7

Existential Problems

7.1 Some Questions

The topics of this chapter can be motivated from several points of view. The first one is straightforward enough: what kind of existence theorems can be counterparts of the uniqueness statement that is the Borel conjecture?

Another point of view is that, given the Borel conjecture says that the fundamental group of an aspherical manifold determines it, it should follow that all properties of aspherical manifolds should be properties of their fundamental groups alone, and it is interesting to then inquire to what extent we can create such a dictionary.

A third point of view, like in Chapter 6, is to take the geometrical reasoning of the Borel conjecture seriously, and try to elaborate on the connection between the Riemannian geometry and topology.

Let me now be more specific, starting, as we often do, with some theorems of Borel (see Borel, 1983) regarding compact group actions on compact aspherical manifolds. First, he provided a lot of information about the identity component in the following theorem.

Theorem 7.1 *If M is a closed aspherical manifold, and G is a connected Lie group acting effectively on M, then $\pi_1(G) \to \pi_1(M)$, induced by the inclusion of an orbit, is injective, with image lying in the center of $\pi_1(M)$.*

Corollary 7.2 *Under the above conditions G is a torus, and if, in addition, $\pi_1 M$ is centerless, it must be trivial.*

For, in a noncommutative compact Lie group, the maximal torus does not inject on the fundamental group. And, the centrality of the image of the orbit is because there is a continuous map $G \times M \to M$.

This theorem concentrates one's attention on finite groups.

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Theorem 7.3 *If $\pi_1 M$ is centerless, then for any G-action on M the map $G \to \mathrm{Out}(\pi_1(M))$ is injective.*

Note that this also includes the previous corollary (since $\mathrm{Out}(\pi_1(M))$ is discrete). Note that Mostow rigidity implies that, for $M = K\backslash G/\Gamma$, we have $\mathrm{Isom}(M) = \mathrm{Out}(\pi_1(M))$. So the isometry group in this case is as large as can be. Of course, since this is a finite group, there are many other metrics whose isometry groups are as large as this one.

Among questions that we will focus on in this chapter are the following:

(1) If the fundamental group of an aspherical manifold M has center, does the manifold admit a circle action? (Conner–Raymond conjecture.)
(2) If G is a finite subgroup of $\mathrm{Out}(\pi_1(M))$ for an aspherical manifold whose fundamental group is centerless, is G realized by a group of homeomorphisms of M? (Nielsen realization problem.)
(3) In what senses is the symmetric metric on $K\backslash G/\Gamma$ the most symmetric one?

The discussion can start on general spaces. The first important negative examples are due to Raymond and Scott (1977); they show that for certain three-dimensional nilmanifolds, there are finite subgroups of $\mathrm{Out}(\pi_1(M))$ that are not realized by any group actions at all on any space with the given fundamental group!

These are based on an algebraic obstruction: they show that (in their situation) there is no group extension

$$1 \to \Gamma \to \pi \to G \to 1$$

(where G has the given action on Γ) because of an obstruction that lies in $H^3(G, Z(\Gamma))$. This prevents the group action from being realized on any space with fundamental group Γ and has nothing to do with asphericity!

If the center of Γ is trivial, then an extension always exists, and the phenomenon lies deeper. For example, there is always then a G-action on some space homotopy equivalent to the $K(\pi, 1)$ on which G acts in the desired way (i.e. the regular G-fold cover of $K(\pi, 1)$). However, this will typically be an infinite-dimensional space – e.g. if π has elements of finite order.

This is the result of building the group action as a free one rather than allowing fixed points. (If π has torsion, this will be infinite-dimensional, although, by taking a skeleton, we can always avoid that.)

When there is a global fixed point to the action, then the map $G \to \mathrm{Out}(\pi_1(M))$ lifts to $\mathrm{Aut}(\pi_1(M))$. For actions that are "as aspherical as possible" (i.e. associated to the action of G on $\underline{E\pi}/\Gamma$, i.e. so that on the universal cover all finite subgroups have contractible fixed sets), the converse holds.[1]

The opposite extreme is where *no nontrivial element of G lifts to* Aut(Γ). In that case the group[2] π automatically satisfies Poincaré duality. The Nielsen conjecture being true would then boil down to the following statement.

Conjecture 7.4 *An aspherical Poincaré complex is homotopy-equivalent to a manifold if it has a finite sheeted cover that is.*

(We have tacitly used the fact that a finite complex satisfies Poincaré duality iff a finite sheeted cover does.)

This is a special case of a conjecture of Wall.[3]

Conjecture 7.5 *If π is a group satisfying Poincaré duality, then there is a closed aspherical manifold with fundamental group π.*

This is very natural from the point of view that every uniqueness statement, like the Borel conjecture, should have an existence statement that goes along with it. Thus, one should ask whether every $K(\pi, 1)$ which could conceivably be a manifold is one.[4,5]

A similar question would be:

Conjecture 7.6 *An aspherical manifold is a product of two manifolds iff its fundamental group is a nontrivial product.*

This is a consequence of the Wall conjecture, as we leave as an exercise.

One might be so bold as to make similar conjectures about fiber bundles and so on. The stage is set for the problems we plan to address in this chapter. Section 7.2 begins with the Wall conjecture.

1 One can construct actions on finite aspherical complexes where there is a map $G \to \text{Aut}\left(\pi_1(M)\right)$, but there are no fixed points for the action. This requires G not to be a p-group. The manifold case is more difficult. In recent work with Cappell and Yan (Cappell *et al.*, 2020) we show that indeed, for each non-p-group, one can find actions without fixed points that lift to Aut(Γ).

2 That is, $K(\pi, 1)$ does.

3 Wall did not conjecture this: he asked it as a question (Wall, 1979). We are here commiting a standard historical crime of attributing the positive answer to a question as a conjecture of the proponent if it lasts more than a few minutes. This is especially venal in my case, since I do not believe this conjecture.

4 I had originally planned on using Wall's conjecture as the first existential problem, but I decided instead to follow Borel's trail.

5 Exercise: Show that the question of whether the universal cover of an aspherical manifold is $\mathbb{R}^n$ only depends on the fundamental group.

7.2 The "Wall Conjecture" and Variants

As Poincaré duality would follow from the existence of an aspherical homology manifold, it is much more reasonable[6] to conjecture that that is what exists in the presence of Poincaré duality.

Expanding the Borel conjecture in this way, i.e. in the uniqueness statement, is actually equivalent to the version if one allows ANR DDP (Absolute Neighborhood Retract Disjoint Disk Property) homology manifolds in the class of objects among which the manifold is unique. (If there were a non-resolvable homology manifold homotopy-equivalent to M, namely a $K(\pi, 1)$ manifold, then there would be a manifold homotopy-equivalent to $M \times \mathbb{T}^4$ with different p_1.) And, if one works in this setting, one would at least expect to get the uniqueness of the homology manifold up to s-cobordism.[7]

However, the Nielsen part would be expected, because the DDP homology manifold would have a cover that is (s-cobordant to) a manifold, and that would make it a manifold. The following seems reasonable to me.[8]

Conjecture 7.7 *The question of whether a torsion-free group is the fundamental group of a closed aspherical manifold only depends on the coarse quasi-isometry type of the group.*

This seems to me quite believable, at least modulo the Borel conjecture. I will explain some of the evidence below and give some heuristic. In particular, we will see, following Bartels *et al.* (2010), that it's true for hyperbolic groups.

For hyperbolic groups, something slightly stronger is suspected (the following is an analogue of the "Cannon conjecture", which we will soon get to).

