20

Sullivan h-Conformal Measures for Compactly Nonrecurrent Elliptic Functions

In this chapter, we deal systematically with one of the primary concepts of the book, namely that of (Sullivan) h-conformal (as always, $h = \mathrm{HD}(J(f))$) measures for compactly nonrecurrent elliptic functions. We will prove their existence for this class of elliptic functions. In Section 20.3, we will introduce an important class of regular compactly nonrecurrent elliptic functions. For this class of elliptic functions, we will prove the uniqueness and atomlessness of h-conformal measures along with their first basic stochastic properties such as ergodicity and conservativity. We will then assume an elliptic function to be compactly nonrecurrent regular throughout the entire book, unless explicitly stated otherwise.

We have already met the concept of conformal measures in Sections 10.1 and 10.2, where we treated them in a very general setting in the former of these two sections and in the latter in a setting and spirit quite close to the one we will be dealing with in the current chapter. In particular, we will frequently use the results of these two sections in the current chapter.

We gave, in Section 10.1, quite an extended historical account of the concept of conformal measures, particularly the Sullivan ones. We repeat a part of it here for the sake of completeness and for the convenience of the reader.

Conformal measures were first defined and introduced by Patterson in his seminal paper [**Pat1**] (see also [**Pat2**]) in the context of Fuchsian groups. Sullivan extended this concept to all Kleinian groups in [**Su2**] and [**Su4**]. He then, in papers [**Su5**] and [**Su7**], defined conformal measures for all rational functions of the Riemann sphere $\widehat{\mathbb{C}}$; he also proved their existence therein. Both Patterson and Sullivan came up with conformal measures in order to get an understanding of geometric measures, i.e., Hausdorff and packing measures. Although Sullivan had already noticed that there are conformal measures for Kleinian groups that are not equal, nor even equivalent to any Hausdorff or packing (generalized) measure, the main purpose of dealing with them

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is to understand Hausdorff and packing measures. Chapter 11 in Volume I, Section 17.6, and, especially, the current Part VI of our book provide good evidence.

Conformal measures, in the sense of Sullivan, have been studied in the context of rational functions in greater detail in [**DU3**], where, in particular, the structure of the set of their exponents was examined.

Since then, conformal measures in the context of rational functions have been studied in numerous research works. We list here only a very few of them that appeared in the early stages of the development of their theory: [**DU1**], [**DU5**], [**DU6**]. Subsequently, the concept of conformal measures, in the sense of Sullivan, has been extended to countable alphabet iterated function systems in [**MU1**] and to conformal graph directed Markov systems in [**MU2**]. These were treated at length in Chapter 11. This was, furthermore, extended to some transcendental meromorphic dynamics in [**KU2**], [**UZ1**], and [**MyU3**]; see also [**UZ2**], [**MyU4**], and [**BKZ1**]. Our current construction fits well with this line of development.

Last, the concept of conformal measures also found its place in random dynamics; we cite only [**MSU**].

20.1 Existence of Conformal Measures for Compactly Nonrecurrent Elliptic Functions

In this section, we prove the existence of h-conformal measures for compactly nonrecurrent elliptic functions. We also locate their potential atoms.

As a fairly straightforward application of Theorem 17.6.7, we shall prove the following main result of this section.

Theorem 20.1.1 *If $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is a compactly nonrecurrent elliptic function, then*

$$\mathrm{DD_h}(J(f)) = \mathrm{DD}_\chi(J(f)) = \mathrm{HD}\big(J_{er}(f)\big) = \mathrm{HD}\big(J_r(f)\big) = h = \mathrm{HD}(J(f)) \tag{20.1}$$

and there exists an h-conformal measure m_h for f (remember that its spherical version is, as are all spherical conformal measures considered in this book, finite; as a matter of fact, probabilistic) all of whose atoms are contained in the set

$$Z_f := \bigcup_{n=0}^{\infty} f^{-n}\big(\mathrm{Crit}(J(f))\big) \cup \bigcup_{n=1}^{\infty} f^{-n}(\infty).$$

Furthermore, m_h is the measure produced in Claim 2° stated in the proof of Theorem 17.6.7.

In addition, if $m_h(\mathrm{Crit}(J(f)) = 0$, then all atoms of m_h are contained in the set

$$I_-(f) = \bigcup_{n=1}^{\infty} f^{-n}(\infty).$$

Proof Let m_h be the measure m produced in Theorem 17.6.7. In fact, because of Corollary 18.3.6, we can, and we do, take the measure m_h as produced in Claim 2° stated in the proof of Theorem 17.6.7. In view of Proposition 18.3.4, we have that $J(f)\backslash J_r(f) \subseteq \mathrm{Sing}^-(f)$. Since the set $\mathrm{Sing}^-(f)$ is countable, it follows that $h = \mathrm{HD}(J(f)) = \mathrm{HD}(J_r(f)) = s_f$. Thus, by virtue of Theorem 17.6.7, (20.1) holds and the measure m_h is h-conformal.

Now by applying h-conformality of the measure m_h, it follows from (17.32) of Lemma 17.6.6 and Corollary 18.3.6 that if $z \in J(f)\backslash \mathrm{Sing}^-(f)$, then

$$m_h(\{z\}) = 0.$$

Since, by Theorem 17.6.7, $m_h(\Omega(f)) = 0$, we conclude that if $z \in f^{-n}(\Omega(f))$ with some integer $n \geq 0$, and $m(\{z\}) \neq 0$, then

$$z \in \bigcup_{n=0}^{\infty} f^{-n}\big(\mathrm{Crit}(J(f))\big).$$

Thus (note also that, by Theorem 17.6.7, $m_h(\infty) = 0$), all atoms of m_h are contained in the set

$$Z_f = \bigcup_{n=0}^{\infty} f^{-n}\big(\mathrm{Crit}(J(f))\big) \cup \bigcup_{n=1}^{\infty} f^{-n}(\infty).$$

In order to prove the last assertion of our theorem, assume that $m_h(\mathrm{Crit}(J(f)) = 0$, $z \in J(f)$, and $f^n(z) \in \mathrm{Crit}(J(f))$ for some integer $n \geq 0$. Then let $0 \leq k \leq n$ be the least integer such that $f^k(z) \in \mathrm{Crit}(J(f))$. Since $m_h(\{f^k(z)\}) = 0$, we then conclude, by h-conformality of m_h, that $m_h(\{z\}) = 0$. The proof of Theorem 20.1.1 is, thus, complete. ∎

20.2 Conformal Measures for Compactly Nonrecurrent Elliptic Functions and Holomorphic Inverse Branches

In this section, we keep $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$, a compactly nonrecurrent elliptic function. Let m be an almost t-conformal measure and m_e be its Euclidean version.

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The upper estimability and strongly lower estimability will be considered in this section with respect to the measure m_e. When we speak about lower estimability, we will make a stronger assumption, namely that the measure m is t-conformal. Since the number of parabolic points is finite, passing to an appropriate iteration, we assume without loss of generality, in this and the next section, that all parabolic periodic points of f are simple.

Consider a closed forward f-invariant subset E of $\mathbb{C}$ such that

$$\|f'\|_E := \sup\{|f'(z)|: z \in E\} < +\infty.$$

Such sets will be called f-pseudo-compact. Obviously, each f-invariant compact subset E of $\mathbb{C}$ is f-pseudo-compact. Recall that $\theta = \theta(f) > 0$ was defined in (18.19), $\beta = \beta_f > 0$ was defined in (18.21), $\alpha_t(\omega)$ in Lemma 15.4.1, and that $\tau > 0$ is so small as required in Lemma 15.3.2.

The proofs of Propositions 4.15 and 4.16 from [**KU4**] translate verbatim to our current case. We present them now.

Proposition 20.2.1 *Let $f: \mathbb{C} \to \widehat{\mathbb{C}}$ be a compactly nonrecurrent elliptic function. Fix an f-pseudo-compact subset E of $J(f)$. Let $z \in E$, $\lambda > 0$, and $0 < r \leq \tau\theta \min\{1, \|f'\|_E^{-1}\}\lambda^{-1}$ be a real number. Suppose that at least one of the following two conditions is satisfied:*

$$z \in E \setminus \bigcup_{n \geq 0} f^{-n}(\mathrm{Crit}(J(f)))$$

or

$$z \in E \quad \text{and} \quad r > \tau\theta \min\left\{1, \|f'\|_E^{-1}\right\}\lambda^{-1} \inf\{|(f^n)'(z)|^{-1}: n = 1, 2, \ldots\right\}.$$

Then there exists an integer $u = u(\lambda, r, z) \geq 0$ such that

$$r|(f^j)'(z)| \leq \lambda^{-1}\theta\tau$$

for all $0 \leq j \leq u$ and the following four conditions are satisfied:

$$\mathrm{diam}_e\big(\mathrm{Comp}(f^j(z), f^{u-j}, \lambda r|(f^u)'(z)|)\big) \leq \beta = \beta_f \tag{20.2}$$

for every $j = 0, 1, \ldots, u$.

