above Proposition 1.21. The third point is obtained by the following ‘differential graded Lie’ type computation

$$d\Omega = \tfrac{1}{2}d[\omega,\omega] = [d\omega,\omega] = \left[\Omega - \tfrac{1}{2}[\omega,\omega],\omega\right] = [\Omega,\omega],$$

since the differential is a derivation (second equality) and by the Jacobi identity (last equality). □

Definition 1.23 (Flat connection) A connection ω on a principal bundle is called *flat* when its curvature is trivial:

$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega] = 0.$$

This is the global form of the Maurer–Cartan equation mentioned above. Geometrically, this property is equivalent to the integrability of the horizontal vector fields.

Theorem 1.24 *A connection on a principal bundle is flat if and only if the horizontal vector fields are preserved by the Lie bracket, that is*

$$\left[X^h, Y^h\right] = [X,Y]^h, \quad \textit{for any } X, Y \in \Gamma(TP).$$

Proof This is a straightforward consequence of the formula

$$\Omega(X,Y) = -\omega\left([X,Y]\right),$$

for horizontal vector fields $X, Y \in \Gamma\left(T^hP\right)$. We refer the reader to [117, Section 3.1] for complete details. □

The following construction allows us to come back to vector bundles.

Definition 1.25 (Associated vector bundle) The *vector bundle associated to a principal bundle P and to a (finite-dimensional) representation* $\rho\colon G \to \mathrm{GL}(V)$ is defined by the coequaliser $P \times_\rho V$, that is the quotient of $P \times V$ under the equivalence relation $(p \cdot g, v) \sim (p, \rho_g(v))$, for any $p \in P$, $v \in V$, and $g \in G$.

It is straightforward to check that the associated bundle $P \times_\rho V$ defines a vector bundle over M with fibres isomorphic to V.

Example 1.26

(i) The vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ associated to the adjoint representation $\mathrm{Ad}\colon \mathrm{G} \to \mathrm{GL}(\mathfrak{g})$ is called the *adjoint bundle*.

(ii) The vector bundle associated to the frame bundle $\mathrm{Fr}(E)$ and the identity representation is isomorphic to the original vector bundle E. This shows that any vector bundle is a vector bundle associated to a principal bundle.

The V-valued differential forms on P, that is $\Omega^\bullet(P, V) \coloneqq \Omega^\bullet(P) \otimes V$, coincide with the differential forms on P with values in the trivial vector bundle $P \times V$, that is $\Omega^\bullet(P, P \times V)$ under the notation introduced at the beginning of this section. Let us now make explicit the differential forms on M with values in the associated bundle $E \coloneqq P \times_\rho V$ in terms of the V-valued differential forms on P.

Definition 1.27 (Tensorial differential forms) A V-valued differential form $\alpha \in \Omega^k(P, V)$ on a principal bundle P is called *tensorial* when it is:

HORIZONTAL: for any $p \in P$, we have $\alpha_p(u_1, \ldots, u_k) = 0$, when at least one tangent vector is vertical, that is $u_i \in T_p^v P$, for some $1 \leqslant i \leqslant k$,

EQUIVARIANT: $\rho_g\left(\mathrm{R}_g^* \alpha\right) = \alpha$, for any $g \in G$.

We denote the graded vector space of tensorial differential forms by $\Omega_\rho^\bullet(P, V)$.

Example 1.28

(i) Theorem 1.22 shows that the curvature of a connection on a principal bundle is tensorial with respect to the adjoint representation, that is $\Omega \in \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g})$.

(ii) The set of connections on a principal bundle forms an affine space with $\Omega^1_{\mathrm{Ad}}(P, \mathfrak{g})$ as associated vector space.

Proposition 1.29 *The graded vector space of tensorial V-valued differential* *forms on P is isomorphic to the graded vector space of E-valued differential forms on M:*

$$\Omega^\bullet_\rho(P, V) \cong \Omega^\bullet(M, E),$$

where $E = P \times_\rho V$ is the associated vector bundle.

Proof The isomorphism from left to right is given as follows. Let $\alpha \in \Omega^k_\rho(P, V)$, $x \in M$, and $v_1, \dots, v_k \in T_x M$. We choose a point $p \in P_x$ in the fibre above x and we choose lifts $u_1, \dots, u_k \in T_p P$ for the vectors $v_1, \dots, v_k$, that is $\mathrm{D}_p\pi(u_i) = v_i$, for $1 \leqslant i \leqslant k$. We denote by $f_p \colon V \to E_x$ the linear isomorphisms defined by $v \mapsto \overline{(p, v)}$. The image of the tensorial differential form α is given by

$$(v_1, \dots, v_k) \mapsto f_p\left(\alpha_p(u_1, \dots, u_k)\right),$$

which lives in $\Omega^k(M, E)$. Since α is equivariant, this definition does not depend on the choice of the point $p \in P_x$ and since α is horizontal, this definition does not depend on the choices of the lifts $u_1, \dots, u_k \in T_p P$.

In this other way round, let $\beta \in \Omega^k(M, E)$, $p \in P$, and $u_1, \dots, u_k \in T_p P$. The image of the E-valued differential form β on M under the reverse isomorphism is given by

$$f_p^{-1}\left(\beta_{\pi(p)}\left(\mathrm{D}_p\pi(u_1), \dots, \mathrm{D}_p\pi(u_k)\right)\right),$$

which is clearly a tensorial V-valued differential form on P. □

Since the differential d fails to preserve horizontal V-valued differential forms on P, one is led to the following definition.

Definition 1.30 (Covariant derivative) The *covariant derivative* associated to a connection ω on a principal bundle is defined by

$$d^\omega(\alpha)(X_1, \dots, X_{k+1}) \coloneqq (d\alpha)\left(X_1^h, \dots, X_{k+1}^h\right),$$

for $\alpha \in \Omega^k(P, V)$ and $X_1, \dots, X_{k+1} \in \Gamma(TP)$.

Example 1.31 In terms of the covariant derivative, the first point of Theorem 1.22 asserts that $\Omega = d^\omega(\omega)$.