Conjecture 7.8 *The question of whether a torsion-free hyperbolic group is the fundamental group of a closed aspherical manifold only depends on the boundary of the group.*

Let me remind the reader a little about the theory of hyperbolic groups[9] (Gromov, 1987). The property of a group being hyperbolic is a property of its Cayley graph. Perhaps the simplest description would be that all closed curves in the Cayley graph bound "disks" of area that grows linearly in the length of the curve.[10]

[6] At least in my view.

[7] Unfortunately, the surgery exact sequence in Bryant *et al.* (1993) is only completely proved there for homology manifolds that satisfy an orientability condition; I believe it is true in full generality. But, in any case, we would not have uniqueness up to homeomorphism because we don't have any s-cobordism theorem for homology manifolds.

[8] At least in dimensions other than 4, where I have no feeling for what is reasonable.

[9] We already had a brief orientation on this in §2.4.

[10] Or, equivalently, for any Riemannian metric on any compact manifold with that fundamental

However, the more traditional, and probably more intuitive, definition is in terms of "thin triangles." Every triangle in the graph (i.e. a union of three geodesics) is uniformly thin: there is a constant δ such that each side is within δ of the union of the other two sides.

This condition is typical of trees and (the universal cover) of negatively curved Riemannian manifolds. Euclidean spaces of dimension greater than 1 are not hyperbolic, and, for example, hyperbolic groups never contain a $\mathbb{Z}^2$. Part of their joy is that they exist in great profusion. One can add large random relations to non-elementary (i.e. not virtually $\mathbb{Z}$) hyperbolic groups to get new ones; there are gluing or combination theorems for certain amalgamated free products. And, from some point of view, almost all groups are hyperbolic.[11]

The boundary of a hyperbolic group consists of equivalence classes of geodesic rays. An important property of hyperbolic metric spaces is that any "quasi-geodesic," that is, a path that is uniformly embedded in the space, is a finite distance from a geodesic. As a result, this notion of boundary is a coarse quasi-isometry invariant. For trees, this is just the end-point compactification (and usually consists of a Cantor set). For the usual examples of co-compact lattices, the boundary is a topological sphere.

For torsion-free word hyperbolic groups, the boundary is one dimension smaller than the group. Its homology is usually, as in the case of the free group, infinitely generated in at least one dimension. If it is not, then one can show that the group is actually $\mathbb{Z}$ in one dimension, k, and the hyperbolic group is then a Poincaré duality group of dimension $k + 1$.

In dimension 2, the above conjecture is known to be true. According to Eckmann (1986)), all two-dimensional Poincaré duality groups are fundamental groups of surfaces. In dimension 3, in light of the geometricization theorem, this problem is closely related to Cannon's conjecture that torsion-free hyperbolic groups whose boundaries are S^2 are fundamental groups of closed hyperbolic 3-manifolds. Once one gets the 3-manifold in the above statement, the hyperbolic structure will be automatic.[12]

Dimension 4 is out of reach, but in higher dimensions this last conjecture is true.

group, any null-homotopic curve bounds a disk with area linearly bounded by the length of the curve.

11 This last makes one extremely pleased with the result that the Baum–Connes conjecture is true for hyperbolic groups (Lafforgue, 2002; Mineyev and Yu, 2002), even with coefficients (Lafforgue, 2012) and so is the Borel conjecture – and indeed the whole package – (Bartels and Lück, 2012a).

12 One cannot hope in dimension 4 to hyperbolize aspherical manifolds with boundary an S^3 because of complex hyperbolic manifolds and the Gromov–Thurston examples (discussed in §2.3).

Theorem 7.9 (Bartels *et al.*, 2010) *Two torsion-free hyperbolic groups with the same boundary of dimension greater than* 5 *are simultaneously fundamental groups of closed aspherical manifolds or are not.*[13]

Corollary 7.10 *So, if the boundary is a sphere, the hyperbolic group is the fundamental group of a closed aspherical manifold.*

This is as per the conjecture by Wall. (Consider the other group to the fundamental group of a closed hyperbolic manifold.)

To see this, cross with a circle, and then use the Borel conjecture to guarantee that there exists a homology manifold realizing these Poincaré complexes. (This uses the total surgery obstruction, and the result of Bryant *et al.* (1993) that, whenever the total surgery obstruction vanishes, there is a homology manifold realizing the object if one is in dimension greater than 5.) One can then take the cover corresponding to the group $\mathbb{Z}$ (as subgroup) and compactify by gluing to each of these homology manifolds the common boundary. It is not hard to prove that there is stratified homotopy equivalence between these homology-manifold-stratified spaces and thus an element of the relative to the singularity set structures. That group vanishes (consider $S^{\text{strat}}(\mathcal{S}^{n+2} \text{rel}\, \mathcal{S}^n)$ as a typical example), so these covers are h-cobordant, and thus have the same local index.[14]

Is there any good reason to believe Wall's conjecture regarding its aspect that goes beyond the Borel conjecture?

One can try to guess the analogue of pseudo-equivalence (see §6.7) and then consider the Borel and Wall conjectures in this setting. The following seems like a reasonable choice (to me).

Definition 7.11 A space X is "haspherical" if the map $X \to \mathbf{B}\pi_1(X)$ is a $\mathbb{Z}$-homology isomorphism.

Note that the Borel conjecture implies that, if X is haspherical and π is torsion-free, then X is rigid, i.e. has vanishing structure set if it's a manifold (and if it has a boundary, then working relative to the boundary).

So, we now can ask the Wall question: if X is a Poincaré complex and haspherical, is X homotopy-equivalent to a manifold?

[13] If the boundary is a sphere, then the $L^*(\mathbb{Z})$ orientability holds automatically (and indeed a normal invariant can be constructed from the action of the group on the boundary). In general, the Quinn invariant is detected by the controlled at infinity homotopy type of the universal cover of $B\Gamma$. Recent work of Ferry *et al.* (2019) gives information in low dimensions: in particular, it also handles the case of $\mathcal{S}^4$ as boundary.

[14] The original proof, which loses a dimension, is to consider the universal cover of the homology manifolds, compactify these, and glue them together to obtain a connected homology manifold with each of these universal covers being open subsets. Ferry *et al.* (2019) use a different variant by crossing with $\mathcal{S}^1$.

Proposition 7.12 *Assuming the Borel conjecture for π, then X, as above, is homotopy-equivalent to an ANR homology manifold, but not necessarily a manifold.*

It is an interesting question to inquire if any π has an haspherical X that is not a manifold. For π a product of fundamental groups of surfaces of high genus, the answer is no (using the Atiyah–Kodaira fibration and their application to the Novikov conjecture, just as we did in §5.3). However, we will presently see that it's doable with $\pi_1 X$ free abelian, so that the space X is an integral homology torus.

That X is homotopy-equivalent to a homology manifold follows from the argument above. Now, for the counterexample, we start with a torus $\mathbb{T}^k$. We take a regular neighborhood of the 2-skeleton and do a Wall realization applied to an element of the form $x \otimes \mathbb{T}^k$ for a nontrivial element x of $L_0(e)$ on the boundary, ∂, of this regular neighborhood. (This produces a normal cobordism $V\colon \partial \to \partial'$ whose surgery obstruction – as a map to $\partial \times [0,1]$ – is $x \otimes \mathbb{T}^k$ in $L_k(\mathbb{Z}^k)$.) Split the torus along ∂ and glue in a copy of V, by a homeomorphism to the boundary of the regular neighborhood, and by a homotopy equivalence to the complement.