Let m be an almost t-conformal measure. Then, for every $\eta > 0$, there exists a continuous function $[0, \infty) \ni t \longmapsto B_t = B_t(\lambda, \eta) > 0$ (independent of z, n, and r) such that if $f^u(z) \in B_e(\omega, \theta)$ for some $\omega \in \Omega(f)$, then

$$f^u(z) \ \text{ is } \ (\eta r|(f^u)'(z)|, B_t) - \alpha_t(\omega)\text{-u.e.} \tag{20.3}$$

with respect to the almost t-conformal measure m, and there exists a function $W_t = W_t(\lambda, \eta): (0, 1] \longrightarrow (0, 1]$ (independent of z, n, and r) such that if $f^u(z) \in B_e(\omega, \theta)$ for some $\omega \in \Omega(f)$, then, for every $\sigma \in (0, 1]$,

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$$f^u(z) \ \textit{is} \ (\eta r|(f^u)'(z)|, \sigma, W_t(\sigma)) - \alpha_t(\omega)\text{-}s.l.e. \tag{20.4}$$

with respect to the almost t-conformal measure m. If $f^u(z) \notin B_e(\Omega(f),\theta)$, *then* (20.3) *and* (20.4) *are also true with*

$$\alpha_t(\omega) \ \textit{replaced by} \ t. \tag{20.5}$$

Proof Suppose, first, that

$$\sup\{\lambda r|(f^j)'(z)|\colon j \geq 0\} > \theta\tau \min\{1, \|f'\|_E^{-1}\}. \tag{20.6}$$

Let $n = n(\lambda, z, r) \geq 0$ be the least integer for which

$$\lambda r|(f^n)'(z)| > \theta\tau \min\{1, \|f'\|_E^{-1}\}. \tag{20.7}$$

Then $n \geq 1$ (owing to the assumption imposed on r),

$$\lambda r|(f^j)'(z)| \leq \theta\tau \min\{1, \|f'\|_E^{-1}\} \tag{20.8}$$

for all $0 \leq j \leq n-1$, and also

$$\lambda r|(f^n)'(z)| \leq \theta\tau. \tag{20.9}$$

If $f^n(z) \notin B_e(\Omega(f),\theta)$, set $u = u(\lambda, r, z) := n$. Then items (20.3)–(20.5) are obvious in view of (20.7) and (20.9), while (20.2) follows from (18.22) and (18.23) along with (20.9). Thus, we are done in this subcase.

So, suppose that

$$f^n(z) \in B_e(\Omega(f),\theta), \tag{20.10}$$

say $f^n(z) \in B_e(\omega,\theta)$, $\omega \in \Omega(f)$. Let $0 \leq k = k(\lambda, z, r) \leq n$ be the least integer such that $f^j(z) \in B_e(\Omega(f),\theta)$ for every $j = k, k+1, \ldots, n$. Consider all the numbers

$$r_i := |f^i(z) - \omega||(f^i)'(z)|^{-1},$$

where $i = k, k+1, \ldots, n$. Put

$$\|f'\|_E^+ := \max\{1, \|f'\|_E\}.$$

By (20.7), we have that

$$r_n = |f^n(z) - \omega||(f^n)'(z)|^{-1} \leq \theta\|f'\|_E^+\theta^{-1}\tau^{-1}\lambda r = \|f'\|_E^+\tau^{-1}\lambda r;$$

therefore, there exists a minimal $k \leq u = u(\lambda, r, z) \leq n$ such that $r_u \leq \|f'\|_E^+\tau^{-1}\lambda r$. In other words,

$$|f^u(z) - \omega| \leq \|f'\|_E^+\tau^{-1}\lambda r|(f^u)'(z)|. \tag{20.11}$$

Now suppose that

$$\sup\{\lambda r|(f^j)'(z)|:\ j \geq 0\} \leq \theta\tau \min\{1, \|f'\|_E^{-1}\}. \tag{20.12}$$

Then it follows from Corollary 18.3.6 and our hypotheses that

$$z \in \bigcup_{j=0}^{\infty} f^{-j}(\Omega(f)).$$

Define then the three numbers $u(\lambda, z, r)$, $k(\lambda, z, r)$, and $n(\lambda, z, r)$ to all be equal to the least integer $j \geq 0$ such that $f^j(z) \in \Omega(f)$. Denote

$$\omega = f^u(z).$$

Notice that, in this case, (20.9) and (20.11) are also satisfied. Our further considerations are valid in both cases (20.6) with (20.10), and (20.12). First note that, by (20.11), we have that

$$B_e(f^u(z), \eta r|(f^u)'(z)|) \subseteq B_e(\omega, (1 + \|f'\|_E^+\tau^{-1}\eta^{-1}\lambda)\eta r|(f^u)'(z)|). \tag{20.13}$$

In view of Lemma 15.4.1 along with (20.8) and (20.9), we get that

$$m_e\big(B_e(f^u(z), \eta r|(f^u)'(z)|)\big) \leq C(1 + \|f'\|_E^+\tau^{-1}\eta^{-1}\lambda)^{\alpha_t(\omega)}(\eta r|(f^u)'(z)|)^{\alpha_t(\omega)}.$$

So, item (20.3) is proved. Also applying (20.11), Lemmas 15.4.5 and 10.4.4, and (20.9), we see that the point $f^u(z)$ is

$$\begin{aligned}&\big(\|f'\|_E^+\tau^{-1}\lambda r|(f^u)'(z)|, \sigma\tau\|f'\|_E^{-1}\eta\lambda^{-1}, 2^{\alpha_t(\omega)}L(\omega, 2\|f'\|_E^+\theta,\\ &\quad \sigma\tau(2\|f'\|_E^+)^{-1}\eta\lambda^{-1})\big) - \alpha_t(\omega)\text{-s.l.e.}\end{aligned}$$

So, if $\|f'\|_E^+\tau^{-1}\lambda \geq \eta$, then, by Lemma 10.4.5, $f^u(z)$ is

$$\begin{aligned}&\big(\eta r|(f^u)'(z)|, \sigma, (2\|f'\|_E^+\tau^{-1}\lambda\eta^{-1})^{\alpha_t(\omega)}L(\omega, 2\|f'\|_E^+\theta,\\ &\quad \sigma\tau(2\|f'\|_E)^{-1}\eta)\lambda^{-1}\big) - \alpha_t(\omega)\text{-s.l.e.}\end{aligned}$$

If, instead, $\|f'\|_E\tau^{-1}\lambda \leq \eta$, then, again, it follows from (20.11), Lemmas 15.4.5 and 10.4.4, and (20.9) that the point $f^u(z)$ is

$$\big(\eta r|(f^u)'(z)|, \sigma, 2^{\alpha_t(\omega)}L(\omega, 2\theta\tau\lambda^{-1}\eta, \sigma/2)\big) - \alpha_t(\omega)\text{-s.l.e.}$$

So, part (20.4) is also proved.

In order to prove (20.2), suppose, first, that $u = k$. In particular, this is the case if $z \in \bigcup_{j\geq 0} f^{-j}(\Omega(f))$. If $k = 0$, we are done since $\lambda r \leq \tau\theta$ by our hypotheses, while $\tau\theta \leq \beta_f$ by (18.23). So, suppose that $k \geq 1$. Since $0 \leq u \leq n$, it then follows from (20.8) and (20.9) that

$$\text{Comp}(f^{k-1}(z), f, r|(f^u)'(z)|) \subseteq \text{Comp}(f^{k-1}(z), f, \theta\tau),$$

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and, by the choice of k and (15.58), we have that $f^{k-1}(z) \notin B_e(\Omega(f),\theta)$. Therefore, (20.2) follows from (18.22)) and (18.23).

If $u > k$ (so, we are in the case of (20.6) and (20.10)), then $r_{u-1} > \|f'\|_E^+\tau^{-1}\lambda r \geq \|f'\|_E\tau^{-1}\lambda r$. Also using (15.58), we get that

$$r_u = \frac{|f^u(z)-\omega|}{|f^{u-1}(z)-\omega|}|f'(f^{u-1}(z))|^{-1}r_{u-1} \geq \|f'\|_E^{-1}r_{u-1} \geq \tau^{-1}\lambda r.$$

Hence, $\lambda r|(f^u)'(z)| \leq \tau|f^u(z)-\omega|$ and, applying Lemma 15.3.2 and (15.58) $u-k$ times, we conclude that, for every $k \leq j \leq u$,

$$\operatorname{diam}_e\big(\operatorname{Comp}(f^j(z), f^{u-j}, \lambda r|(f^u)'(z)|)\big) \leq \theta\tau < \beta_f.$$

And now, for all $j = k-1, k-2, \ldots, 1, 0$, the same argument as in the case of $u = k$ finishes the proof. ■

Proposition 20.2.2 *Let $f\colon \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be a compactly nonrecurrent elliptic function. Fix an f-pseudo-compact subset E of $J(f)$. Let both ε and λ be positive numbers such that $\varepsilon < \lambda\min\{1,\tau^{-1},\theta^{-1}\tau^{-1}\gamma\}$. If $0 < r < \tau\theta\min\{1,\|f'\|_E^{-1}\}\lambda^{-1}$ and $z \in E\backslash\operatorname{Crit}(J(f))$, then there exists an integer $s = s(\lambda,\varepsilon,r,z) \geq 1$ with the following three properties:*

$$|(f^s)'(z)| \neq 0. \tag{20.14}$$

If $u = u(\lambda,r,z)$ of Proposition 20.2.1 is well defined, then $s \leq u(\lambda,r,z)$.

If either u is not defined or $s < u$, then there exists a critical point $c \in \operatorname{Crit}(f)$ such that

$$|f^s(z)-c| \leq \varepsilon r|(f^s)'(z)|. \tag{20.15}$$

In any case,

$$\operatorname{Comp}\big(z, f^s, (KA^2)^{-1}2^{-N_f}\varepsilon r|(f^s)'(z)|\big) \cap \operatorname{Crit}(f^s) = \emptyset, \tag{20.16}$$

where A was defined in (18.20).

Proof Since $z \notin \operatorname{Crit}(J(f))$ and in view of Proposition 20.2.1, there exists a minimal number $s = s(\lambda,\varepsilon,r,z)$ for which at least one of the following two conditions is satisfied:

$$|f^s(z)-c| \leq \varepsilon r|(f^s)'(z)| \tag{20.17}$$

for some $c \in \operatorname{Crit}(J(f))$ or

$$u(\lambda,r,z) \text{ is well defined} \quad \text{and} \quad s(\lambda,\varepsilon,r,z) = u(\lambda,r,z). \tag{20.18}$$

Since $|(f^s)'(z)| \neq 0$, the parts (20.14) and (20.15) are proved.