Lemma 1.32 *The covariant derivative restricts to tensorial V-valued differential forms on P:*

$$d^\omega \colon \Omega^\bullet_\rho(P, V) \to \Omega^{\bullet+1}_\rho(P, V).$$

Proof It is enough to check that the differential d preserves equivariant V-valued differential forms on P. □

Proposition 1.29 implies that the covariant derivative induces a degree 1 linear operator on $\Omega^\bullet(M, E)$. From the explicit isomorphisms given in the above proof, one can see that the image of $d^\omega\colon \Omega^0_\rho(P, V) \to \Omega^1_\rho(P, V)$ gives a map $\nabla_\rho\colon \Omega^0(M, E) \to \Omega^1(M, E)$, which satisfies the Leibniz rule (Definition 1.1); so it defines a connection on the associated vector bundle. Furthermore, one can see that the covariant derivative d^ω corresponds to the twisted de Rham differential d^{∇_ρ}. These two differentials can be expressed in Lie theoretical terms as follows:

The infinitesimal version of the group representation $\rho\colon G \to \mathrm{GL}(V)$ produces a Lie algebra representation $\mathrm{D}_e\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. This latter one defines an action of the differential graded Lie algebra $\Omega^\bullet(P, \mathfrak{g})$ on the graded vector space $\Omega^\bullet(P, V)$ under the formula:

$$(\tau \cdot \alpha)_p\,(v_1, \ldots, v_{k+l}) \coloneqq \frac{1}{k!l!} \sum_{\sigma\in\mathbb{S}_{k+l}} \mathrm{sgn}(\sigma)\,\mathrm{D}_e\,\rho\Big(\tau_p\,(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\Big)\Big(\alpha_p\,(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})\Big),$$

for $\tau \in \Omega^k(P, \mathfrak{g})$, $\alpha \in \Omega^l(P, V)$, $p \in P$, and $v_1, \ldots, v_{k+l} \in T_pP$.

Proposition 1.33 *On tensorial V-valued differential forms $\alpha \in \Omega^\bullet_\rho(P, V)$ on P, the covariant derivative is equal to*

$$d^\omega(\alpha) = d\alpha + \omega \cdot \alpha.$$

Proof We refer the reader to the proof of [144, Theorem 31.19]. □

In the case of the adjoint bundle, the infinitesimal Lie action is given by

$$\mathrm{ad} \coloneqq \mathrm{D}_e\mathrm{Ad}\colon \quad \begin{array}{ccc} \mathfrak{g} & \to & \mathfrak{gl}(\mathfrak{g}) \\ x & \mapsto & [x, -]. \end{array}$$

So we get the formula

$$d^\omega(\alpha) = d\alpha + [\omega, \alpha],$$

which is the global form of the example of a differential twisted by a Maurer–Cartan element.

Example 1.34 Using this property, the Bianchi identity of Theorem 1.22 amounts to $d^\omega(\Omega) = 0$.

Corollary 1.35 *When the connection ω is flat, the covariant derivative d^ω squares to zero.*

Proof This follows from the following formula:

$$\begin{aligned}(d^\omega \circ d^\omega)(\alpha) &= d^\omega (d\alpha + \omega \cdot \alpha) = d(\omega \cdot \alpha) + \omega \cdot (d\alpha + \omega \cdot \alpha) \\ &= d\omega \cdot \alpha - \omega \cdot d\alpha + \omega \cdot d\alpha + \tfrac{1}{2}[\omega, \omega] \cdot \alpha \\ &= (d\omega) \cdot \alpha + \tfrac{1}{2}[\omega, \omega] \cdot \alpha \\ &= \Omega \cdot \alpha,\end{aligned}$$

since the action $\cdot$ is by a differential graded Lie algebra (second line). □

Corollary 1.35 is the exact analogue for the covariant derivative of principal bundles of Proposition 1.6 for the twisted de Rham differential of vector bundles. It implies the following result: when a connection ω of a principal bundle is flat, then so is the connection ∇_ρ on any associated vector bundle.

Let us now study the group of symmetries of a principal bundle and its action on the above-mentioned notions. Its name is motivated by its applications in physics.

Definition 1.36 (Gauge group) The *gauge group* $\mathscr{G}(P)$ of a principal bundle $\pi\colon P \to M$ is the group consisting of all fibre-preserving and equivariant diffeomorphisms of P, called *gauge transformations*:

$$\mathscr{G}(P) := \left\{ f\colon P \xrightarrow{\cong} P \mid f \circ \pi = \pi\ ;\ f\left(\mathrm{R}_g(p)\right) = \mathrm{R}_g(f(p)), \forall p \in P, \forall g \in G \right\}.$$

Remark 1.37 In the physics literature, the structure group G is called the *gauge group*. In the mathematical literature, the above mentioned group $\mathscr{G}(P)$ is also called the *group of gauge transformations*. We chose the present terminology in order to match with the general definition of the gauge group given in the next Section 1.2.

Proposition 1.38 *The gauge group is isomorphic to the group of equivariant G-valued functions on P:*

$$\mathscr{G}(P) \cong C^\infty(P, G)^G.$$

Proof We consider here the conjugation action on G, so an equivariant G-valued function on P is a smooth map $\sigma\colon P \to G$ such that

$$\sigma(\mathrm{R}_g(p)) = \mathrm{C}_{g^{-1}}(\sigma(p)) = g^{-1}\sigma(p)g,$$

for any $p \in P$ and $g \in G$. Given a gauge transformation $f\colon P \to P$, one considers the equivariant G-valued function $\sigma_f\colon P \to G$ defined by

$$F(p) = \mathrm{R}_{\sigma_F(p)}(p).$$

□

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The gauge group is in general infinite dimensional and it acts on $\mathfrak{g}$-valued differential forms $\alpha \in \Omega^\bullet(P, \mathfrak{g})$ on the left by pullback:

$$f.\,\alpha := \left(f^{-1}\right)^* \alpha.$$

This action restricts naturally to (flat) connections. Let us give it a more explicit description.

Theorem 1.39 *The action of a gauge transformation $f \colon P \to P$ on a (flat) connection ω is given by*

$$f.\,\omega = \mathrm{Ad}_{\sigma_f} \circ \omega + \left(\sigma_f^{-1}\right)^* \omega_G.$$

Proof We refer the reader to [79, Chapter II] for details. □

This is the global form of the 'gauge action' formula given in Proposition 1.11.