This is almost X. The trouble is that we have not controlled its integral homology. If we could arrange for our normal cobordism V to be a $\mathbb{Z}$-homology equivalence, we would be done. But that is exactly what the Cappell and Shaneson (1974) homology surgery theory is for. We exactly need to know for this that our element vanishes in $\Gamma_k(\mathbb{Z}[\mathbb{Z}^k] \to \mathbb{Z})$. For k odd, this is trivial, because that group is trivial for general reasons about odd Γ groups (they inject in the L-group of their target, which is here $L_{\text{odd}}(e) = 0$). After arranging for the normal cobordism to be an integral homology h-cobordism, we obtain the desired haspherical Poincaré complex. It obviously has a vanishing total surgery obstruction, and is homotopy-equivalent to a homology manifold, but it is not resolvable.

Once we have examples for odd-dimensional tori, we can cross with a circle and get examples in even dimensions as well.

Remark 7.13 One could have asked a different question that might seem more natural: *If a group π satisfies Poincaré duality over $\mathbb{Z}$, is it the fundamental group of a haspherical (homology) manifold?*

The answer to this is no. Apply the Baumslag *et al.* (1980) construction to a finite simply connected Poincaré complex that does not have any normal invariants (i.e. whose Spivak fibration does not have a lifting – see §3.8). This will be an aspherical $\mathbb{Z}$-Poincaré complex, which cannot be a homology

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manifold, because, if it were, you can see that the Spivak fibration would have to be reducible.

Another analogue of the Wall conjecture was suggested by Mike Davis (2000). Accepting the idea that one should only ask for $\mathbb{Z}$-homology manifolds from $\mathbb{Z}$-Poincaré duality, the question becomes[15] *If π is a group (with suitable finiteness properties) that satisfies $R\pi$ Poincaré duality, for a ring R that is a subring of $\mathbb{Q}$, then is there an R-homology manifold with fundamental group π whose universal cover is R-acyclic?*

This conjecture has at least one thing to recommend it over Wall's: Wall's conjecture is so hard[16] partly because it is currently very hard to come up with new $\mathbb{Z}$-Poincaré duality groups that are not manifolds by their very construction. For $\mathbb{Q}$ there is a very natural source – namely, any uniform lattice that has torsion. (The action of π on G/K has finite isotropy, but inverting the orders of these groups restores the Poincaré duality that one would have had in the free situation.)

The bad news is that this conjecture is very badly false.

Theorem 7.14 (Fowler, 2009) *If π is a non-torsion-free uniform lattice, and π contains an element of odd order ($\neq 1$), then there is no ANR $\mathbb{Q}$-homology manifold with fundamental group π and $\mathbb{Q}$-acyclic universal cover.*

It is an open problem whether the same holds in the presence of only 2-torsion. (You will soon see the issue when we sketch the argument in the next paragraph.) However, lest one conjecture that Davis's conjecture is missing a torsion-free hypothesis, Fowler has given examples where the $\mathbb{Q}$-homology manifold exists despite the existence of torsion. Interestingly enough, his construction is a Davis construction (see §2.3).

Here's a sketch of why Fowler's theorem is true. Suppose X^n is a $\mathbb{Q}$-Poincaré complex. Then it has a symmetric signature[17] in $\mathrm{L}^n(\mathbb{Q}\pi)$. This is a rational homotopy invariant.[18] If X were a $\mathbb{Q}$-homology manifold, then one could lift the symmetric signature back to $\mathrm{H}_n(\mathrm{B}\pi, L^*(\mathbb{Q}))$ under the assembly map:

$$\mathrm{H}_n(\mathrm{B}\pi, L^*(\mathbb{Q})) \to \mathrm{L}^n(\mathbb{Q}\pi).$$

[15] Davis actually asked a slightly different question, and only for torsion-free groups, that Fowler (2012) disproved.

[16] If it's false, that is!

[17] The reader might wish to review some of the discussion in §6.7 at this point.

[18] Actually, there is a slight technical issue. In the $\mathbb{Z}$ case, the symmetric signature is defined up to sign unless one chooses an orientation. We have let this go without saying. In the case of $\mathbb{Q}$, the fundamental class can be sent to any nonzero multiple under a rational homotopy equivalence. This can change, e.g. the 1×1 quadratic form (1) to (k) for some positive integer k. This can change the symmetric signature in $\mathrm{L}^*(\mathbb{Q})$ by an element of order 2 or 4. One can live with this issue (say by ignoring the prime 2) or avoid it by keeping track of fundamental classes (like careful people keep track of orientations).

The point is this. When π is torsion-free, this map should be an isomorphism[19] (as part of the Borel package), but when π is not, the right-hand side has additional elements coming from

$$H_n(\underline{E}\pi/\pi, L^*(\mathbb{Q}\pi_x)).$$

The question, then, is where does $\sigma^*(X)$ lie with respect to these extra pieces? Actually it's pretty clear what element it is: we have the equivariant symmetric signature of G/K that lies in the same group and is clearly equal to a lift of $\sigma^*(X)$. At that point, we can use localization theorem technology borrowed from Atiyah and Segal (1968) or (what Fowler does) use a proof of the equivariant Novikov conjecture in this case by going to a g-equivariant symmetric signature of the universal cover G/K in its "bounded L-theory." One gets an obstruction in this way from the ρ-invariant (see §4.10) of the lens space that is normal to a generic point in a stratum of $\pi\backslash G/K$. It is here that one needs a condition. For the free involution on a sphere, the ρ-invariant happens to vanish and the proof breaks down, but for odd primes, the formula in Atiyah and Bott (1967, 1968) shows that the ρ-invariant is never zero and the proof is complete.

Remark 7.15 In many cases where there is 2-primary torsion, the above argument can be applied as well. (In many cases, a more elementary argument using algebraic K-theory suffices Fowler (2012) – a necessary condition is that all the singular strata corresponding to cyclic subgroups have Euler characteristic equal to 0.)

Even the torsion-free case of the Davis question strikes me as unlikely, despite the failure of the ideas above to disprove it.[20] My main reason for hesitation is that we do not have a good theory for surgery on R-homology manifolds for $R \neq \mathbb{Z}$. The assembly map being an isomorphism does not imply that there is a unique homology h-cobordism class of homology manifolds with the given R-homotopy type (as far as I know). Even for manifolds, local surgery theory has a more complicated normal invariant set than ordinary surgery theory (see Taylor and Williams, 1979b).

Despite the falsity of these many variations on the Wall conjecture,[21] we will continue to exploit and expand these ideas in the following section.

[19] And this is a case that is actually known, again by the remarkable paper of Bartels and Lück (2012a).

[20] Obviously, when the Borel package is in place, the method above gives no restriction.

[21] By the way, there is a form that is as well founded as the Borel conjecture: *Suppose X is a Poincaré space whose non-empty boundary is a manifold M (i.e. (X, M) is a Poincaré pair). Then if X is (h)aspherical, it is homotopy-equivalent relative to the boundary to a (unique) manifold,* rel M.

7.3 The Nielsen Problem and the Conner–Raymond Conjecture

The best evidence for the free Nielsen problem comes from Borel conjecture via the Wall conjecture. When there are fixed points, the situation is much more complicated.[22]

We will, following Block and Weinberger (2008) and Cappell *et al.* (2013), concentrate on $G = \mathbb{Z}_2$. In this case, Smith theory determines the 2-adic equivariant homotopy type of the action. Indeed, any action will be pseudo-equivalent to the action of $\mathbb{Z}_2$ on $\underline{\mathrm{E}\pi}/\Gamma$ (in the notation of §7.1).