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In order to prove (20.16), notice, first, that no matter which of the two numbers s is, in view of Proposition 20.2.1, we always have that

$$\varepsilon r|(f^s)'(z)| \le \varepsilon\lambda^{-1}\theta\tau. \tag{20.19}$$

Let us now argue that, for every $0 \le j \le s$,

$$\mathrm{diam}_e\big(\mathrm{Comp}(f^{s-j}(z), f^j, \varepsilon r|(f^s)'(z)|)\big) \le \beta_f. \tag{20.20}$$

Indeed, if $s = u$, it follows immediately from Proposition 20.2.1 and (20.2) since $\varepsilon \le \lambda$. Otherwise, $|f^s(z) - c| \le \varepsilon r|(f^s)'(z)| \le \varepsilon\lambda^{-1}\theta\tau < \theta$; therefore, by (18.19), $f^s(z) \notin B_e(\Omega(f), \theta)$. Thus, (20.20) follows from (18.22).

Now, by (20.20) and Lemma 18.1.11, there exist $0 \le p \le N_f$, an increasing sequence of integers $1 \le k_1 < k_2 < \cdots < k_p \le s$, and mutually distinct critical points $c_1, c_2, \ldots, c_p$ of f such that

$$\{c_l\} = \mathrm{Comp}\big(f^{s-k_l}(z), f^{k_l}, \varepsilon r|(f^s)'(z)|\big) \cap \mathrm{Crit}(f), \tag{20.21}$$

for every $l = 1, 2, \ldots, p$, and if $j \notin \{k_1, k_2, \ldots, k_p\}$, then

$$\mathrm{Comp}\big(f^{s-j}(z), f^j, \varepsilon r|(f^s)'(z)|\big) \cap \mathrm{Crit}(f) = \emptyset. \tag{20.22}$$

Setting $k_0 = 0$, we shall show by induction that, for every $0 \le l \le p$,

$$\mathrm{Comp}\big(f^{s-k_l}(z), f^{k_l}, (KA^2)^{-1}2^{-l}\varepsilon r|(f^s)'(z)|\big) \cap \mathrm{Crit}(f^{k_l}) = \emptyset. \tag{20.23}$$

Indeed, for $l = 0$, there is nothing to prove. So, suppose that (20.23) is true for some $0 \le l \le p - 1$. Then, by (20.22),

$$\mathrm{Comp}\big(f^{s-(k_{l+1}-1)}(z), f^{k_{l+1}-1}, (KA^2)^{-1}2^{-l}\varepsilon r|(f^s)'(z)|\big) \cap \mathrm{Crit}(f^{k_{l+1}-1}) = \emptyset.$$

So, if

$$c_{l+1} \in \mathrm{Comp}(f^{s-k_{l+1}}(z), f^{k_{l+1}}, (KA^2)^{-1}2^{-(l+1)}\varepsilon r|(f^s)'(z)|),$$

then, by Lemma 8.4.3 applied for holomorphic maps $H = f$, $Q = f^{k_{l+1}-1}$, and the radius $R = (KA^2)^{-1}2^{-(l+1)}\varepsilon r|(f^s)'(z)| < \gamma$, we get that

$$\begin{aligned}
&|f^{s-k_{l+1}}(z) - c_{l+1}| \\
&\quad\le KA^2|(f^{k_{l+1}})'(f^{s-k_{l+1}}(z))|^{-1}(KA^2)^{-1}2^{-(l+1)}\varepsilon r|(f^s)'(z)| \\
&\quad= 2^{-(l+1)}\varepsilon r|(f^{s-k_{l+1}}(z))'| \\
&\quad\le \varepsilon r|(f^{s-k_{l+1}})'(z)|,
\end{aligned}$$

which contradicts the definition of s and proves (20.23) for $l + 1$. In particular, it follows from (20.23) that

$$\mathrm{Comp}\big(z, f^s, (KA^2)^{-1}2^{-N_f}\varepsilon r|(f^s)'(z)|\big) \cap \mathrm{Crit}(f^s) = \emptyset.$$

The proof is finished. ■

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We will also need the following similar result.

Lemma 20.2.3 *Let $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be a compactly nonrecurrent elliptic function. Assume that $\Omega(f) = \emptyset$. Then there exist two constants $a, \xi > 0$ such that the following holds. Suppose that*

$$z \in J(f)\setminus \bigcup_{n=0}^{\infty} f^{-n}(\{\infty\} \cup \mathrm{Crit}(f)).$$

Suppose also that $r \in (0, \gamma(a\xi)^{-1})$, where $\gamma > 0$ was defined in (18.22).

Then there exists an integer $s \geq 0$ with the following properties:

(a) *$ra\xi|(f^s)'(z)| \geq \gamma$, or*
(b) *$ra\xi|(f^s)'(z)| < \gamma$,*
and
(c) *there exists a critical point $c \in \mathrm{Crit}(J(f))$ such that $|(f^s)(z) - c| < r\xi|(f^s)'(z)|$, or*
(d) *there exists a pole $b \in f^{-1}(\infty)$ such that $|(f^s)(z) - b| < r\xi|(f^s)'(z)|$.*

In either case,

$$\mathrm{Comp}\big(z, f^s, 2\xi r|(f^s)'(z)|\big) \cap \mathrm{Crit}(f^s) = \emptyset.$$

Proof Put $a = 2KA^2 2^{N_f}$, where A was defined in (18.20). Fix $\rho \in (0, 1/2)$ so small that, for every $w \in \mathbb{C}\setminus(\mathrm{Crit}(f) \cup f^{-1}(\infty))$, the map f restricted to the set

$$B_e\big(w, 2\rho\, \mathrm{dist}_e\big(w, \mathrm{Crit}(f) \cup f^{-1}(\infty)\big)\big)$$

is one-to-one. Set $\xi = 2^{-4}\rho$. Take $\lambda > 0$ in Proposition 20.2.2 such that $\varepsilon > 0$ appearing there can be taken to be equal to $a\xi$. In view of Corollary 18.3.6, there exists a least integer $n \geq 0$ such that $ra\xi|(f^n)'(z)| \geq \gamma$. Since $r < \gamma(a\xi)^{-1}$, we see that $n \geq 1$. If there exists an integer $0 \leq j \leq n-1$ satisfying (c) or (d), take s to be the least one. Otherwise, take $s = n$. By the definition of n, it follows from (18.22) that

$$\mathrm{diam}_e(\mathrm{Comp}(z, f^k, 2\xi r|(f^k)'(z)|)) < \beta_f$$

for all $k = 0, \ldots, n-1$. Thus, we see that (20.20) is satisfied if $s \leq n-1$ and the proof of the last formula in our lemma is complete by verbatim repetition of the fragment of the proof of Lemma 20.2.2 from "Now, by (20.20)" to its end. If $s = n$, the same argument shows that

$$\mathrm{Comp}\big(z, f^{n-1}, 2\xi r|(f^{n-1})'(z)|\big) \cap \mathrm{Crit}(f^{n-1}) = \emptyset. \tag{20.24}$$

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By the choice of ξ and the definition of n, we also know that the map f^{n-1} restricted to the ball $B_e(f^{n-1}(z), 16\xi r|(f^{n-1})'(z)|)$ is injective. Thus, by the $\frac{1}{4}$-Koebe Theorem (Theorem 8.3.3),

$$f\big(B_e(f^{n-1}(z), 16\xi r|(f^{n-1})'(z)|)\big) \supset B_e(f^n(z), 4\xi r|(f^n)'(z)|);$$

therefore,

$$\mathrm{Comp}\big(f^{n-1}(z), f, 2\xi r|(f^{n-1})'(z)|\big) \subseteq B_e(f^{n-1}(z), 16\xi r|(f^{n-1})'(z)|).$$

Combining this with (20.24) and injectivity of f restricted to

$$B_e(f^{n-1}(z), 16\xi r|(f^{n-1})'(z)|),$$

we conclude that

$$\mathrm{Comp}\big(z, f^n, 2\xi r|(f^n)'(z)|\big) \cap \mathrm{Crit}(f^n) = \emptyset.$$

We are done. ■

20.3 Conformal Measures for Compactly Nonrecurrent Regular Elliptic Functions: Atomlessness, Uniqueness, Ergodicity, and Conservativity

In this section, we continue dealing with conformal measures. We already have their existence, and our goal now is to prove their uniqueness, atomlessness, ergodicity, and conservativity. This will require stronger hypotheses than mere compact nonrecurrence. In fact, it will require one more hypothesis. This hypothesis is the regularity of a compactly nonrecurrent elliptic function introduced at the beginning of Section 18.4; see, especially, (18.51). First, we need it to be able to show that the h-conformal measure constructed in Theorem 20.1.1 is atomless. This, in turn, is a prerequisite for, essentially all, our considerations concerning geometric measures (Hausdorff and packing) and measurable dynamics with respect to the measure class generated by the conformal measure m_h. In this book, we need regularity from the proof of Lemma 20.3.9 onward. Let us record the following immediate observation.

Observation 20.3.1 Every compactly nonrecurrent elliptic function $f\colon \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ with $\mathrm{Crit}_\infty(f) = \emptyset$ is regular.

This simple observation starkly indicates that the class of all regular nonrecurrent elliptic functions is large indeed; see also the entire Section 19 devoted to examples of nonrecurrent elliptic functions. As an immediate consequence of Observation 20.3.1, we have the following corollary.

Corollary 20.3.2 *Every expanding and parabolic elliptic function is regular.*

Another sufficient condition, immediately following from Theorem 17.3.1 for a nonrecurrent elliptic function to be regular, is this.