Let us denote the affine space of connections on the principal bundle P by $\mathrm{C}(P)$ and the set of flat connections by $\mathrm{MC}(P)$. The associated moduli spaces

$$\mathscr{C}(P) := \mathrm{C}(P)/\mathscr{G}(P) \quad \text{and} \quad \mathscr{MC}(P) := \mathrm{MC}(P)/\mathscr{G}(P)$$

of (flat) connections under the gauge group are of fundamental importance in mathematics [4, 5, 39, 40] and in physics [67, 103]. In mathematics, the moduli space of flat connections is isomorphic to the character variety $\mathrm{Hom}(\pi_1(M), G)/G$, where $\pi_1(M)$ stands for the fundamental group of M. In physics terminology, connections are called *gauge fields* and their moduli spaces represent the configuration spaces of quantum field theories on which Feynman path integrals are 'defined' and 'evaluated'.

1.2 Maurer–Cartan Equation in Differential Graded Lie Algebras

The expression $d\omega + \frac{1}{2}[\omega, \omega]$ for the curvature of the $\mathfrak{g}$-valued 1-form defining a connection ω in a principal bundle P makes sense since the space $\Omega^\bullet(P, \mathfrak{g})$ of all $\mathfrak{g}$-valued forms has a richer structure than merely a chain complex: it is a differential graded Lie algebra. In this section, we recall the general formalism for studying the Maurer–Cartan equation and the symmetries of its solutions in differential graded Lie algebras. From now on, we switch from the cohomological degree convention to the homological degree convention, and work over a ground field $\mathbb{k}$ of characteristic 0.

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Definition 1.40 (Differential graded Lie algebra) A *differential graded (dg) Lie algebra* is the data $\mathfrak{g} = (A, d, [\,,])$ of a chain complex, that is a collection of vector spaces $A = \{A_n\}_{n\in\mathbb{Z}}$ with linear maps $d\colon A_n \to A_{n-1}$ of degree -1 satisfying $d^2 = 0$, equipped with a degree-preserving linear map $[\,,] : A^{\otimes 2} \to A$, called the *Lie bracket*, satisfying the following properties

DERIVATION: $d([x, y]) = [dx, y] + (-1)^{|x|}[x, dy]$,

SKEW SYMMETRY: $[x, y] = -(-1)^{|x||y|}[y, x]$,

JACOBI IDENTITY: $[[x, y], z] + (-1)^{|x|(|y|+|z|)}[[y, z], x] + (-1)^{|z|(|x|+|y|)}[[z, x], y] = 0$,

where the notation $|x|$ stands for the degree of homogeneous elements $x \in A_{|x|}$.

Definition 1.41 (Maurer–Cartan element/equation) Let $\mathfrak{g} = (A, d, [\,,])$ be a dg Lie algebra. A *Maurer–Cartan element* ω if an element of A_{-1} that is a solution to the *Maurer–Cartan equation*

$$d\omega + \tfrac{1}{2}[\omega, \omega] = 0. \tag{1.2}$$

The set of Maurer–Cartan elements in $\mathfrak{g}$ is denoted by $\mathrm{MC}(\mathfrak{g})$.

The intuition behind the formulas that we are about to write comes from the situation where the dg Lie algebra $\mathfrak{g}$ is finite dimensional, so that the Maurer–Cartan equation is actually a finite collection of actual polynomial equations in a finite-dimensional vector space, and the Maurer–Cartan set is a variety (intersection of quadrics).

Lemma 1.42 *In a (finite dimensional) dg Lie algebra $\mathfrak{g}$, the tangent space $T_\omega(\mathrm{MC}(\mathfrak{g}))$ of the Maurer–Cartan variety $\mathrm{MC}(\mathfrak{g})$ at a point ω consists of elements $\eta \in A_{-1}$ satisfying*

$$d^\omega(\eta) \coloneqq d\eta + [\omega, \eta] = 0. \tag{1.3}$$

Proof The Maurer–Cartan variety is the 0 locus of the curvature function

$$\Omega\colon \omega \in A_{-1} \mapsto d\omega + \tfrac{1}{2}[\omega, \omega] \in A_{-2}.$$

Its derivative at ω is equal to $\mathrm{D}_\omega \Omega\,(\eta) = d\eta + [\omega, \eta]$. Finally, the tangent space of $\mathrm{MC}(\mathfrak{g})$ at ω is the zero locus of this derivative. □

Considering the *adjoint operator*

$$\mathrm{ad}_\omega \coloneqq [\omega, -],$$

the abovementioned map is equal to the sum $d^\omega = d + \mathrm{ad}_\omega$.

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Proposition 1.43 *For any Maurer–Cartan element $\omega \in \mathrm{MC}(\mathfrak{g})$ of a dg Lie algebra $\mathfrak{g}$, the map d^ω is a derivation satisfying*

$$d^\omega \circ d^\omega = 0.$$

Proof The Jacobi identity is equivalent to the fact that the adjoint operator is a derivation:

$$\begin{aligned}\mathrm{ad}_\omega([x,y]) = [\omega,[x,y]] &= (-1)^{1+|y|(|x|+1)}[y,[\omega,x] + (-1)^{1+|x|+|y|}[x,[y,\omega]] \\ &= [\mathrm{ad}_\omega\, x, y] + (-1)^{|x|}[x, \mathrm{ad}_\omega\, y].\end{aligned}$$

The linear map $d^\omega = d + \mathrm{ad}_\omega$ is a derivation as a sum of derivations. Then, the image of any element η of A under the composite $d^\omega \circ d^\omega$ is given by

$$d^\omega \circ d^\omega(\eta) = \underbrace{d^2(\eta)}_{=0} + \underbrace{d([\omega,\eta]) + [\omega, d\eta]}_{=[d\omega,\eta]} + \underbrace{[\omega,[\omega,\eta]]}_{=\frac{1}{2}[[\omega,\omega],\eta]} = \Big[\underbrace{d\omega + \tfrac{1}{2}[\omega,\omega]}_{=0}, \eta\Big] = 0,$$

using the properties of a dg Lie algebra (differential, derivation, and Jacobi identity) and the Maurer–Cartan equation. □

Definition 1.44 (Twisted differential and twisted dg Lie algebra) For any dg Lie algebra $\mathfrak{g}$ and any Maurer–Cartan element $\omega \in \mathrm{MC}(\mathfrak{g})$, the differential

$$d^\omega = d + \mathrm{ad}_\omega = d + [\omega, -]$$

is called the *twisted differential*. The associated dg Lie algebra

$$\mathfrak{g}^\omega := (A, d^\omega, [\,,\,]).$$

is called the *twisted* dg Lie algebra.