(This observation explains why, whenever G is a p-group, a lifting to $\mathrm{Aut}(\Gamma)$ guarantees a global fixed point, because the lift to $\underline{\mathrm{E}\pi}$ has a fixed point by Smith's theorem. However, for $\mathbb{Z}_n$, with n composite, there are fixed-point free actions on Euclidean space (see Bredon, 1972) and this argument fails.[23])

In Block and Weinberger (2008) it is observed that, for very low-dimensional fixed-point sets, the fact that the fixed set is an $\mathbb{F}_2$-homology manifold implies that it's a manifold (see Bredon, 1972) and that the map is an equivariant homotopy equivalence. Cappell *et al.* (2013) deal more directly with the fact that it's a pseudo-equivalence. By being more careful, we can achieve:

Theorem 7.16 *There are closed manifolds W with word hyperbolic group fundamental group Γ such that $\mathbb{Z}_2 \subset \mathrm{Aut}(\Gamma)$ but for which there is no involution realizing this homotopy involution. Indeed, this involution is not realized on any closed ANR homology manifold homotopy-equivalent to W.*

We take our inspiration from the Gromov and Piatetski-Shapiro method[24] (see method three in §2.2.3). In other words, we will build two involutions on aspherical manifolds with (incompressible[25] aspherical) boundary W_1 and W_2 so that we have an equivariant homotopy equivalence

$$h\colon \partial W_1 \to \partial W_2.$$

[22] See Farrell and Lafont (2004) for examples of fixed sets of automorphisms of aspherical manifolds that don't have integral Poincaré duality, so the extension does not correspond to an aspherical orbifold.

[23] Thus, for X a finite-dimensional aspherical complex, the lifting condition suffices for prime powers. And, as we mentioned in Footnote 1, conversely, for $\mathbb{Z}_{pq}$, with p and q distinct primes, in Cappell *et al.* (2020), we construct an aspherical manifold with $\mathbb{Z}_{pq}$ in $\mathrm{Aut}(\pi)$, but which has a fixed-point free action by combining the argument in Bredon (1972) with a Davis construction.

[24] We could have been inspired by the way we produced haspherical homology manifolds that are not homotopy-equivalent to manifolds by realizing a surgery obstruction by gluing, or the construction of non-resolvable homology manifolds (Bryant *et al.*, 1993). These, in turn, were inspired by the work Jones (1973) on patch-space decompositions for Poincaré spaces. Of course, we have to recognize a common thread in all these examples.

[25] i.e. π_1-injective

We will arrange that h is homotopic to a homeomorphism h' (this would be automatic if the Borel conjecture were true), so we can form the manifold

$$W = W_1 \cup_{h'} W_2.$$

It is homotopy-equivalent to the result of gluing using h – but that will only give a Poincaré complex with an involution, not a manifold. Since h is not equivariantly homotopic to a homeomorphism, there isn't an obvious reason why W should have an involution in this pseudo-equivalence class (i.e. realizing the same automorphism of π_1) and it will be our problem to eliminate this unlikely possibility.

The involutions and equivariant-homotopy equivalence $h\colon \partial W_1 \to \partial W_2$ will be the UNil counterexamples to the Borel conjecture discussed in the previous chapter (see §6.5). Then the W_i will be built using cobordism theory and relative hyperbolization (see §§2.3 and 2.4).

The UNil obstruction to the Borel uniqueness for ∂W_1 gives rise to a UNil obstruction to the existence of an involution for W. By choosing the initial ∂W_1 to be hyperbolic, the remaining parts of the construction can be done carefully enough to give word hyperbolicity of W.

Remark 7.17 We can use pseudo-equivalences on the boundary to similar effect in making this construction. We have so far only examined carefully examples that come out of equivariant homotopy equivalences and using UNil. Presumably there are also examples that come from Nil (i.e. the simple homotopy condition) or via embedding theory. Possibly pseudo-equivalence allows for phenomena where one would get actions on CW-complexes that cannot be realized on manifolds because there is no $\mathbb{F}_p$ homology manifold $\mathbb{F}_p[\pi]$-homology-equivalent to the fixed set. Among other advantages, these should give rise to examples for p odd, for example.

Now for a few more details.

To obtain an aspherical manifold, one can start with an affine involution on the torus.[26] Such a manifold always bounds equivariantly. We can even make it bound explicitly an equivariant aspherical manifold, so that it is incompressible. This is W_1. We then do the equivariant Wall realization to the free part (as in §6.5). This will produce a smooth involution on the torus with an equivariant null-cobordism (gluing on the null-cobordism of the affine torus). This can be hyperbolized relative to the boundary (Davis *et al.*, 2001) to produce the null-cobordism with involution W_2.

It is a diagram chase involving the equivariant total surgery obstruction

[26] Note that this always obstructs the word hyperbolicity of $\pi_1 W$.

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(and Cappell's splitting theorem) to see that this equivariant homotopy cannot be realized by an involution on an ANR homology manifold; see Block and Weinberger (2008). If the fixed set is of dimension ≤ 2, this suffices.[27]

However, one can get around this by considering the algebraic mapping cone of the pseudo-equivalence $M \to W$. It gives an element in $S^{\text{alg}}(W \times_{\mathbb{Z}_2} \mathrm{E}\mathbb{Z}_2)$. (One should be careful – this mapping cylinder often is not chain equivalent to a finitely generated free chain complex. But, using Wall's (1965) homological criteria for finiteness, it is projective.) The existence analogue will be in the delooped version of this – which naturally has a map to $S_{w-1}(\mathrm{B}\pi)$ which has a map to the UNil, as before. (See Cappell *et al.*, 2013 for some more discussion.)

To achieve word hyperbolicity, one starts with an involution on a hyperbolic manifold M inducing the correct orientation character and with fixed set of codimension greater than 2. This can be done as in §2.2 (together with the observations in §6.7 about the splitting off of UNil factors in this situation). (See especially §2.2.3 on grafting, where involutions with codimension-1 fixed sets are constructed; it is easy to modify this to increase the codimension.)

Then we consider $W_1 = M \times [0, 1]$ hyperbolized, to achieve acylindricalness. We then build W_2 as before (see Belegradek, 2006, for why this is incompressible and relatively hyperbolic, with the boundary as the maximal "parabolic"). Then, as in Belegradek (2006), the gluing theorem of Dahamani (2003), or even the more basic one of Bestvina and Feighn (1992), shows that W has word hyperbolic fundamental group.

Problem 7.18 Does there exist a counterexample to Nielsen when W is genuinely negatively curved? Even non-positively curved is not obvious to me.

Now let us turn to the Conner–Raymond conjecture. Recall that this is the question of whether every closed aspherical manifold X, with nontrivial center, $\mathbb{Z}(\pi_1 X) \not\approx 0$, has a topological circle action.

The X is quite simple: note that W as constructed above has a homeomorphism H inducing the relevant involution on π. Then X is the mapping torus of H,

$$\pi_1(X) \cong \pi_1(W) \rtimes \mathbb{Z},$$

where the automorphism of $\pi_1(W)$ is H_*. Let t be the generator of $\mathbb{Z}$.