Proposition 20.3.3 *If* $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ *is a compactly nonrecurrent elliptic function and*

$$\frac{2q_{\max}(f)}{q_{\max}(f)+1} > \frac{2l_\infty}{l_\infty + 1},$$

then f *is regular.*

Now we derive from (18.51) a technical condition, (20.27), which will be directly needed in our considerations involving the continuity of conformal measures. It immediately follows from (18.51) that, for every $c \in \mathrm{Crit}_\infty(f)$, $h > \frac{2p_cq_c}{p_cq_c+1}$. Hence,

$$\frac{p_c - 1}{p_c}h < (q_c+1)h - 2q_c.$$

So there exists $h_- \in (1,h)$ such that

$$\frac{p_c - 1}{p_c}h_- < (q_c+1)h_- - 2q_c; \tag{20.25}$$

therefore, there exists $\kappa_c > 0$ such that

$$\frac{p_c - 1}{p_c}h_- < \kappa_c < (q_c+1)h_- - 2q_c. \tag{20.26}$$

The right-hand side of this formula is equivalent to the following:

$$\left(\frac{h_- - \kappa_c}{2-\kappa_c}\right)\left(\frac{q_c+1}{q_c}\right) > 1. \tag{20.27}$$

We now pass to more general considerations. Let m_s be a Borel probability measure on $\mathbb{C}$ and m_e be its Euclidean version, i.e.,

$$\frac{dm_e}{dm_s}(z) := (1+|z|^2)^t.$$

We shall prove the following.

Lemma 20.3.4 *If* $z \in \mathbb{C}$, $r_n \searrow 0$, *and there are two constants* $\underline{M} \le \overline{M}$ *such that*

$$\underline{M} \le \liminf_{n\to\infty} \frac{m_e(B_e(z,r_n))}{r_n^t} \le \limsup_{n\to\infty} \frac{m_e(B_e(z,r_n))}{r_n^t} \le \overline{M},$$

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then

$$\limsup_{n\to\infty} \frac{m_s\big(B_s(z,(2(1+|z|^2))^{-1}r_n)\big)}{((2(1+|z|^2))^{-1}r_n)^t} \le 2^t\overline{M}$$

and

$$\liminf_{n\to\infty} \frac{m_s\big(B_s(z,2(1+|z|^2)^{-1}r_n)\big)}{(2(1+|z|^2)^{-1}r_n)^t} \ge 2^{-t}\underline{M}.$$

Proof Since, for every $r > 0$ sufficiently small,

$$B_e(z,2^{-1}(1+|z|^2)^{-1}r) \subseteq B_s(z,r) \subseteq B_e(z,2(1+|z|^2)r)$$

and since

$$\lim_{r\searrow 0} \frac{m_e(B_e(z,r))}{m_s(B_e(z,r))} = (1+|z|^2)^t,$$

we get that

$$\limsup_{n\to\infty} \frac{m_s\big(B_s(z,(2(1+|z|^2))^{-1}r_n)\big)}{((2(1+|z|^2))^{-1}r_n)^t} \le \lim_{n\to\infty} \frac{m_s(B_e(z,r_n))}{2^{-t}(1+|z|^2)^{-t}r_n^t} = 2^t\overline{M}$$

and

$$\liminf_{n\to\infty} \frac{m\big(B_s(z,2(1+|z|^2)^{-1}r_n)\big)}{(2(1+|z|^2)^{-1}r_n)^t} \ge \lim_{n\to\infty} \frac{m_s(B_e(z,r_n))}{2^t(1+|z|^2)^{-t}r_n^t} = 2^{-t}\underline{M}.$$

We are done. ■

Assuming that the compactly nonrecurrent elliptic function $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is regular, our first goal is to show that the h-conformal measure m proven to exist in Theorem 20.1.1 is atomless and that

$$\mathrm{H}_s^h(J(f)) = 0$$

whenever $h < 2$. The regularity assumption will be needed only from Lemma 20.3.9 onward. We will now consider for f almost t-conformal measures ν with $t \ge 1$. The notion of upper estimability introduced in Definition 10.4.2 is considered with respect to the Euclidean almost t-conformal measure ν_e. Recall that $l = l(f) \ge 1$ is the integer produced in Lemma 18.2.15 and put

$$\begin{aligned} R_l(f) &:= \inf\left\{R(f^j,c)\colon c \in \mathrm{Crit}(f) \quad\text{and}\quad 1 \le j \le l(f)\right\} \\ &= \min\left\{R(f^j,c)\colon c \in \mathrm{Crit}(f)\cap\mathcal{R} \quad\text{and}\quad 1 \le j \le l(f)\right\} < \infty \end{aligned} \tag{20.28}$$

and

$$\begin{aligned} A_l(f) &:= \sup\left\{A(f^j,c)\colon c \in \mathrm{Crit}(f) \quad\text{and}\quad 1 \le j \le l(f)\right\} \\ &= \max\left\{A(f^j,c)\colon c \in \mathrm{Crit}(f)\cap\mathcal{R} \quad\text{and}\quad 1 \le j \le l(f)\right\}, \end{aligned} \tag{20.29}$$

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where the numbers $R(f^j, c)$ and $A(f^j, c)$ are defined in Section 8.4. Since

$$\overline{O_+(f(\mathrm{Crit}_c(J(f))))}$$

is a compact f-invariant subset of $\mathbb{C}$ (so disjoint from $f^{-1}(\infty)$) and since

$$\overline{\mathrm{PC}_c^0(f)} = \overline{O_+(\mathrm{Crit}_c(J(f)))} = \mathrm{Crit}_c(J(f)) \cup \overline{O_+(f(\mathrm{Crit}_c(J(f))))},$$

we have the following straightforward but useful fact.

Lemma 20.3.5 *If $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is a compactly nonrecurrent elliptic function, then the set $\overline{\mathrm{PC}_c^0(f)}$ is f-pseudo-compact.*

Recall, for the purpose of proving the next two lemma, that the sequence $\{Cr_i(f)\}_{i=1}^p$ was defined inductively by (18.33) and the sequence $\{S_i(f)\}_{i=1}^p$ was defined by (18.35), while the number p, here and in what follow in this section, comes from Lemma 18.2.11(c).

Since the number N_f of equivalence classes of the relation $\sim_f$ between critical points of an elliptic function $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is finite, looking at Lemmas 18.2.15 and 17.6.6, the following lemma follows immediately from Lemma 10.4.10.

Lemma 20.3.6 *Let $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be a compactly nonrecurrent elliptic function. Fix an integer $1 \le i \le p-1$. If $R_i^{(u)} > 0$ is a positive constant and $t \longmapsto C_{t,i}^{(u)} \in (0, \infty)$, $t \in [1, \infty)$, is a continuous function such that all points $z \in \overline{\mathrm{PC}_c^0(f)}_i$ are $(r, C_{t,i}^{(u)})$-t-u.e. with respect to some Euclidean almost t-conformal measure ν_e (with $t \ge 1$) for all $0 < r \le R_i^{(u)}$, then there exists a continuous function $t \longmapsto \tilde{C}_{t,i}^{(u)} > 0$, $t \in [1, \infty)$, such that all critical points $c \in Cr_{i+1}(f)$ are $(r, \tilde{C}_{t,i}^{(u)})$-$t$-u.e. with respect to the measure ν_e for all $0 < r \le A_l^{-1} R_i^{(u)}$.*

In the above lemma, the superscript u stands for "upper." In the lemma below, it has the same connotation. The number u is also used to denote the value of the function $u(\lambda, r, z)$ defined in Proposition 20.2.1. This should not cause any confusion.

Lemma 20.3.7 *Let $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be a compactly nonrecurrent elliptic function. Fix an integer $1 \le i \le p$. If $R_{i,1}^{(u)} > 0$ is a positive constant and $[1, +\infty) \ni \longmapsto C_{t,i,1}^{(u)} \in (0, \infty)$, is a continuous function such that all critical points $c \in S_i(f)$ are $(r, C_{t,i,1}^{(u)})$-t-u.e. with respect to some Euclidean almost t-conformal measure ν_e (with $t \ge 1$) for all $0 < r \le R_{i,1}^{(u)}$, then there exist a continuous function $[1, +\infty) \ni \longmapsto \tilde{C}_{t,i,1}^{(u)} > 0$, and $\tilde{R}_{i,1}^{(u)} > 0$ such that all*

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points $z \in \overline{\mathrm{PC}_c^0(f)_i}$ *are* $(r, \tilde{C}_{t,i,1}^{(u)})$*-*$t$*-u.e. with respect to the measure* ν_e *(with* $t \geq 1$*) for all* $0 < r \leq \tilde{R}_{i,1}^{(u)}$.

Proof Put

$$\varepsilon := 2K(KA^2)2^{N_f},$$

where $A \geq 1$ was defined in (18.20). Then fix $\lambda > 0$ so large that

$$\varepsilon < \lambda \min\left\{1, \tau^{-1}, \theta^{-1}\tau^{-1} \min\{\rho, R_{i,1}^{(u)}/2\}\right\}, \tag{20.30}$$

where ρ was defined in (18.38). We shall show that one can take

$$\tilde{R}_{i,1}^{(u)} := \min\left\{\tau\theta\lambda^{-1} \min\left\{1, \|f'\|_{\overline{\mathrm{PC}_c^0(f)_i}}^{-1}\right\}, R_{i,1}^{(u)}, 1\right\}$$

and

$$\tilde{C}_{t,i,1}^{(u)} := \max\{K^2 2^t C_{t,i,1}^{(u)}, K^{2t} B_t\},$$

where $B_t = B_t(\lambda, \eta) > 0$ comes from Proposition 20.2.1 with $\eta = 2K$.