Remark 1.45 The terminology was chosen by analogy with the twisted de Rham differential mentioned in Section 1.1.

Lemma 1.42 and Proposition 1.43 show that, for each $\omega \in \mathrm{MC}(\mathfrak{g})$ and each $\lambda \in A_0$, the element $d^\omega(\lambda) = d\lambda + [\omega, \lambda] \in T_\omega(\mathrm{MC}(\mathfrak{g}))$ lives in the tangent space at ω. In other words, any element $\lambda \in A_0$ induces to a vector field

$$\Upsilon_\lambda \in \Gamma(T(\mathrm{MC}(\mathfrak{g})))$$

given by

$$\Upsilon_\lambda(\omega) := d\lambda + [\omega, \lambda].$$

Definition 1.46 (Gauge symmetries of Maurer–Cartan elements) The flows associated to vector fields Υ_λ for $\lambda \in A_0$ are called the *gauge symmetries* of Maurer–Cartan elements. Two Maurer–Cartan elements α and β are said to be *gauge equivalent* if there exists an element $\lambda \in A_0$ for which the flow of the vector field Υ_λ relates α to β in finite time.

We shall now introduce a useful method for simplifying calculations in dg Lie algebras, like the explicit expression for the integration of the gauge flow. The differential d of a dg Lie algebra $\mathfrak{g} = (A, d, [\,,\,])$ is said to be *internal* if there exists an element δ in A_{-1} such that $d = \mathrm{ad}_\delta$.

Definition 1.47 (Differential trick) Let $\mathfrak{g} = (A, d, [\,,\,])$ be a dg Lie algebra. The *differential trick* amounts to considering the one-dimensional extension

$$\mathfrak{g}^+ := (A \oplus \Bbbk\delta, d, [\,,\,]),$$

$$\text{where} \quad |\delta| := -1, \quad d(\delta) := 0, \quad [\delta, x] := dx, \quad \text{and} \quad [\delta, \delta] := 0.$$

The dg Lie algebra $\mathfrak{g}^+$ is the universal extension of $\mathfrak{g}$, which makes its differential an inner derivation. The canonical map

$$\begin{aligned} \iota \colon \mathfrak{g} &\hookrightarrow \mathfrak{g}^+ \\ x &\longmapsto \begin{cases} \delta + x, & \text{for } |x| = -1, \\ x & \text{for } |x| \neq -1, \end{cases} \end{aligned}$$

embeds $\mathfrak{g}$ inside $\mathfrak{g}^+$. The fact that this map is not a morphism of dg Lie algebras is surprisingly useful: as we shall now see, as a consequence, it alters and simplifies various equations and formulas. The main simplification is that an element $\omega \in A_{-1}$ is a Maurer–Cartan element in $\mathfrak{g}$ if and only if $\delta + \omega$ is a *square-zero element*

$$[\delta + \omega, \delta + \omega] = 0$$

in the extension $\mathfrak{g}^+$. Let us denote by $\mathrm{Sq}(\mathfrak{g}^+)$ the set of degree -1 square-zero elements in $\mathfrak{g}^+$ of the form $\delta + \omega$. The above embedding provides us with a canonical identification of the sets (or varieties in the finite-dimensional case)

$$\mathrm{MC}(\mathfrak{g}) \cong \mathrm{Sq}(\mathfrak{g}^+)$$

and thus of tangent spaces. However, the formulas in the latter case are much simpler: the arguments given above show that $\mathrm{ad}_{\delta+\omega} = [\delta + \omega, -]$ is a square-zero derivation of $\mathfrak{g}^+$, for any $\delta + \omega \in \mathrm{Sq}(\mathfrak{g}^+)$, and that the tangent space at this point is given by

$$T_\omega \mathrm{MC}(\mathfrak{g}) \cong T_{\delta+\omega}\mathrm{Sq}(\mathfrak{g}^+) = \{\eta \in A_{-1} \mid [\delta + \omega, \eta] = 0\}.$$

In the extension $\mathfrak{g}^+$, the formula for the vector field induced by any $\lambda \in A_0$ is

$$\Upsilon^+_\lambda(\delta + \omega) := [\delta + \omega, \lambda] = \mathrm{ad}_{-\lambda}(\delta + \omega).$$

Definition 1.48 (Nilpotent dg Lie algebra) A dg Lie algebra $\mathfrak{g} = (A, d, [\ ,\ ])$ is called *nilpotent* when there exists an integer $n \in \mathbb{N}$ such that

$$[x_1, [x_2, \ldots, [x_{n-1}, x_n] \ldots]] = 0$$

for any $x_1, \ldots, x_n \in A$.

This nilpotency condition is enough to ensure that all the infinite series of brackets we will now consider make sense. Such a condition is too strong to cover all the examples that we have in mind: in Chapter 2, we will settle the more general framework of complete dg Lie algebras inside which fit all our examples and for which all the results below also hold.

Proposition 1.49 *Let* $\mathfrak{g} = (A, d, [\ ,\ ])$ *be a nilpotent dg Lie algebra. The integration of the flow associated to the vector fields* $\Upsilon_{-\lambda}$*, for* $\lambda \in A_0$*, starting at* $\alpha \in \mathrm{MC}(\mathfrak{g})$ *gives at time t:*

$$\frac{\mathrm{id} - \exp(t\,\mathrm{ad}_\lambda)}{t\,\mathrm{ad}_\lambda}(t\,d\lambda) + \exp(t\,\mathrm{ad}_\lambda)(\alpha).$$

Proof We use the differential trick and work in the extension $\mathfrak{g}^+$. The differential equation

$$\frac{d(\delta + \gamma(t))}{dt} = \Upsilon^+_{-\lambda}(\delta + \gamma(t)) = \mathrm{ad}_\lambda(\delta + \gamma(t))$$

associated to the flow $\Upsilon^+_{-\lambda}$ is then easy to solve since there is now no more constant term. The adjoint operator being a linear map, the solution to this differential equation is given by the following exponential:

$$\begin{aligned}
\exp(t\,\mathrm{ad}_\lambda)(\delta + \alpha) &= \exp(t\,\mathrm{ad}_\lambda)(\delta) + \exp(t\,\mathrm{ad}_\lambda)(\alpha) \\
&= \delta + (\exp(t\,\mathrm{ad}_\lambda) - \mathrm{id})(\delta) + \exp(t\,\mathrm{ad}_\lambda)(\alpha) \\
&= \delta + \frac{\mathrm{id} - \exp(t\,\mathrm{ad}_\lambda)}{t\,\mathrm{ad}_\lambda}(t\,d\lambda) + \exp(t\,\mathrm{ad}_\lambda)(\alpha).
\end{aligned}$$

□

Remark 1.50 This formula is the algebraic counterpart of the formula for the action of the gauge transformations on connections of principal bundles given in Section 1.1.