Proposition 7.19 *The center $Z(\pi_1 X) \cong \mathbb{Z}$ generated by t^2.*

[27] Strictly speaking, one should use the equivariant analogue of taming theory (see Ancel and Cannon, 1979; Ferry, 1992) to replace the action by one where the fixed set is embedded locally flatly. Then the action would be necessarily equivariant homotopy-equivalent to the action of the Poincaré complex W.

As a result, because of the work of Borel (explained in the introduction) any circle action on X must have orbits in the homology class of some nontrivial even power of t. In particular, they must represent nontrivial one-dimensional rational homology classes.

We claim that, in fact, X has no circle actions. Our proof will be based on a lovely theorem of Conner and Raymond.[28]

Theorem 7.20 (Conner and Raymond, 1971) *If X is a connected space*[29] *with a circle action so that the orbits are nontrivial in $H_1(X;\mathbb{Q})$, then there is a space Y with a $\mathbb{Z}_n$-action such that X can be identified with $(Y \times \mathbb{R})/n\mathbb{Z}$, where $\mathbb{Z}$ acts diagonally on $Y \times \mathbb{R}$ factoring through the $\mathbb{Z}_n$-action on Y and by translation on $\mathbb{R}$. The action of the circle on X is via the left action of $\mathbb{R}/n\mathbb{Z}$.*

The n is related to how divisible the orbits are as elements of $H_1(X;\mathbb{Z})$. Note that one should be a bit careful: the orbits do not all represent the same element $H_1(X;\mathbb{Z})$ unless one views them as immersed circles (i.e. as being given via the orbit map $\mathcal{S}^1 \to X$) rather than just as subsets.

To get a feeling for the theorem, let's just consider the case of free actions. (The orbit condition implies that all isotropy is finite, so this is not far off.) In that case X can be described as a principal $\mathcal{S}^1$ bundle over $X/\mathcal{S}^1$. However, these bundles frequently have the homologically trivial fibers. The condition for which this is not the case is that the Euler class in $H^2(X/\mathcal{S}^1;\mathbb{Z})$ must be of finite order (i.e. vanish rationally). This leads to a finite cyclic cover of $X/\mathcal{S}^1$ on which the bundle is trivial. That finite cover is Y, and $X/\mathcal{S}^1$ is $Y/\mathbb{Z}_n$.

We note that this theorem is extremely general, and does not apply, for example, to the setting of manifolds. After all, $X/\mathcal{S}^1$ can well be a non-manifold (recall the examples of Bing (1959) mentioned in §6.1), so we will be forced to allow Y to be an ANR homology manifold. (Note that $Y \times \mathbb{R}$ is a cover of X, so we do obtain that Y is an ANR homology manifold from the hypothesis that X is a (ANR homology) manifold.

That is why we modified the Nielsen problem in our treatment above to exclude the action of $\mathbb{Z}_2$ on any ANR homology manifold in the homotopy type. The proof of the theorem comes about by eliminating any other possibilities

[28] Frequently the proofs and disproofs of conjectures are based on the work of the ones who formulate the problem. This might engender a feeling of irony in those of a competitive spirit, yet for those of us who think of mathematics as a magnificent cooperative endeavor, nothing is more natural. Surely, the milestones that are marked by being able to confirm or refute the beliefs of those who have thought profoundly about a subject should be the result of walking further down the road that their insights paved. In this case, Cappell, Yan, and I were surely "walking in the footsteps of giants."

[29] We will suppress the point-set topological hypotheses in this theorem; suffice it to say, that the theorem holds in great generality.

of what the $\mathbb{Z}_n$-action in the Conner–Raymond theorem can look like for the manifold X. These details are not particularly hard.

We note that the manifold X has the following interesting property:

Remark 7.21 As constructed, X has Riemannian metrics g_n so that the indices $[\mathrm{Isom}(\tilde{X}, \tilde{g}_n), \pi_1(X)] \to \infty$ but no metric for which this index is infinite.

This is simply because X has arbitrarily large self-covers (in the topological category; i.e. not by Riemannian "self"-covers) associated to odd-order cyclic quotients of the HNN map $\pi_1(X) \to \mathbb{Z}$ (because odd powers of H are pseudo-isotopic to H).

We will later see that this is indeed unusual: for M homeomorphic to a compact locally symmetric manifold (of non-positive curvature) there is a[30] $C(M)$ so that any Riemannian metric g on M with $[\mathrm{Isom}(\tilde{M}, \tilde{g}), \pi_1(M)] > C(M)$ is actually isometric to a locally symmetric metric (and therefore has G as its isometry group, so the index is uncountable). Farb and I had conjectured[31] that for quite general aspherical manifolds there is a "magic number theorem" (see §7.7 below), but this remark puts an upper bound on the extent to which one can reasonably conjecture that phenomenon (e.g. it might be good to assume that π_1(M) is centerless).

Question 7.22 If a closed aspherical manifold X has fundamental group with nontrivial center, can it have a sequence of (Lipschitz) Riemannian metrics g_n, so that $[\mathrm{Isom}(\tilde{X}, \tilde{g}_n), \pi_1(X)] \to \infty$?

In §7.6 we will continue this discussion.

One can also ask whether the Conner–Raymond conjecture is virtually true? The examples that are constructed using failures of Nielsen realization and the Conner–Raymond theorem are virtually products with circles, so for them this is trivially true.

Is there any form of the Conner–Raymond conjecture that is closer to the Borel conjecture? Here is one that I know.

Conjecture 7.23 *An aspherical manifold M has nontrivial center in its fundamental group iff there is a connected topological group G that acts on M so that the orbits are not null-homotopic (i.e. the map $G \to M$ is not null-homotopic).*

That the center is nontrivial if there is such an action is obvious.

Regarding the converse, there is a universal case of this conjecture, namely that $G = \mathrm{Homeo}_0(M)$ (where the subscript 0 indicates the identity component).

30 Which only depends on the volume of M in its standard locally symmetric metric.
31 Farb and Weinberger (2008).

By the Borel conjecture, the blocked version of this space would be homotopy-equivalent to the space of self-homotopy equivalences of M, whose identity component is a $K(Z(\pi_1 M), 1)$ (presumably a homotopy torus[32]).

So the question is whether $\mathrm{Homeo}(M) \to \mathrm{Homeo}(\tilde{M})$, comparing essentially fiber bundles and block bundles, is, for example, a $\mathbb{Q}$ homotopy equivalence, or at least as far as π_1. Farrell and Hsiang (1978a) explain why this should be true[33] (at least in a stable range, using work of Waldhausen) if one knows that assembly maps in algebraic K-theory are isomorphisms.

Although the above heuristic does not make sense in low dimensions, e.g. dimension less than 7 or 8, nevertheless, I have no reason to doubt (and no good reason to believe) the conclusion.

7.4 Products: On the Difference that a Group Action Makes

Once one gets used to the Borel conjecture for manifolds, and the even larger Borel package extending its reach in various algebraic and geometric directions, one gets used to things like the following:

- Two simply connected manifolds are homeomorphic iff the results of crossing them with any compact aspherical manifold are.
- And that the simple connectivity we assumed is just to avoid algebraic K-theory difficulties (such as Whitehead torsion issues).

We shall see that frequently such statements are indeed consequences of the Borel conjecture, but that it is not quite true for all aspherical "objects." In particular, we will see that this is not true for aspherical homology manifolds that are not resolvable, if there are any. (Or, to vary the point somewhat: it is true for haspherical manifolds but not haspherical homology manifolds. We can cross with the "fake homology tori" constructed in §7.2 and create some interesting homeomorphisms.)