Consider $0 < r \leq \tilde{R}_{i,1}^{(u)}$ and $z \in \overline{\mathrm{PC}_c^0(f)_i}$. If $z \in \mathrm{Crit}(J(f))$, then $z \in \mathrm{Crit}_c(J(f))$ and $z \in S_i(f)$, and we are, therefore, done. Thus, we may assume that $z \notin \mathrm{Crit}(J(f))$. Let $s = s(\lambda, \varepsilon, r, z)$. By the definition of ε,

$$2Kr|(f^s)'(z)| = (KA^2)^{-1}2^{-N_f}\varepsilon r|(f^s)'(z)|. \tag{20.31}$$

Suppose first that $u(\lambda, r, z)$ is well defined and $s = u(\lambda, r, z)$. Then, by item (20.3) in Proposition 20.2.1 or by item (20.5) in Proposition 20.2.1, we see that the point $f^s(z)$ is $(2Kr|(f^s)'(z)|, B_t)$-t-u.e. Using (20.31), we obtain, from item (20.16) in Proposition 20.2.2 and Lemma 10.4.7, that the point z is $(r, K^{2h}B_t)$-t-u.e..

If either u is not defined or $s < u(\lambda, r, z)$, then, in view of item (20.16) in Proposition 20.2.2, there exists a critical point $c \in \mathrm{Crit}_c(J(f))$ such that $|f^s(z) - c| \leq \varepsilon r|(f^s)'(z)|$. Since $s \leq u$, by Proposition 20.2.1 and (20.30), we get that

$$2Kr|(f^s)'(z)| \leq \varepsilon r|(f^s)'(z)| < \min\{\rho, R_{i,1}^{(u)}/2\}. \tag{20.32}$$

Since $z \in \overline{\mathrm{PC}_c^0(f)_i}$, this implies that $c \in S_i(f)$. Therefore, using (20.32), the assumptions of Lemma 20.3.7, and (20.31) and then applying item (20.16) in Proposition 20.2.2 (remember that, by Lemma 20.3.5, the set $\overline{\mathrm{PC}_c^0(f)}$ is f-pseudo-compact) and Lemma 10.4.7, we conclude that z is $(r, K^2 2^t C_{t,i,1}^{(u)})$-$t$-u.e. The proof is complete. ∎

Given an arbitrary integer $k \geq 1$, recall that, for any pole b of f^k, the number q_b denotes its multiplicity and $B_b^k(R)$ is the connected component of $f^{-k}(B_\infty^*(R))$ containing b. We have proved Lemma 4.21 in [**KU4**] with no constraints imposed on the elliptic function f. In fact, the following more general lemma is true (with the same proof), where f^{-1} is replaced by f^{-k}.

Lemma 20.3.8 *Let $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be an elliptic function. Fix an integer $k \geq 1$ and a point $b \in f^{-k}(\infty)$.*

If ν_e is a Euclidean almost t-conformal measure with $t > \frac{2q_b}{q_b+1}$ such that $\nu_e(b) = 0$, and if m is the h-conformal measure proven to exist in Theorem 20.1.1, then

$$\nu_e(B_b^k(R)) \preceq R^{2-\frac{q_b+1}{q_b}t}$$

and

$$m_e(B(b,r)) \succeq r^{(q_b+1)h-2q_b}$$

for all sufficiently small radii $0 < r \leq 1$.

Proof It follows from Lemma 17.6.6 that $m_e(\{z \in \mathbb{C} \colon R \leq |z| < 2R\}) \asymp R^2$ and $\nu_e(\{z \in \mathbb{C} \colon R \leq |z| < 2R\}) \preceq R^2$ for all $R > 0$ large enough. It, therefore, follows from (17.13) that

$$m_e\big(B_b^k(R) \backslash \overline{B_b^k(2R)}\big) \asymp R^2 R^{-\frac{q_b+1}{q_b}h} \tag{20.33}$$

and

$$\nu_e\big(B_b^k(R) \backslash \overline{B_b^k(2R)}\big) \preceq R^2 R^{-\frac{q_b+1}{q_b}t}. \tag{20.34}$$

Now fix $r > 0$ so small that $R = (r/L_k)^{-q_b}$ is large enough for (20.33) and (20.34) to hold. Using (17.16) and (20.34), we get that

$$\begin{aligned}
\nu_e(B_b^k(R)) &= \nu_e\left(\bigcup_{j \geq 0} (B_b^k(2^j R) \backslash \overline{B_b^k(2^{j+1} R)})\right) \\
&= \sum_{j=0}^{\infty} \nu_e(B_b^k(2^j R) \backslash \overline{B_b^k(2^{j+1} R)}) \\
&\preceq \sum_{j=0}^{\infty} (2^j R)^2 (2^j R)^{-\frac{q_b+1}{q_b}t} \\
&= R^{2-\frac{q_b+1}{q_b}t} \sum_{j=0}^{\infty} 2^{j\left(2-\frac{q_b+1}{q_b}t\right)}
\end{aligned}$$

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$$= L_k^{q_b\left(2-\frac{q_b+1}{q_b}t\right)} r^{(q_b+1)t-2q_b} \sum_{j=0}^{\infty} 2^{j\left(2-\frac{q_b+1}{q_b}t\right)}$$

$$\asymp r^{(q_b+1)t-2q_b}, \tag{20.35}$$

where the last comparability sign holds since $\frac{q_b+1}{q_b}t > 2$. We are done with the first part of our lemma.

Now replace ν_e by m_e and t by h (which is greater than $\frac{2q_b}{q_b+1}$ because of Theorem 17.3.1) in the above formula. In this case, the "$\preceq$" sign in (20.35) can, by virtue of (20.33), be replaced by the comparability sign "$\asymp$." Since the first equality sign in (20.35) becomes "$\geq$" (we have not ruled out the possibility that $m_e(b) > 0$ yet) and $m_e(B(b,r)) \geq m_e(B_b(R))$, we are also done in this case. ■

From now onward, in all our considerations in this chapter, we assume that $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ to be a compactly nonrecurrent regular elliptic function.
We shall now prove the following.

Lemma 20.3.9 *If $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is a compactly nonrecurrent regular elliptic function, then the h-conformal measure m_h, for $f: J(f) \longrightarrow J(f) \cup \{\infty\}$, proven to exist in Theorem 20.1.1, is atomless.*

Proof By induction on $i = 0, 1, \ldots, p$ (remember that p comes from Lemma 18.2.11(c)), it follows from Lemma 20.3.7 (this lemma provides the base of induction as $S_0(f) = \emptyset$ and, simultaneously, contributes to the inductive step), Lemma 20.3.6, and Lemma 18.2.14 that there exists a continuous function $t \longmapsto C_t \in (0, \infty)$, $t \in [1, \infty)$, such that if ν is an arbitrary almost t-conformal measure on $J(f)$, then

$$\nu_e(B(x,r)) \leq C_t r^t \tag{20.36}$$

for all $x \in \overline{\mathrm{PC}_c^0(f)}$ and all $r \leq r_0$ for some $r_0 > 0$ sufficiently small. Consider now the almost s_j-conformal measures

$$m_j^s := m_{V_j}, \quad j \geq 1,$$

and their Euclidean versions

$$m_j^e := \left(m_{V_j}\right)_e,$$

both introduced at the beginning of the proof of Theorem 17.6.7, where the numbers $s_j = s(V_j)$ also come from the proof of Theorem 17.6.7. Letting $j \to \infty$ and recalling that, according to Theorem 20.1.1, $m_{h,s}$ is a weak limit, coming from Claim 2° stated in the proof of Theorem 17.6.7, of measures m_j^s, $j \geq 1$, we see from (20.36) that

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$$m_{h,e}(B(x,r)) \le C_h r^h \tag{20.37}$$

for all $x \in \overline{\mathrm{PC}_c^0(f)}$ and all $r \le r_0$. It now follows from Lemma 20.3.4 that

$$\limsup_{r \searrow 0} \frac{m_s(B(x,r))}{r^h} \le 2^h C_h \tag{20.38}$$

for all $x \in \overline{\mathrm{PC}_c^0(f)}$. In particular,

$$m_{h,s}(\mathrm{Crit}_c(f)) = 0. \tag{20.39}$$

Now fix $k \ge 1$, $b \in f^{-k}(\infty)$, and $u \in \left(\frac{2q_b}{q_b+1}, h\right)$. Consider all integers $j \ge 1$ so large that $s_j \ge u$. Since $m_j^e(f^{-k}(\infty)) \le m_j^e(f^{-k}(V_j)) = 0$, it follows from Lemma 20.3.8 that

$$m_j^e(B_b^k(R)) \preceq R^{2-\frac{q_b+1}{q_b}s_j} \le R^{2-\frac{q_b+1}{q_b}u}.$$

Hence, $m_{h,e}(b) = 0$. Since $m_{h,s}$ and $m_{h,e}$ are equivalent on $\mathbb{C}$, this gives $m_{h,s}(b) = 0$. Consequently,

$$m_{h,s}\left(\bigcup_{n\ge 1} f^{-n}(\infty)\right) = 0. \tag{20.40}$$

In particular,

$$m_{h,s}(\mathrm{Crit}_p(f)) = 0. \tag{20.41}$$

We now move on to dealing with the set $\mathrm{Crit}_\infty(f)$. Since $s_j \nearrow h$ and since $h_- < h$ (h_- was defined in (20.25)), disregarding finitely many js, we may assume without loss of generality that

$$s_j > h_- \tag{20.42}$$

for all $j \ge 1$.

Fix $c \in \mathrm{Crit}_\infty(f)$. Fix also $j \ge 1$ and put

$$t := s_j.$$

Since $\lim_{n\to\infty} f^n(c) = \infty$, there exists an integer $k \ge 1$ such that $q_{b_n} \le q_c$ (where $b_n \in f^{-1}(\infty)$, defined in (18.48), is near $f^n(c)$, and q_c was defined in (18.49)) and

$$|f^n(c)| > \max\left\{1, 2\mathrm{Dist}_e(0, f(\mathrm{Crit}(f)))\right\} \tag{20.43}$$

for all $n \ge k$. We may need in the course of the proof the integer $k \ge 1$ to be appropriately bigger. Put

$$a := f^k(c).$$

We recall that κ_c was defined in (20.26). We shall prove the following.