The universal formula underlying the integration of finite-dimensional real Lie algebras into Lie groups is the following one.

Definition 1.51 (Baker–Campbell–Hausdorff formula) The *Baker–Campbell–Hausdorff (BCH) formula* is the element in the associative algebra of formal power series on two variables x and y given by

$$\mathrm{BCH}(x, y) := \ln\left(e^x e^y\right).$$

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It is straightforward to notice that the BCH formula is associative and unital:

$$\mathrm{BCH}(\mathrm{BCH}(x,y),z) = \mathrm{BCH}(x,\mathrm{BCH}(y,z)) \quad \text{and} \quad \mathrm{BCH}(x,0) = x = \mathrm{BCH}(0,x).$$

The celebrated theorem of Baker, Campbell, and Hausdorff, see [13], states that this formula can be written using only the commutators $[a,b] = a \otimes b - b \otimes a$:

$$\mathrm{BCH}(x,y) = x + y + \tfrac{1}{2}[x,y] + \tfrac{1}{12}[x,[x,y]] + \tfrac{1}{12}[y,[x,y]] + \cdots.$$

It can thus be applied to any nilpotent Lie algebra.

Definition 1.52 (Gauge group) The *gauge group* associated to a nilpotent dg Lie algebra $\mathfrak{g} = (A, d, [\,,\,])$ is the group obtained from A_0 via the Baker–Campbell–Hausdorff formula:

$$\Gamma := (A_0, x \cdot y := \mathrm{BCH}(x,y), 0.),$$

The name 'gauge group' is justified by the following proposition.

Theorem 1.53 *Let* $\mathfrak{g} = (A, d, [\,,\,])$ *be a nilpotent dg Lie algebra. The formula*

$$\lambda.\alpha := \frac{\mathrm{id} - \exp(\mathrm{ad}_\lambda)}{\mathrm{ad}_\lambda}(d\lambda) + \exp(\mathrm{ad}_\lambda)(\alpha)$$

for the gauge action defines a left action of the gauge group Γ *on* $\mathrm{MC}(\mathfrak{g})$.

Proof We use the differential trick to simplify the calculations: we have to show that the assignment

$$\lambda.(\delta + \alpha) := \exp(\mathrm{ad}_\lambda)(\delta + \alpha)$$

defines an action of the gauge group Γ on $\mathrm{Sq}(\mathfrak{g}^+)$. Since ad_λ is a derivation, then $\exp(\mathrm{ad}_\lambda)$ is a morphism of graded Lie algebras. This implies that $\lambda.(\delta+\alpha)$ is again a square-zero element:

$$\begin{aligned}
[\lambda.(\delta + \alpha), \lambda.(\delta + \alpha)] &= [\exp(\mathrm{ad}_\lambda)(\delta + \alpha), \exp(\mathrm{ad}_\lambda)(\delta + \alpha)] \\
&= \exp(\mathrm{ad}_\lambda)([\delta + \alpha, \delta + \alpha]) = 0.
\end{aligned}$$

It is straightforward to check that the action of 0 is trivial

$$0.(\delta + \alpha) = \exp(\mathrm{ad}_0)(\delta + \alpha) = \mathrm{id}(\delta + \alpha) = \delta + \alpha.$$

The BCH formula satisfies the relation

$$\exp(\mathrm{ad}_{\mathrm{BCH}(x,y)}) = \exp(\mathrm{ad}_x) \circ \exp(\mathrm{ad}_y).$$

(The BCH formula is actually characterised by this relation, see [132, Proposition 5.14].) To prove it, let us work in the associative algebra of formal power series in three variables x, y, and z. Notice that for any element a, the adjoint operator $\mathrm{ad}_a = l_a - r_a$ is equal to the difference of the left multiplication by a

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with the right multiplication by a. Since the underlying algebra is associative, these two linear maps commute and thus

$$\exp(\mathrm{ad}_a)(b) = \exp(l_a) \circ \exp(r_{-a})(b) = e^a \, b \, e^{-a}.$$

Using this, we get

$$\begin{aligned} \exp(\mathrm{ad}_{\mathrm{BCH}(x,y)})(z) &= e^{\mathrm{BCH}(x,y)} \, z \, e^{-\mathrm{BCH}(x,y)} \\ &= (e^x e^y) \, z \, (e^x e^y)^{-1} = e^x (e^y \, z \, e^{-y}) e^{-x} \\ &= \exp(\mathrm{ad}_x) \circ \exp(\mathrm{ad}_y)(z). \end{aligned}$$

Finally, with this relation implies

$$\begin{aligned} \lambda.(\mu.(\delta + \alpha)) &= \exp(\mathrm{ad}_\lambda)(\exp(\mathrm{ad}_\mu)(\delta + \alpha)) \\ &= (\exp(\mathrm{ad}_\lambda) \circ \exp(\mathrm{ad}_\mu))(\delta + \alpha) \\ &= \exp(\mathrm{ad}_{\mathrm{BCH}(\lambda,\mu)})(\delta + \alpha) \\ &= (\lambda \cdot \mu).(\delta + \alpha). \end{aligned}$$

□

Definition 1.54 (Moduli space of Maurer–Cartan elements) The *moduli space of Maurer–Cartan elements* is the set of equivalence classes of Maurer–Cartan elements under the gauge group action:

$$\mathscr{MC}(\mathfrak{g}) := \mathrm{MC}(\mathfrak{g})/\Gamma.$$

This moduli space loses the data provided by the gauge symmetries themselves. One may instead consider the following main protagonist of deformation theory.