And, the point then becomes even more evident and significant in the equivariant setting. Odd-order locally linear group actions behave like manifolds, but beyond the locally linear setting, or when there's 2-torsion, in some ways these orbifolds act like non-resolvable homology manifolds (or even more extremely).

[32] I do not know whether the center can be a group like, for example, $\mathbb{Q}$.

[33] See the discussion referred to in §5.5.3 (and the notes in §6.11). Roughly, the reasoning goes like this: The A-theory assembly map governs topological concordance space theory (which forms the obstructions on a simplex-by-simplex basis to turning a block bundle into a fiber bundle). Rationally, that assembly map is equivalent to the algebraic K-theory assembly map, which is an isomorphism assuming the Borel conjecture in K-theory.

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Let's put a little flesh on this skeleton.

Shaneson's thesis from the modern perspective,[34], a restatement of Farrell's:

$$S^s(M \times \mathcal{S}^1) \cong S^s(M \times [0,1]) \times S^h(M),$$
$$L^s_{n+1}(M \times \mathcal{S}^1) \cong L^s_{n+1}(M) \times L^h_n(M),$$

where the last statement, modulo decorations, is the isomorphism

$$H_n(\mathcal{S}^1; L(\pi_1 M)) \cong L_n(\mathbb{Z} \times \pi_1 M),$$

which looks just the Borel conjecture for $\mathbb{Z}$ (with coefficients in the group ring $\mathbb{Z}[\pi_1 M]$). The composition

$$S^h(M) \to S^S(M \times \mathcal{S}^1) \to S^h(M)$$

(the left arrow is induced by taking the product with $\mathcal{S}^1$, and the right is applying Farrell's fibering theorem) is the identity, which establishes the injectivity (aside from K-theory) of $\times\mathcal{S}^1$. We can apply this n times to get an injectivity statement for taking the product with $\mathbb{T}^n$.

These results are steps in (and analogues of) the relations[35] of $S^{\mathrm{Bdd}}(M \times \mathbb{R}^n \downarrow \mathbb{R}^n)$ to one another via taking the product with $\mathbb{R}$. Essentially all that is affected is the decoration. Aside from the K-theory issues (that effect only the prime 2) these maps are all isomorphisms.[36]

Corollary 7.24 *If Z is a non-positively curved manifold,*[37] *then taking a product with Z is injective on structure sets (aside from change of decoration*[38]*)*

We use the diagram

$$\begin{array}{ccccc} S(M) & \to & S(M \times Z) & & \\ & & \downarrow & & \\ & & S^{\mathrm{Bdd}}(M \times \tilde{Z} \downarrow \tilde{Z}) & \to & S^{\mathrm{Bdd}}(M \times \mathbb{R}^n \downarrow \mathbb{R}^n) \end{array}$$

where the bottom arrow uses the inverse of the exponential map to see that the top arrow is an injection (modulo decoration).

34 What a wonderful example of terrible history! Shaneson's thesis essentially helped create the modern perspective wherein statements about structure sets and L-groups are viewed as essentially equivalent. Perhaps the best thinking covers itself up (in this way) and (is so successful it) becomes invisible.

35 Chapman proved that a bounded structure over $\mathbb{R}^n$ can always be "wrapped over a torus" and then is transfer invariant, i.e. isomorphic to any of its finite covers.

36 Note also that bounded structures are the same as controlled structures, and then by the yoga of controlled topology one should get $\mathrm{H}^{\mathrm{lf}}_*(\mathbb{R}^n; S(M))$ – except that this only works with $-\infty$ decoration.

37 The proof uses nonpositive curvature very weakly: it just requires a Lipschitz homeomorphism h of the universal cover with $\mathbb{R}^n$ that has the property that $d(x, y)$ can be bounded in terms of $||h(x) - h(y)||$ (i.e. h must be "effectively proper" or equivalently a "uniform embedding").

38 And for Z an n-manifold, one loses no more than one does for the n-torus.

The attentive reader might have noticed that this "logarithm" was the key to the proof of the Novikov conjecture for such Z, and therefore come to the conclusion that this injectivity is part and parcel of this package.

And, indeed for manifolds, it is.

Let's think about the bottom line in the diagram where Z is now just the universal cover of an aspherical homology manifold. One might not be able to find the relevant kind of logarithm map, but it still is reasonable to believe that $S^{\mathrm{Bdd}}(M \times Z \downarrow Z) \cong H_z^{\mathrm{lf}}(Z; S(M) \cong S(M)$ (with a shift of decoration).

However, what is unreasonable is to expect that this isomorphism is implemented by taking a product with Z. In the manifold case we saw this by unpeeling one R at a time.

In the homology manifold this can't go on all the way down to a point(!), for then it would be a manifold. And, indeed what happens is this. The effect of crossing with a (homology) manifold X in surgery is governed by the symmetric signature $\sigma^*(\mathrm{X})$. It is the image under assembly of a controlled symmetric signature that lives in a group isomorphic to $H_z^{\mathrm{lf}}(X; L^*(\mathbb{Z}))$.

The usual way to see that multiplying with something is injective is to show that that thing is a unit, or maps to a unit under some map. For a manifold, at least, the image of controlled symmetric signature in $H_z^{\mathrm{lf}}(X, X - x; L^*(e)) \cong L^\circ(\mathbb{Z}) \cong \mathbb{Z}$ is 1, and therefore if the map from controlled to uncontrolled has good enough properties; that is, assuming the Novikov conjecture, one can expect this product phenomenon.

However, when X is a homology manifold, then this image of the controlled symmetric signature is some number that is $1 \bmod 8$, and it determines whether or not X is resolvable. So, if X has local index equal to 9, crossing with X can kill 3-torsion in a structure set. And, if the local index is 17, then 17-torsion can die, but the 3-torsion is safe, and so on.

In the setting of $\mathbb{Z}$-homology manifolds then, crossing with something aspherical doesn't have to be integrally injective (modulo decorations), but it does have to be (assuming the Novikov conjecture[39]) rationally.

It's interesting to ponder a Poincaré complex P whose total surgery obstruction is of order 17 and that we cross with an aspherical homology manifold X with local index 17 whose canonical Ferry–Pedersen reduction is a stably trivial bundle. Then $p \times X$ will exist, and, if one believes the Borel package,

39 Suspect the sanity of someone who wants to start by considering $\mathbb{Z}_2$ as the first nontrivial example in studying group actions: it is frequently much more difficult than any odd-order group, as we had noted in Fowler's theorem (§7.2), for example, or the nonlinear similarity problem (§6.7). And, indeed, the issue here is quite similar! Needless to say, I can imagine some situations where $\mathbb{Z}_2$ is "the first case" (because one wants only one singular stratum, and vanishing K-groups, etc.; or among people for whom nontrivial means "the first case not yet handled through the efforts of all mathematicians over the course of the previous millennia").

it will approximately fiber over X; however, the local structure will not be a product. (The same thing, of course, happens in the manifold setting when one has a non-simply connected Poincaré complex, which is only finitely dominated, and its total surgery obstruction vanishes in $\mathrm{BS}^{-\infty}$ and crosses with a high-dimensional torus.)

Now, let's turn to the equivariant situation. We have so far seen that this venue is richer in phenomena because of Nil and UNil, and, if we choose to be equivariant rather than isovariant, also because of embedding theory. But now we will see that the situation is richer for yet another reason: the local structures that are present are richer (and more geometrically apprehensible) than what occurs in (homology) manifold theory and we can lose the rational injectivity of crossing.[40]

So, let's think about the smallest group, $\mathbb{Z}_2$, and the smallest nontrivial aspherical universal cover R with the involution $X \to -X$. We will denote this by $(\mathbb{R}, -)$. What happens here?