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Claim 1°. There exists a constant $c_1 \geq 1$, independent of j, such that

$$m_j^e(B_e(a,r)) \leq c_1 r^{\kappa_c}$$

for all $r > 0$ small enough independently of j.

Proof Put $q = q_c$. In view of (20.43) and Theorem 17.1.8, for every $n \geq 1$, there exists a unique holomorphic inverse branch

$$f_n^{-1} \colon B_e\left(f^n(a), \frac{1}{2}|f^n(a)|\right) \longrightarrow \mathbb{C}$$

of f sending $f^n(a)$ to $f^{n-1}(a)$. Then, by Lemma 8.3.13 and (18.48), we have, for every $n \geq k$, that

$$\begin{aligned} f_n^{-1}\left(B_e\left(f^n(a), \frac{1}{4}|f^n(a)|\right)\right) &\subset B_e\left(f^{n-1}(a), \frac{K}{4}|f^n(a)| \cdot |f'(f^{n-1}(a))|^{-1}\right) \\ &\subseteq B_e\left(f^{n-1}(a), C|f^n(a)| \cdot |f^n(a)|^{-\frac{q+1}{q}}\right) \\ &= B_e\left(f^{n-1}(a), C|f^n(a)|^{-\frac{1}{q}}\right) \\ &\subseteq B_e\left(f^{n-1}(a), \frac{1}{2}|f^{n-1}(a)|\right) \end{aligned}$$

with some constant $C > 0$, where the last inclusion was written assuming that $|f^{n-1}(a)| \geq 2C|f^n(a)|^{-\frac{1}{q}}$, which holds if the integer k is taken large enough. So, the composition

$$f_a^{-n} = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_n^{-1} \colon B_e\left(f^n(a), \frac{1}{4}|f^n(a)|\right) \longrightarrow \mathbb{C},$$

sending $f^n(a)$ to a, is well defined and forms a holomorphic branch of f^{-n}. Take $0 < r < \frac{1}{16}|a|$ and let $n + 1 \geq 1$ be the least integer such that

$$r|(f^{n+1})'(a)| \geq \frac{1}{16}|f^{n+1}(a)|.$$

Such an integer exists since $|f'(z)| \asymp |f(z)|^{\frac{q_b+1}{q_b}}$ if z is near a pole b. By definition $n \geq 0$ and since $r < \frac{1}{16}|a|$, we have that

$$r|(f^n)'(a)| < \frac{1}{16}|(f^n)(a)|.$$

Then, by the $\frac{1}{4}$-Koebe Distortion Theorem (Theorem 8.3.3), we have that

$$B_e(a,r) \subseteq f_a^{-n}\big(B_e(f^n(a), 4r|(f^n)'(a)|)\big). \tag{20.44}$$

Now we consider three cases determined by the value of $r|(f^n)'(a)|$.

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Case 1. $\delta(f^{-1}(\infty)) \le r|(f^n)'(a)| < \frac{1}{16}|f^n(a)|$, where $\delta(f^{-1}(\infty))$ comes from (18.47).

In view of (20.44) and the Koebe Distortion Theorem along with almost conformality of the measure m_j^e, we get that

$$\begin{aligned} m_j^e(B_e(a,r)) &\le K^t|(f^n)'(a)|^{-t} m_j^e(B_e(f^n(a), 4r|(f^n)'(a)|)) \\ &\preceq K^t|(f^n)'(a)|^{-t}(4r|(f^n)'(a)|)^2 \\ &\asymp r^2|(f^n)'(a)|^{2-t}. \end{aligned} \tag{20.45}$$

Put

$$q_n := q_{b_n}.$$

Since $t > h_-$ (see (20.42)) and $q_n \le q_c$, it follows from (20.27) that

$$\left(\frac{t-\kappa_c}{2-\kappa_c}\right)\left(\frac{q_n+1}{q_n}\right) > 1.$$

Hence,

$$\begin{aligned} |f^n(a)| < |f^n(a)|^{\frac{t-\kappa_c}{2-\kappa_c}\frac{q_n+1}{q_n}} &\asymp |f'(f^{n-1}(a))|^{\frac{t-\kappa_c}{2-\kappa_c}} \\ &\preceq |(f^n)'(a)|^{\frac{t-\kappa_c}{2-\kappa_c}} = |(f^n)'(a)||(f^n)'(a)|^{\frac{t-2}{2-\kappa_c}}. \end{aligned}$$

Combining this and the Case 1 assumption, we get that

$$r < \frac{1}{16}|(f^n)'(a)|^{-1}|f^n(a)| \preceq |(f^n)'(a)|^{\frac{t-2}{2-\kappa_c}}.$$

Therefore, $r^{2-\kappa_c} \preceq |(f^n)'(a)|^{t-2}$, or, equivalently, $r^2|(f^n)'(a)|^{2-t} \preceq r^{\kappa_c}$. Together with (20.45), we obtain that

$$m_j^e(B(a,r)) \preceq r^{\kappa_c}. \tag{20.46}$$

Case 2. $|f^n(a) - b_n| \le 32A^{\frac{q_{\min}+1}{q_{\min}}} r|(f^n)'(a)| < 32A^{\frac{q_{\min}+1}{q_{\min}}}\delta(f^{-1}(\infty))$, where $A > 0$ was defined in (18.20) and $q_{\min}$ is the minimal order of all critical points and poles.

Put $\alpha := 32A^{\frac{q_{\min}+1}{q_{\min}}}$. Then

$$\begin{aligned} B_e(f^n(a), 4r|(f^n)'(a)|) &\subseteq B_e(b_n, (4+c)r|(f^n)'(a)|) \\ &\subseteq B_e(b_n, (4+\alpha)\delta(f^{-1}(\infty))) \end{aligned}$$

and it follows from Lemma 20.3.8 that

$$m_j^e(B_e(f^n(a), 4r|(f^n)'(a)|)) \preceq (4r|(f^n)'(a)|)^{(q_n+1)t-2q_n}.$$

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Thus,

$$\begin{aligned} m_j^e(B_e(a,r)) &\le K^t|(f^n)'(a)|^{-t}(4r|(f^n)'(a)|)^{(q_n+1)t-2q_n} \\ &\asymp r^{(q_n+1)t-2q_n}|(f^n)'(a)|^{(t-2)q_n} \\ &\le r^{(q_n+1)t-2q_n}. \end{aligned}$$

But, as $q_n \le q_c$ and $t > h_-$, it follows from (20.26) that

$$(q_n+1)t - 2q_n \ge (q_n+1)t - 2q_c > \kappa_c;$$

therefore,

$$m_j^e(B(a,r)) \le r^{\kappa_c}.$$

It remains for us to consider the following.

Case 3. $r|(f^n)'(a)| < \frac{1}{32}A^{-\frac{q_{\min}+1}{q_{\min}}}|f^n(a) - b_n|$.

But then

$$\begin{aligned} r|(f^{n+1})'(a)| &= r|(f^n)'(a)||f'(f^n(a))| \\ &< \frac{1}{32}A^{-\frac{q_{\min}+1}{q_{\min}}}|f^n(a) - b_n|(A|f^{n+1}(a)|)^{\frac{q_n+1}{q_n}} \\ &\le \frac{1}{32}A^{-\frac{q_{\min}+1}{q_{\min}}}A^{\frac{1}{q_n}+1}|f^{n+1}(a)| \\ &\le \frac{1}{32}|f^{n+1}(a)| \\ &\le \frac{1}{16}|f^{n+1}(a)| \end{aligned}$$

contrary to the definition of n. So, Claim 1° is proved. ■

The last step of our proof is to demonstrate the following.

Claim 2°. There exist $c_2 \ge$ and $R > 0$, both independent of j, such that

$$m_j^e(B_e(c,r)) \le c_2 r^{p_c\kappa_c + h(1-p_c)}$$

for all $j \ge 1$ and for all $r \le R$, where p_c is the order of critical point c of the map f^k.

Proof Let $p := p_c \ge 2$. There exists $R > 0$ so small that

$$f^k(B_e(c),R) \subseteq B_e(f^k(c), 2^{-4}|f^k(c)|)$$

and that there exists $M \ge 1$ such that

$$M^{-1}|z-c|^p \le |f^k(z) - f^k(c)| \le M|z-c|^p$$

and

$$M^{-1}|z-c|^{p-1} \le |(f^k)'(z)| \le M|z-c|^{p-1}$$

for all $z \in B_e(c,R)$. Thus, for all $k \ge 0$ and all $r \le R$,

$$f^k(A(c;2^{-(l+1)}r, 2^{-l}r)) \subseteq A\big(f^k(c); M^{-1}r^p 2^{-p(l+1)}, Mr^p 2^{-pl}\big).$$

Since the map $f^k_{|B_e(c,R)}$ is p-to-one, using almost conformality of the measure m_j^e and the right-hand side of (20.26), we get that

$$\begin{aligned} m_j^e\Big(A\Big(f^k(c);\ M^{-1}r^p 2^{-p(l+1)}, Mr^p 2^{-pl}\Big)\Big) & \\ &\ge \frac{1}{p}M^{-h}(2^{-(l+1)}r)^{t(p-1)} m_j^e\left(A(c;2^{-(l+1)}r, 2^{-l}r)\right) \\ &\ge p^{-1}M^{-h}(2^{-(l+1)}r)^{h(p-1)} m_j^e\left(A(c;2^{-(l+1)}r, 2^{-l}r)\right). \end{aligned}$$