Definition 1.55 (Deligne groupoid) Let $\mathfrak{g} = (A, d, [\,,\,])$ be a nilpotent dg Lie algebra. The *Deligne groupoid* associated to $\mathfrak{g}$ has the Maurer–Cartan set $\mathrm{MC}(\mathfrak{g})$ as its set of objects, and the gauge symmetries λ such that $\lambda.\alpha = \beta$ as the set of (iso)morphisms from α to β.

1.3 Deformation Theory with Differential Graded Lie Algebras

The purpose of this section is to explain how one can study deformation theory with dg Lie algebras using Maurer–Cartan elements and their gauge symmetries. This will provide us with a transition from this chapter to the next chapter as infinitesimal deformations are controlled by nilpotent objects while formal deformations are controlled by complete objects.

As mentioned in Section 1.2, one can twist a dg Lie algebra $\mathfrak{g} = (A, d, [\,,\,])$ with any Maurer–Cartan element

$$\varphi \in \mathrm{MC}(\mathfrak{g}) := \left\{\varphi \in A_{-1} \mid d\varphi + \tfrac{1}{2}[\varphi, \varphi] = 0\right\}$$

to produce a twisted dg Lie algebra

$$\mathfrak{g}^{\varphi} := (A, d^{\varphi} := d + [\varphi, \,], [\,,\,]).$$

The relationship between dg Lie algebras and deformation theory relies ultimately on the following key property, which says that deformations of a Maurer–Cartan element coincide with Maurer–Cartan elements of the twisted dg Lie algebra.

Lemma 1.56 *Let $\mathfrak{g}$ be a dg Lie algebra and let $\varphi \in \mathrm{MC}(\mathfrak{g})$ be a Maurer–Cartan element. The following equivalence holds*

$$\alpha \in \mathrm{MC}(\mathfrak{g}^{\varphi}) \iff \varphi + \alpha \in \mathrm{MC}(\mathfrak{g}).$$

Proof While we shall see a conceptual explanation in Lemma 4.9, see also Corollary 4.10, it is easy to prove this result directly by showing that both conditions are equivalent to

$$d\alpha + [\varphi, \alpha] + \tfrac{1}{2}[\alpha, \alpha] = 0.$$

□

The fundamental theorem of deformation theory recently proved by J.P. Pridham [128] and J. Lurie [98] claims that any deformation problem over a field $\mathbb{k}$ of characteristic 0 can be encoded by a dg Lie algebra. We shall give its precise statement in Section 7.1, but let us now explain what it means heuristically. Given an underlying 'space' V, a type of structure $\mathcal{P}$ that it can support, and an equivalence relation on $\mathcal{P}$-structures on V, there should exist a dg Lie algebra $\mathfrak{g} = (A, d, [\,,\,])$ such that its set of Maurer–Cartan elements is in one-to-one correspondence with the set of $\mathcal{P}$-structures on V and such that the action of the gauge group Γ on $\mathrm{MC}(\mathfrak{g})$ coincides with the equivalence relation considered on $\mathcal{P}$-structures.

Example 1.57 When V is a finite-dimensional vector space and when $\mathcal{P}$ stands for associative algebra structures up to isomorphisms, the deformation dg Lie algebra is given by the Hochschild cochain complex

$$C^{\bullet}(V, V) := \left(\prod_{n \geq 1} \mathrm{Hom}\,(V^{\otimes n}, V), 0, [\,,\,]\right),$$

where the Lie bracket is the one defined by Gerstenhaber [60] (see Section 4.1) and where the homological degree of the factor $\operatorname{Hom}(V^{\otimes n}, V)$ is equal to $1-n$. In this case, a Maurer–Cartan element is precisely a binary associative product, and gauge equivalence is an isomorphism.

There are two possible viewpoints one can adopt here: one can either ensure convergence of deformations by working *locally* with complete 'spaces' V and complete dg Lie algebras $\mathfrak{g}$ (as will be done in Chapters 3 and 4) or work *globally*, inspired by the functors of points in algebraic geometry, as follows. Let $\mathfrak{R}$ be a local (commutative) ring with the maximal ideal $\mathfrak{m}$ and with the residue field $\mathbb{k}$, that is $\mathfrak{R} \cong \mathbb{k} \oplus \mathfrak{m}$. Given any dg Lie algebra $\mathfrak{g} = (A, d, [\,,\,])$, one can consider its $\mathfrak{R}$-extension defined by

$$\mathfrak{g} \otimes \mathfrak{R} := (A \otimes \mathfrak{R}, d, [\,,\,]),$$

where $d(\zeta \otimes x) := d\zeta \otimes x$ and where $[\zeta \otimes x, \xi \otimes y] := [\zeta, \xi] \otimes xy$. The other way round, one recovers the original dg Lie algebra $\mathfrak{g}$ from its $\mathfrak{R}$-extension $\mathfrak{g} \otimes \mathfrak{R} \cong \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{m}$ by reducing modulo $\mathfrak{m}$.

Definition 1.58 ($\mathfrak{R}$-deformation) An $\mathfrak{R}$-*deformation* of a Maurer–Cartan element $\varphi \in \mathrm{MC}(\mathfrak{g})$ is a Maurer–Cartan element $\Phi \in \mathrm{MC}(\mathfrak{g} \otimes \mathfrak{R})$ of the $\mathfrak{R}$-extension of $\mathfrak{g}$ such that its reduction modulo $\mathfrak{m}$ is equal to φ. The set of such deformations is denoted by $\mathrm{Def}_\varphi(\mathfrak{R})$.