The issue occurs almost immediately.

Let's consider $\mathcal{S}^n$ with a free involution. Then taking the product with $(\mathbb{R}, -)$ can be thought of as (at least, it seems closely related to) the "suspension map"

$$\begin{array}{ccccc} S^{\mathbb{Z}_2}(\mathcal{S}^n) & \to & S^{\mathbb{Z}_2}(\mathcal{S}^n \times (\mathbb{R}, -)) & \cong & S^{\mathrm{Bdd}}(\mathbb{RP}^{n+1} - \text{a point}) \\ \downarrow & & & & \downarrow \\ S(\mathbb{RP}^n) & & \longrightarrow & & S(\mathbb{RP}^{n+1}). \end{array}$$

The boundedness in the top line is over $\mathbb{R}$ and $[0, \infty)$. Boundedness over these turns out to be equivalent to propriety. The vertical lines are isomorphisms (any proper homotopy $\mathbb{RP}^{n+1}$ – a point can be compactified, and the boundary is necessarily a homotopy sphere, and therefore a sphere).

This "suspension map" was analyzed directly by Browder and Livesay (1973) in the early days of surgery. They were interested in the kernel and cokernel in the bottom line to get an approach to the structures of $\mathbb{RP}^n$. In any case we know that $S(\mathbb{RP}^n)$ is finite[41] iff $n \not\equiv 3 \bmod 4$, and is the sum of $\mathbb{Z}$ and finite group for $n = 3 \bmod 4$. In any case, it surely is not rationally injective.

Bolstered by this we can decide to directly compute

$$S^{\mathbb{Z}_2}(\mathcal{S}^n) \to S^{\mathbb{Z}_2}(\mathcal{S}^n \times (\mathcal{S}^1, -)).$$

Note that $S^{\mathbb{Z}_2}(\mathcal{S}^n \times (\mathcal{S}^1, -)) \cong S(\mathbb{RP}^{n+1} \# \mathbb{RP}^{n+1})$, so there is a very significant

[40] Despite expecting rational injectivity of appropriate assembly maps!

[41] For simplicity and connections to the classical literature in this paragraph and the next, S denotes a topological manifold structure, rather than homology manifold structures. (Thus $S(\mathcal{S}^n)$ is trivial, not $\mathbb{Z}$.)

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lack of surjectivity because of UNil, but also the $\mathbb{Z}$ coming from $L_0(\mathbb{Z}_2)$ for $n = 3 \bmod 4$ also dies.[42]

This "anomalous" product is the key to at least two interesting geometric phenomena.

The first is nonlinear similarity (Cappell and Shaneson, 1981), discussed earlier in this book – the fact that for some even-order groups, like $\mathbb{Z}_{4k}$, when $k > 1$, there are distinct linear representations that *are* conjugate via homeomorphisms. They begin with representations that are not conjugate because of ρ-invariants, and then after crossing with $(\mathbb{R}, -)$ they become conjugate.

The second is a very nice result of Hambleton and Pedersen (1991) related to the classical spherical spaceform problem: namely, which groups act freely on some sphere. The answer, due to Madsen *et al.* (1976), is that a finite group so acts iff all subgroups of order p^2 and $2p$ are cyclic. The first condition is homotopy-theoretic, and follows from Poincaré duality of the putative quotient, but the second condition, due to Milnor, is essentially surgery-theoretic.[43]

The question then arises for groups that act freely, properly discontinuously, and cocompactly on $\mathbb{R}^n \times S^k$, must every finite subgroup satisfy the $2p$ condition? (The p^2 condition is indeed automatic since it comes out of cohomological considerations: see Cartan and Eilenberg, 1956.) Here the answer is affirmative, and basically the reason also involves crossing with $(\mathbb{R}, -)$ a few times.

Formally, the point for group actions is this. When one does a similar analysis for a controlled symmetric signature, now what arises is a more complicated local group rather than $L^*(\mathbb{Z})$; the isotropy enters, and one has – at least! – things like ρ-invariants entering. Away from 2 the information is essentially the same as the equivariant signature operator, and that has the property that for odd-order groups (acting smoothly – so we are dealing with a representation theory problem!) one obtains a unit, but for even-order groups, it frequently is a 0-divisor, with the kinds of implications just mentioned.

In summary: usually homology manifolds and manifolds behave very similarly, but they don't with respect to transversality problems. Bundle structures are much rarer.

Similar issues arise in the orbifold setting. For odd-order groups the equivariant signature (for locally linear actions) is locally a unit, i.e. is an orientation – but for even-order groups it's frequently a 0-divisor. For ANR homology manifolds that are not resolvable, the symmetric signature is neither a unit nor a 0-divisor, so one tends to see phenomena that are intermediate between the differences between odd-order and even-order group actions.

42 Taking the cover associated to $\mathbb{Z}_2 \subset \mathbb{Z}_2 * \mathbb{Z}_2$ gives a computational proof of the bounded vanishing mentioned in the previous paragraph.

43 In its modern formulation: see J. Davis (1983).

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7.5 Fibering[44]

We recall a theorem of Browder and Levine (1966) that was the predecessor of Farrell's thesis:

Theorem 7.25 *A closed manifold M with $\pi_1(M) = \mathbb{Z}$ is a fiber bundle over $\mathcal{S}^1$ iff its universal cover has finitely generated homology.*[45]

Farrell's theorem gives one generalization of this – what happens for fibration over $\mathcal{S}^1$ if π_1 is not $\mathbb{Z}$.

But an alternative generalization asks what we can say more generally about the structure of manifolds whose universal covers have finitely generated homology?

Note that for many fundamental groups there are no such manifolds.[46]

Example 7.26 If M is a closed manifold and $\pi_1 M^n$ has infinitely many ends (e.g. is a nontrivial free product other than $\mathbb{Z}_2 * \mathbb{Z}_2$), then H_{n-1} of the universal cover of M is infinitely generated.

Indeed the universal cover has infinitely many compact separating codimension-1 submanifolds that are not homologous (as in Figure 7.1).

More generally, the following is a consequence of Quinn (1972, 1982b,c, 1986) – and see Block and Weinberger (1997).[47]

Proposition 7.27 *If* Bπ *is a finitely dominated complex and M is any closed manifold with fundamental group π and whose universal cover has finitely generated homology, then (i)* Bπ *is a Poincaré complex and (ii) the universal cover of M is a Poincaré complex.*

It is obviously not necessary for Bπ to be finitely dominated for such an M to exist. Any finite group is a counterexample! Nevertheless, let us provisionally make this assumption.

[44] See also our discussion above in §5.1

[45] They proved this theorem in the smooth and PL categories in dimension greater than 5, but it is now known to be true in the topological category in all dimensions (for the "usual reasons").

[46] For some groups there are no such finite complexes or even having finitely generated homology through some fixed dimension. This is a variation of the usual FP_k hierarchy (see e.g. Brown, 1982) for groups, which seems worth further study.

[47] The argument is basically this. A finite complex X is a Poincaré complex iff the map from the boundary of the regular neighborhood of X in a high-dimensional Euclidean space is a homotopy sphere. This implies that, for a fibration of finite complexes, $F \to E \to B$, as these fibers for E are the join of the fibers for E and B, and it therefore follows that E is Poincaré iff both F and B are. In our situation, $E = M$, and the fibration is associated to the classifying map of the fundamental group, and we have assumed that all three are finite complexes. Thus, all three must be Poincaré complexes. Since M is a manifold, its universal cover automatically has a normal invariant, and consequently the fiber is homotopy-equivalent to a topological manifold (in all dimensions).