Applying Claim 1°, we, therefore, get that

$$\begin{aligned} & m_j^e(B_e(c,r)) \\ &= \sum_{l=0}^{\infty} m_j^e(A(c, 2^{-(l+1)}r, 2^{-l}r)) \\ &\le pM^h r^{h(1-p)} \sum_{l=0}^{\infty} 2^{h(p-1)(l+1)} m_j^e\big(A\big(f^k(c); M^{-1}r^p 2^{-p(l+1)}, Mr^p 2^{-pl}\big)\big) \\ &\le pM^h c_1 2^{h(p-1)} r^{h(1-p)} \sum_{p=0}^{\infty} 2^{h(p-1)l}(Mr^p 2^{-pl})^{\kappa_c} \\ &= p2^{h(p-1)}c_1 M^{h+\kappa_c} r^{h(1-p)+p\kappa_c} \sum_{l=0}^{\infty} 2^{(h(p-1)-p\kappa_c)l} \\ &= p2^{h(p-1)}c_1 M^{h+\kappa_c}(1-2^{h(p-1)-p\kappa_c})^{-1} r^{p\kappa_c + h(1-p)}, \end{aligned}$$

where writing the last equality sign we used the fact that $p\kappa_c + h(1-p) > 0$ equivalent to the left-hand side of (20.26). Claim 2° is, thus, proved. ■

Repeating again that $p\kappa_c + h(1-p) > 0$, Claim 2° implies that $m_h(c) = 0$. So,

$$m_{h,s}(\mathrm{Crit}_\infty(f)) = 0. \tag{20.47}$$

Along with (20.39)–(20.41) and Theorem 20.1.1, this shows that the measure m_h is atomless and the proof of Lemma 20.3.9 is complete. ■

The argument from the beginning of the proof of this lemma, based on Lemmas 20.3.7 and 20.3.6, gives the following,

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Lemma 20.3.10 *If* $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ *is a compactly nonrecurrent regular elliptic function, then the set* $\overline{\mathrm{PC}_c^0(f)}$ *is uniformly h-upper estimable with respect to the measure* m_h *constructed in Theorem 20.1.1.*

Denote by $\mathrm{Tr}(f) \subseteq J(f)$ the set of all transitive points of f, i.e., the set of points in $J(f)$ such that $\omega(z) = J(f)$. The main and last result of this section is the following.

Theorem 20.3.11 *If* $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ *is a compactly nonrecurrent regular elliptic function, then*

(1)

$$\mathrm{DD_h}(J(f)) = \mathrm{DD}_\chi(J(f)) = \mathrm{HD}\big(J_{er}(f)\big) = \mathrm{HD}\big(J_r(f)\big) = h = \mathrm{HD}(J(f)) \tag{20.48}$$

and there exists a unique spherical h-conformal probability measure m_h *for* $f: J(f) \longrightarrow J(f) \cup \{\infty\}$. *This measure is atomless.*

(2) *The spherical h-conformal measure* m_h *is weakly metrically exact, in particular ergodic and conservative.*

(3) *All other conformal measures are purely atomic, supported on* $\mathrm{Sing}^-(f)$ *with exponents larger than h.*

(4) $m_h(\mathrm{Tr}(f)) = 1$.

In what follows, the h-conformal measure m, either spherical m_s *or its Euclidean version* m_e*, will be denoted by* m_h*. Following the convention of this book, the spherical and Euclidean versions of* m_h *will be, respectively, denoted by* $m_{h,s}$ *and* $m_{h,e}$*.*

Proof Formula (20.48) is a part of Theorem 20.1.1. In view of Lemma 20.3.9, there exists an atomless h-conformal measure m_h for $f: J(f) \longrightarrow J(f) \cup \{\infty\}$. So, the existence part of (1) is done.

Continuing the proof, let $R > 0$ be so large that the ball $B_e(0, R)$ contains a fundamental domain of F. For every $w \in \mathbb{C}$, fix $w' \in B(0, R)$ such that

$$w \sim_f w'.$$

Suppose that ν_e is an arbitrary Euclidean t-conformal measure for f and some $t \geq 0$. By Lemma 17.6.4, $t \geq h$. For each

$$z \in J(f) \backslash \mathrm{Sing}^-(f),$$

let $(x_k(z))_{k=1}^\infty$ be the sequence produced in Proposition 18.3.3. Define, for every $l \geq 1$,

$$Z_l := \big\{z \in J(f) \backslash \mathrm{Sing}^-(f) \colon \eta(z) \geq 1/l\big\}.$$

Fix $l \geq 1$ and assume that $z \in Z_l$. Disregarding finitely many terms if needed, assume without loss of generality that

$$\left|f^{n_k}(z) - x_k(z)\right| < \frac{1}{32Kl} \tag{20.49}$$

for all ≥ 1. Then, for each $k \geq 1$,

$$B\left(f^{n_k}(z), \frac{1}{2l}\right) \subseteq B\left(x_k(z), \frac{1}{l}\right)$$

and the holomorphic inverse branch

$$f_z^{-n_k} : B_e\left(f^{n_k}(z), \frac{1}{2l}\right) \longrightarrow \mathbb{C}$$

produced in Proposition 18.3.4, sending $f^{n_k}(z)$ to z, is well defined. Using conformality of the measure ν along with the $\frac{1}{4}$-Koebe Theorem (Theorem 8.3.3), the Koebe Distortion Theorem I, Euclidean version (Theorem 8.3.8), and Proposition 17.6.2, we get the following:

$$\begin{aligned}
\nu_e\left(B_e\left(z, \frac{1}{16l}|(f^{n_k})'(z)|^{-1}\right)\right) &\leq \nu_e\left(f_z^{-n_k}\left(B_e\left(f^{n_k}(z), \frac{1}{4l}\right)\right)\right) \\
&\leq K^t|(f^{n_k})'(z)|^{-t}\nu_e\left(B_e\left(f^{n_k}(z), \frac{1}{4l}\right)\right) \\
&\leq K^t|(f^{n_k})'(z)|^{-t}\nu_e\left(B_e\left(x_k(z), \frac{1}{2l}\right)\right) \\
&= K^t|(f^{n_k})'(z)|^{-t}\nu_e\left(B_e\left(x_k'(z), \frac{1}{2l}\right)\right) \\
&\leq K^t\nu_e(B_e(0, R+1))|(f^{n_k})'(z)|^{-t}.
\end{aligned} \tag{20.50}$$

Likewise, using Lemma 8.3.13, the Koebe Distortion Theorem I, Euclidean version (Theorem 8.3.8), and Corollary 17.6.3, we get the following:

$$\begin{aligned}
\nu_e\left(B_e\left(z, \frac{1}{16l}|(f^{n_k})'(z)|^{-1}\right)\right) &\geq \nu_e\left(f_z^{-n_k}\left(B_e\left(f^{n_k}(z), \frac{1}{16Kl}\right)\right)\right) \\
&\geq K^{-t}|(f^{n_k})'(z)|^{-t}\nu_e\left(B_e\left(f^{n_k}(z), \frac{1}{16Kl}\right)\right) \\
&\geq K^{-t}|(f^{n_k})'(z)|^{-t}\nu_e\left(B_e\left(x_k(z), \frac{1}{32Kl}\right)\right) \\
&= K^{-t}|(f^{n_k})'(z)|^{-t}\nu_e\left(B_e\left(x_k'(z), \frac{1}{32Kl}\right)\right) \\
&\geq K^t M\left(t, \frac{1}{32Kl}\right)|(f^{n_k})'(z)|^{-t},
\end{aligned} \tag{20.51}$$

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where the constant $M\left(t, \frac{1}{32Kl}\right)$ comes from Corollary 17.6.3. Summarizing (20.50) and (20.51), we obtain that

$$B(\nu_e,l)^{-1}|(f^{n_k})'(z)|^{-t} \le \nu_e\left(B_e\left(z, \frac{1}{16l}|(f^{n_k})'(z)|^{-1}\right)\right) \le B(\nu_e,l)|(f^{n_k})'(z)|^{-t}, \tag{20.52}$$

where $B(\nu_e,l) \geq 1$ is some constant depending only on R, ν_e, and l.

Fix now E, an arbitrary bounded Borel set contained in Z_l. Since $m_{h,e}$ is outer regular, for every $x \in E$, there exists a radius $r(x) > 0$ of the form from (20.52) such that

$$m_{h,e}\left(\bigcup_{x\in E} B_e(x,r(x))\backslash E\right) < \varepsilon. \tag{20.53}$$

Now, by the Besicovitch Covering Theorem, i.e., Theorem 1.3.5, we can choose a countable subcover

$$\{B_e(x_i,r(x_i))\}_{i=1}^{\infty},$$

$r(x_i) \le \varepsilon$, from the cover $\{B_e(x,r(x))\}_{x\in E}$ of E, of multiplicity bounded by some constant $C \geq 1$, independent of the cover. Therefore, by (20.52) and (20.53), we obtain that

$$\begin{aligned}\nu_e(E) &\le \sum_{i=1}^{\infty} \nu_e(B_e(x_i,r(x_i))) \le B(\nu_{h,e},l)\sum_{i=1}^{\infty} r(x_i)^t \\ &\le B(\nu_e,l)B(m_{h,e},l)\sum_{i=1}^{\infty} r(x_i)^{t-h} m_{e,h}(B_e(x_i,r(x_i))) \\ &\le B(\nu_e,l)B(m_{h,e},l)C\varepsilon^{t-h} m_{h,e}\left(\bigcup_{i=1}^{\infty} B_e(x_i,r(x_i))\right) \\ &\le CB(\nu_e,l)B(m_{h,e},l)\varepsilon^{t-h}(\varepsilon + m_{h,e}(E)).\end{aligned} \tag{20.54}$$

In the case when $t > h$, letting $\varepsilon \searrow 0$, we obtain that $\nu_e(Z_l) = 0$. Since

$$J(f)\backslash \mathrm{Sing}^-(f) = \bigcup_{l=1}^{\infty} Z_l,$$

we, therefore, get that

$$\nu_e(J(f)\backslash \cup\ \mathrm{Sing}^-(f)) = 0,$$

which means that $\nu_e(\mathrm{Sing}^-(f)) = 1$. Thus, item (3) of our theorem is proved.