Proposition 1.59 *The set of $\mathfrak{R}$-deformations of a Maurer–Cartan $\varphi \in \mathrm{MC}(\mathfrak{g})$ is in natural bijection with the set of Maurer–Cartan elements of the twisted dg Lie algebra $\mathfrak{g}^\varphi \otimes m$:*

$$\mathrm{Def}_\varphi(\mathfrak{R}) \cong \mathrm{MC}(\mathfrak{g}^\varphi \otimes \mathfrak{m}).$$

Proof Notice first that, for any Maurer–Cartan element $\varphi \in \mathrm{MC}(\mathfrak{g})$, the $\mathfrak{R}$-extension of $\mathfrak{g}$ twisted by φ is isomorphic to

$$(\mathfrak{g} \otimes \mathfrak{R})^\varphi \cong \mathfrak{g}^\varphi \otimes \mathfrak{R} \cong \mathfrak{g}^\varphi \oplus \mathfrak{g}^\varphi \otimes \mathfrak{m}.$$

The result is then a direct application of Lemma 1.56: any element

$$\Phi = \varphi + \bar{\Phi} \in \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{m}$$

is an $\mathfrak{R}$-deformation of φ if and only if $\bar{\Phi}$ is Maurer–Cartan element of the twisted dg Lie algebra $\mathfrak{g}^\varphi \otimes \mathfrak{m}$. □

Example 1.60 An $\mathfrak{R}$-deformation of an associative algebra structure φ on V is an $\mathfrak{R}$-linear associative algebra structure Φ on $V \otimes \mathfrak{R}$ whose reduction modulo $\mathfrak{m}$ is equal to φ. This comes from the formula

$$\operatorname{Hom}_{\mathbb{k}}(V^{\otimes n}, V) \otimes \mathfrak{R} \cong \operatorname{Hom}_{\mathfrak{R}}\left((V \otimes \mathfrak{R})^{\otimes n}, V \otimes \mathfrak{R}\right).$$

For the case when the ring $\mathfrak{R}$ is Artinian, that is when there exists $K \in \mathbb{N}$ such that $\mathfrak{m}^K = 0$, all series we ever consider will converge automatically since the dg Lie algebra $\mathfrak{g} \otimes \mathfrak{m}$ is nilpotent. The case of complete algebras considered in Chapter 3 corresponds to the case when $\mathfrak{R}$ is *complete* with respect to its $\mathfrak{m}$-adic topology; it is then a limit of local Artinian rings. In both cases, the BCH formula produces a convergent series, and the gauge group $\bar{\Gamma} := (A_0 \otimes \mathfrak{m}, \mathrm{BCH}, 0)$ is well defined. Its action

$$\lambda.\Phi := \frac{\mathrm{id} - \exp(\mathrm{ad}_\lambda)}{\mathrm{ad}_\lambda}(d\lambda) + \exp(\mathrm{ad}_\lambda)(\Phi)$$

on $\mathfrak{R}$-deformations $\Phi = \varphi + \bar{\Phi}$ is also well defined.

Definition 1.61 (Moduli space of $\mathfrak{R}$-deformations) The *moduli space of* $\mathfrak{R}$-*deformations* is the set of classes

$$\mathscr{D}ef_\varphi(\mathfrak{R}) := \mathrm{Def}_\varphi(\mathfrak{R})/\bar{\Gamma}$$

of $\mathfrak{R}$-deformations modulo the action of the gauge group $\bar{\Gamma}$.

The first seminal example is given by the *algebra of dual numbers*

$$\mathfrak{R} = \mathbb{k}[t]/(t^2),$$

which is an Artinian local ring.

Definition 1.62 (Infinitesimal deformation) An *infinitesimal deformation* of a Maurer–Cartan element $\varphi \in \mathrm{MC}(\mathfrak{g})$ is a Maurer–Cartan element of the form

$$\Phi = \varphi + \bar{\Phi}t \in \mathrm{MC}(\mathfrak{g} \otimes \mathbb{k}[t]/(t^2)).$$

Infinitesimal deformations are related to the homology group of degree -1 of the twisted Lie algebra as follows.

Theorem 1.63 *There are canonical bijections*

$$\mathrm{Def}_\varphi\,(\mathbb{k}[t]/(t^2)) \cong Z_{-1}(\mathfrak{g}^\varphi) \quad \textit{and} \quad \mathscr{D}ef_\varphi(\mathbb{k}[t]/(t^2)) \cong H_{-1}(\mathfrak{g}^\varphi).$$

Proof Any degree -1 element $\Phi = \varphi + \bar{\Phi}t \in \mathfrak{g} \otimes \mathbb{k}[t]/(t^2)$ is a Maurer–Cartan element if and only if is satisfies the equation

$$\underbrace{d\varphi + \tfrac{1}{2}[\varphi, \varphi]}_{=0} + \Big(\underbrace{d(\bar{\Phi}) + [\varphi, \bar{\Phi}]}_{=d^\varphi(\bar{\Phi})}\Big)t = 0.$$

So infinitesimal deformations coincide with cycles of degree -1 in the twisted dg Lie algebra.

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Two infinitesimal deformations $\Phi_1 = \varphi + \bar{\Phi}_1 t$ and $\Phi_2 = \varphi + \bar{\Phi}_2 t$ are equivalent if there exists an element $\lambda \in A_0$ such that

$$\lambda t.\left(\varphi + \bar{\Phi}_1 t\right) = \varphi + \left(d\lambda + [\varphi, \lambda] + \bar{\Phi}_1\right) t = \varphi + \bar{\Phi}_2 t.$$

This latter equation is equivalent to $\bar{\Phi}_2 - \bar{\Phi}_1 = d^\varphi(\lambda)$. This proves that two infinitesimal deformations are equivalent if and only if they are homologous in the twisted dg Lie algebra. □

The other seminal example is given by the *algebra of formal power series*

$$\mathfrak{R} = \Bbbk[[t]],$$

which is a complete local ring.

Definition 1.64 (Formal deformation) A *formal deformation* of a Maurer–Cartan element $\varphi \in \mathrm{MC}(\mathfrak{g})$ is a Maurer–Cartan element of the form

$$\Phi = \varphi + \Phi_1 t + \Phi_2 t^2 + \cdots \in \mathrm{MC}(\mathfrak{g} \otimes \Bbbk[[t]]).$$

The obstructions to formal deformations are related to the homology group of degree -2 of the twisted dg Lie algebra as follows.

Theorem 1.65 *If $H_{-2}(\mathfrak{g}^\varphi) = 0$, then any cycle of degree -1 of g^φ extends to a formal deformation of φ.*

Proof In the present case, the Maurer–Cartan equation $d\Phi + \frac{1}{2}[\Phi, \Phi] = 0$ splits with respect to the power of t as

$$d\Phi_n + [\varphi, \Phi_n] + \tfrac{1}{2}\sum_{k=1}^{n-1} [\Phi_k, \Phi_{n-k}] = 0 \qquad (*)$$

for any $n \geqslant 1$. For $n = 1$, the equation $(*)$ gives

$$d\Phi_1 + [\varphi, \Phi_1] = d^\varphi(\Phi_1) = 0,$$

so the first term of a formal deformation coincides with a cycle of degree -1 of the twisted dg Lie algebra.

Let us now consider such a degree -1 cycle Φ_1, and let us assume that we have $H_{-2}(\mathfrak{g}^\varphi) = 0$. We show, by induction on $n \geqslant 1$, that there exist elements $\Phi_1, \ldots, \Phi_n \in A_{-1}$ satisfying the equations $(*)$ up to n. The case $n = 1$ obviously holds true. Assume that this statement holds true up to $n - 1$. The first two terms of Equation $(*)$ at n are equal to $d^\varphi(\Phi_n)$; let us show that the third

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term is degree -2 cycle with respect to the twisted differential:

$$\begin{aligned} d^{\varphi}\left(\sum_{k=1}^{n-1}[\Phi_k, \Phi_{n-k}]\right) &= \sum_{k=1}^{n-1}\left([d^{\varphi}(\Phi_k), \Phi_{n-k}] - [\Phi_k, d^{\varphi}(\Phi_{n-k})]\right) \\ &= 2\sum_{k=1}^{n-1}[d^{\varphi}(\Phi_k), \Phi_{n-k}] \\ &= -\sum_{k=1}^{n-1}\sum_{l=1}^{k-1}[[\Phi_l, \Phi_{k-l}], \Phi_{n-k}] \\ &= -\sum_{\substack{a+b+c=n \\ a,b,c\geqslant 1}}[[\Phi_a, \Phi_b], \Phi_c] = 0, \end{aligned}$$

by the Jacobi identity. Since $H_{-2}(\mathfrak{g}^{\varphi}) = 0$, there exists $\Phi_n \in A_{-1}$ satisfying Equation $(*)$ at n, which concludes the proof. □

The homology group of degree -1 of the twisted dg Lie algebra detects the Maurer–Cartan elements that are rigid, that is the ones that cannot be deformed nontrivially.

Theorem 1.66 *If $H_{-1}(\mathfrak{g}^{\varphi}) = 0$, then any formal deformation of φ is equivalent to the trivial one.*

Proof We once again use the differential trick and work in the extension

$$(\mathfrak{g} \otimes \Bbbk[[t]])^{+} = \mathfrak{g} \otimes \Bbbk[[t]] \oplus \Bbbk\delta.$$

Given a formal deformation $\Phi = \varphi + \sum_{n\geqslant 1}\Phi_n t^n$ of φ, let us try to find, by induction, an element

$$\lambda = \lambda_1 t + \lambda_2 t^2 + \cdots \in A_0 \otimes \Bbbk[[t]]$$

satisfying

$$\exp(\mathrm{ad}_{\lambda})(\delta + \varphi) = \delta + \Phi \qquad (**)$$

in $(\mathfrak{g} \otimes \Bbbk[[t]])^{+}$. For $n = 1$, the relation satisfied by the coefficients of t in Equation $(**)$ is

$$\mathrm{ad}_{\lambda_1}(\delta + \varphi) = -d^{\varphi}(\lambda_1) = \Phi_1.$$

Recall that Equation $(*)$ in the proof of Theorem 1.65 shows that Φ_1 is cycle of degree -1 for the twisted differential; since $H_{-1}(\mathfrak{g}^{\varphi}) = 0$, it is also a boundary, so we may find such an element $\lambda_1 \in A_0$. Suppose now that there exist elements $\lambda_1, \ldots, \lambda_{n-1} \in A_0$ satisfying Equation $(**)$ modulo t^n and let us look for an element $\lambda_n \in A_0$ satisfying it modulo t^{n+1}. Under the notations

$$\widetilde{\lambda} := \lambda_1 t + \cdots + \lambda_{n-1} t^{n-1} \quad \text{and} \quad \exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi) = \sum_{n=0}^{\infty} \Big(\exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi)\Big)_n t^n,$$

the relation satisfied by the coefficients of t^n in Equation $(**)$ is

$$\mathrm{ad}_{\lambda_n}(\delta + \varphi) + \Big(\exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi)\Big)_n = -d^{\varphi}(\lambda_n) + \Big(\exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi)\Big)_n = \Phi_n.$$

The assumption $H_{-1}(\mathfrak{g}^{\varphi}) = 0$ ensures that such an element λ_n exists provided that Φ_n and $\Big(\exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi)\Big)_n$ have the same image under the twisted differential d^{φ}. We consider the element $\widetilde{\Phi} := \Phi_1 t + \cdots + \Phi_{n-1} t^{n-1}$. Equation $(*)$ guarantees that

$$d^{\varphi}(\Phi_n) = -\tfrac{1}{2}\left(\left[\widetilde{\Phi}, \widetilde{\Phi}\right]\right)_n.$$

On the other hand, Equation $(**)$ implies

$$\begin{aligned}
\left[\widetilde{\Phi}, \widetilde{\Phi}\right] &\equiv \left[\exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi) - (\delta + \varphi), \exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi) - (\delta + \varphi)\right] \pmod{t^{n+1}} \\
&\equiv -2\left[(\delta + \varphi), \exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi)\right] \pmod{t^{n+1}} \\
&\equiv -2\, d^{\varphi}\Big(\exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi)\Big) \pmod{t^{n+1}},
\end{aligned}$$

since $\exp(\mathrm{ad}_{\widetilde{\lambda}})$ is a morphism of Lie algebras. In the end, we get

$$d^{\varphi}(\Phi_n) = -\tfrac{1}{2}\left(\left[\widetilde{\Phi}, \widetilde{\Phi}\right]\right)_n = \Big(d^{\varphi}\Big(\exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi)\Big)\Big)_n = d^{\varphi}\Big(\Big(\exp(\mathrm{ad}_{\widetilde{\lambda}})(\delta + \varphi)\Big)_n\Big),$$

which concludes the proof. □

The present proof shows how the differential trick works heuristically: it states that 'if a property holds true in a graded Lie algebra with trivial differential, then it holds true in any dg Lie algebras'.