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Suppose now that $t = h$. Then, letting $\varepsilon \searrow 0$, (20.54) takes on the form

$$\nu_e(E) \le CB(\nu_e, l)B(m_{h,e}, l)m_{h,e}(E). \tag{20.55}$$

Since this holds for every integer $l \ge 1$, we, thus, conclude that

$$\nu_e|_{J(f)\backslash \mathrm{Sing}^-(f)} \prec m_{h,e}|_{J(f)\backslash \mathrm{Sing}^-(f)} \asymp m_{h,s}|_{J(f)\backslash \mathrm{Sing}^-(f)}.$$

Reversing the roles of $m_{h,e}$ and ν_e, we infer that

$$\nu_e|_{J(f)\backslash \mathrm{Sing}^-(f)} \asymp m_{h,s}|_{J(f)\backslash \mathrm{Sing}^-(f)}. \tag{20.56}$$

Suppose that $\nu_e(\mathrm{Sing}^-(f)) > 0$. Then there exists

$$y \in \mathrm{Crit}(J(f)) \cup \Omega(f) \cup f^{-1}(\infty)$$

such that $\nu_s(y) > 0$. But then

$$\sum_{\xi \in y^-} |(f^{n(\xi)})'(\xi)|_s^{-h} < +\infty,$$

where $y^- = \bigcup_{n \ge 0} f^{-n}(y)$ and, for every $\xi \in y^-$, $n(\xi)$ is the least integer $n \ge 0$ such that $f^n(\xi) = y$. Hence,

$$\nu_y := \frac{\sum_{\xi \in y^-} |(f^{n(\xi)})'(\xi)|_s^{-h} \delta_\xi}{\sum_{\xi \in y^-} |(f^{n(\xi)})'(\xi)|_s^{-h}}$$

is a spherical h-conformal measure supported on $y^- \subseteq \mathrm{Sing}^-(f)$. This contradicts the, already proven (see (20.56)), fact that the measures $\nu_y|_{J(f)\backslash \mathrm{Sing}^-(f)}$ and $m_{h,s}|_{J(f)\backslash \mathrm{Sing}^-(f)}$ are equivalent and $m_{h,s}(J(f)\backslash \mathrm{Sing}^-(f)) = 1$. Thus, ν_e and $m_{h,s}$ are equivalent.

Let us now prove that any probability spherical h-conformal measure ν_s is ergodic. Indeed, suppose, to the contrary, that $f^{-1}(G) = G$ for some Borel set $G \subseteq J(f)$ with $0 < \nu_s(G) < 1$. But then the two conditional measures ν_G and $\nu_{J(f)\backslash G}$

$$\nu_G(B) := \frac{\nu_s(B \cap G)}{\nu_s(G)} \quad \text{and} \quad \nu_{J(f)\backslash G}(B) := \frac{\nu_s(B \cap (J(f)\backslash G))}{\nu_s(J(f)\backslash G)}$$

would be h-conformal and mutually singular; a contradiction.

If now ν_s is again an arbitrary probability spherical h-conformal measure, then, by a simple computation based on the definition of conformal measures, we see that the Radon–Nikodym derivative $\phi := d\nu_s/dm_{h,s}$ is constant on grand orbits of f. Therefore, by ergodicity of $m_{h,s}$, we conclude that ϕ is constant $m_{h,s}$-a.e. As both $m_{h,s}$ and ν_s are probability measures, this implies that $\phi = 1$ a.e.; hence, $\nu_s = m_{h,s}$. Thus, item (1) of our theorem is established.

Let us now show that the probability spherical h-conformal measure $m_{h,s}$ is conservative. We shall prove first that E, any forward invariant ($f(E) \subseteq E$) Borel subset of $J(f)$, is of measure either 0 or 1. Indeed, suppose to the contrary that

$$0 < m_{h,s}(E) < 1.$$

Let

$$\hat{E} := \Lambda_f + E = \{w + y \colon w \in \Lambda_f,\ y \in E\}.$$

Then the set $\hat{E}$ is Λ_f-translation invariant, i.e.,

$$w + \hat{E} = \hat{E} \tag{20.57}$$

for all $w \in \Lambda_f$. Furthermore,

$$E \subseteq \hat{E},\ m_{h,s}(\hat{E}) > 0,$$

and

$$f(\hat{E}) = f(E) \subseteq E \subseteq \hat{E}.$$

Since $m_{h,s}(E) < 1$ and since f maps the sets of measure $m_{h,s}$ equal to zero into sets of measure $m_{h,s}$ equal to zero, it follows from this that

$$m_{h,s}(\hat{E}) < 1.$$

Since

$$m_{h,s}(\mathrm{Sing}^-(f)) = 0,$$

in order to get a contradiction, it suffices to show that

$$m_{h,s}(\hat{E} \backslash \mathrm{Sing}^-(f)) = 0.$$

Fix an arbitrary point $x \in J(f)$ and an arbitrary radius $R > 0$. Seeking contradiction, suppose that

$$m_{h,e}(B_e(x,R) \backslash \hat{E}) = 0.$$

Then also

$$m_{h,s}(B_e(x,R) \backslash \hat{E}) = 0.$$

By conformality of $m_{h,s}$, we have that $m_{h,s}(f(Y)) = 0$ for all Borel sets $Y \subseteq \mathbb{C}$ such that $m_{h,s}(Y) = 0$. Hence, also using the fact that

$$f^n(B_e(x,R) \backslash \hat{E}) \supseteq f^n(B_e(x,R)) \backslash f^n(\hat{E}), \tag{20.58}$$

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we get that

$$\begin{aligned} 0 = m_{h,s}\big(f^n(B_e(x,R)\backslash \hat{E})\big) &\geq m_{h,s}\big(f^n(B_e(x,R))\backslash f^n(\hat{E})\big) \\ &\geq m_{h,s}\big(f^n(B_e(x,R))\backslash \hat{E}\big) \geq m_{h,s}\big(f^n(B_e(x,R))\big) - m_{h,s}(\hat{E}) \end{aligned} \tag{20.59}$$

for all $n \geq 0$. By virtue of Proposition 17.2.6, there exists an integer $l \geq 1$ such that $f^l(B_e(x,R)) = \widehat{\mathbb{C}}$. In particular,

$$m_{h,s}\big(f^l(B_e(x,R))\big) = 1.$$

Then (20.59) implies that $0 \geq 1 - m_{h,s}(\hat{E})$, which is a contradiction. Consequently,

$$m_{h,e}(B_e(x,R)\backslash \hat{E}) > 0. \tag{20.60}$$

Denote by Z the Borel set of all points $z \in E\backslash(I_\infty(f) \cup \text{Sing}^-(f))$ such that

$$\lim_{r\to 0} \frac{m_{h,e}\big(B(z,r) \cap (\hat{E}\backslash(I_\infty(f) \cup \text{Sing}^-(f)))\big)}{m_{h,e}(B(z,r))} = 1. \tag{20.61}$$

In view of the Lebesgue Density Theorem, i.e., of Theorem 1.3.7, we have that $m_{h,s}(Z) = m_{h,s}(\hat{E})$. Since $m_{h,s}(E) > 0$, there exists at least one point $z \in Z$. Since

$$z \in J(f)\backslash(I_\infty(f) \cup \text{Sing}^-(f)),$$

Proposition 18.3.3 applies. Let $(x_j(z))_{j=1}^\infty$, $\eta(z) > 0$, and an increasing sequence $(n_j)_{j=1}^\infty$ be given by this proposition. Put

$$\delta = \eta(z)/8.$$

It then follows from (20.60) and Proposition 18.3.3 that, for every $j \geq 1$ large enough, we have that

$$m_{h,e}\big(B_e(x_j(z),\delta)\backslash \hat{E}\big) > 0. \tag{20.62}$$

Therefore, as $f^{-1}(J(f)\backslash E) \subseteq J(f)\backslash E$, the standard application of Theorem 8.3.8 and Lemma 10.4.7 shows that

$$\limsup_{r\to 0} \frac{m_{h,e}(B(z,r)\backslash \hat{E})}{m_{h,e}(B(z,r))} > 0, \tag{20.63}$$

which contradicts (20.61). Thus, either

$$m_{h,s}(E) = 0 \quad \text{or} \quad m_{h,s}(E) = 1. \tag{20.64}$$

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Now conservativity is straightforward. One needs to prove that, for every Borel set $B \subseteq J(f)$ with $m_{h,s}(B) > 0$, one has $m_{h,s}(G) = 0$, where

$$G := \left\{ x \in J(f) \colon \sum_{n \geq 0} \mathbb{1}_B(f^n(x)) < +\infty \right\}.$$

Indeed, suppose that $m(G) > 0$. For all $n \geq 0$, let

$$\begin{aligned} G_n &:= \left\{ x \in J(f) \colon \sum_{k \geq n} \mathbb{1}_B(f^n(x)) = 0 \right\} \\ &= \left\{ x \in J(f) \colon f^k(x) \notin B \text{ for all } k \geq n \right\}. \end{aligned}$$

Since

$$G = \bigcup_{n \geq 0} G_n,$$

there exists $k \geq 0$ such that $m_{h,s}(G_k) > 0$. Since all the sets G_n are forward invariant, we get from (20.64) that

$$m_h(G_k) = 1.$$

But, on the other hand, all the sets $f^{-n}(B)$, $n \geq k$, are of positive measure and are disjoint from G_k. This contradiction finishes the proof of conservativity of $m_{h,s}$. Item (2) is established. Because of (2) and since $\operatorname{supp}(m_{h,s}) = J(f)$, we have that $m_{h,s}(\operatorname{Tr}(f)) = 1$, i.e., item (4). The proof of Theorem 20.3.11 is complete. ■

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