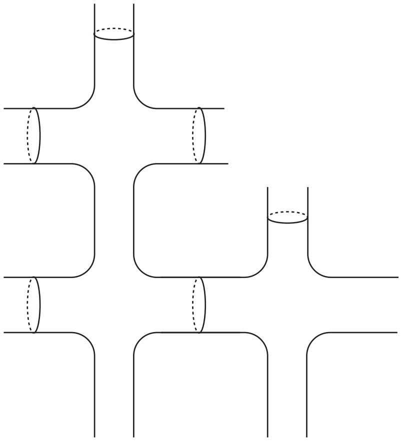

Figure 7.1 Part of the universal cover of a manifold with free fundamental group; a set of separating hypersurfaces as in the picture are linearly independent whenever no subset bounds a compact region.

Conjecture 7.28 *If M is a closed manifold with $\pi_1 M$ of finite type and whose universal cover has finitely generated homology, then there is an aspherical ANR homology manifold X and a* UV^1 *approximate fibration $M \to X$.*

Let's review some of the definitions and motivate the conjecture. We shall see that it is a natural analogue and consequence of the Borel conjecture. (Indeed, it also implies the Borel conjecture, so I guess that means it's equivalent to it.)

The first statement that there should be an aspherical ANR homology manifold with the same fundamental group as M follows from the existence version of the Borel conjecture (with Wall conjecturing that this should even be homotopy-equivalent to a manifold). The manifold is then homotopy-equivalent to a fibration over X with a Poincaré space as fiber.

If X is not resolvable, then obviously M cannot fiber over it. (The local index of M would have to be divisible by the local index of X.)

However, approximate fibration is somewhat less. A map is an approximate fibration if the usual condition for a fibration, namely that for any square

$$\begin{array}{ccc} A & \to & M \\ \cap & & \downarrow \\ A \times [0,1] & \to & X \end{array}$$

there is a diagonal lift, $A \times [0,1] \to M$, but now only demanding that the diagram commute up to ε. (In other words, for each ε we want a lift.) This condition is quite close, when X is a manifold, to being a block bundle (see Quinn, 1982a, 1987b).

Let $E(\tilde{M} \downarrow X)$ be the total space of the universal cover fibration over X, i.e. the product $(\tilde{M} \times \tilde{X})/\pi$ which has a natural fibration structure over X.

The obstruction to homotoping the map $M \to E(\tilde{M} \downarrow X)$ to a controlled homotopy equivalence over X (which, parenthetically, is equivalent to homotoping it to an approximate fibration) is precisely an element of a certain (looping[48] of a) structure set of X (because the homotopy fiber involved is simply connected) by making use of the theorem of Bryant *et al.* (1993). (See §4.9 for a blocked discussion and also §5.1.) Consequently, by the Borel conjecture again, we get the conclusion of the conjecture.

Conversely, if for example X is an aspherical manifold for which the Borel conjecture fails, then gluing together a counterexample in $S(X \times \mathcal{D}^4)$ to $X \times \mathcal{D}^4$ builds a manifold that is homotopy equivalent to $X \times \mathcal{S}^4$ and therefore has finitely generated homology in its universal cover. However, the obstruction to making it approximately fiber over X is exactly the nontrivial element in $S(X \times \mathcal{D}^4)$ that starts the construction, completing the proof. (This requires some thought, perhaps, but it follows from the proof of Siebenmann periodicity given in Cappell and Weinberger, 1987.)

Question 7.29 Is there any natural construction of manifolds whose universal covers have finite type that might be a useful source of groups that satisfy Poincaré duality that are not immediately forced to be fundamental groups of manifolds? Note that any Poincaré BΓ is the fundamental group of such a manifold – but this doesn't count, since we don't have a construction of these, as I've already lamented – the boundary of a regular neighborhood of BΓ embedded in any Euclidean space would be such a manifold.

One source *might be* even a construction of manifolds M where Out$(\pi_1 M)$ might or might not be trivial, and where there is a finite subgroup G for which no element of order p lifts to Aut$(\pi_1 M)$. In that case the nontrivial extension $1 \to \pi_1 M \to \Gamma \to G \to 1$ produces a Poincaré duality group which is not obviously a manifold. On the other hand, if we knew the Novikov conjecture for π (even rationally), this would not be a candidate for a counterexample to the Wall conjecture based on the local index. And, in any case, I don't know any interesting examples of this sort.

Remark 7.30 Note that by striving for a general discussion, we were inexorably led to approximate fibrations. However, when X is a manifold, nothing we have said precludes M from actually fibering over X. Nevertheless, since

[48] By $\dim M - \dim X$.

there are sphere block bundles that do not correspond to sphere bundles, the total space of one of these over a closed aspherical manifold would provide such an example.

Remark 7.31 There is a slightly different line of reasoning that could lead to the above conjecture. (And naturally leads to a version for arbitrary regular covers in place of the universal cover: but this version is obstructed by (a sequence of) Nil groups in general. What follows is a somewhat different recombination of the ingredients in §5.1.)

Let us consider what a stratified version of the Borel conjecture could be – in the first interesting case, where our space has two strata. The question is which of these can be rigid?

This is actually kind of complicated. Here is an example that has nothing to do with the Borel conjecture:

Example 7.32 Suppose that $f\colon M \to N$ is an approximate fibration whose homotopy fiber is $\mathbb{CP}^{2k}$, then $(\mathrm{Cyl}(f), \mathrm{rel}\,\partial)$ is a rigid stratified space.

Interestingly Cappell, Yan, and I observed that this is not true in the slightly more general situation where the homotopy fiber is a general simply connected manifold F with signature equal to 1! There is an interesting contribution of the monodromy map $\pi_1 N \to \mathrm{Aut}(F)$.

So let's consider just the situation relative to a singularity[49] (see §§6.5 and 6.6). There is a fibration:

$$S(X\,\mathrm{rel}\,\Sigma) \to S(X-\Sigma) \to \text{Fiber}\,\big(H\big(\Sigma; L(\text{local holink})\big) \to L\big(\text{global holink}\big)\big).$$

(The loop of this fibration is

$$\begin{aligned}\text{Fiber}\big(H_{+1}\big(\Sigma; L(\text{local holink})\big) &\to L(\text{global holink})\big)\\ &\to S(X\,\mathrm{rel}\,\Sigma) \to S(X-\Sigma),\end{aligned}$$

which just means that the extensions of a structure of $X-\Sigma$ over Σ correspond to the ways of making the end of $X-\Sigma$ into an approximate fibration over Σ.)

It's easy enough to compute $S(X-\Sigma)$: it is essentially a manifold with boundary where one is not working relative to the boundary. In the mapping cylinder case, it vanishes. (It boils down to $S(M\times[0,1)\,\mathrm{rel}\,M\times\{0\})=0$.) This calls attention to the issue of whether

$$H\big(\Sigma; L(\text{local holink})\big) \to L(\text{global holink})$$

[49] Actually, there is a stabilization necessary here when the homotopy link (holink) is not simply connected, because of decoration issues. We shall ignore this.

is an isomorphism. If $\pi_2\Sigma = 0$, then the L-sheaf is "flat," i.e. is a formal consequence of the short exact sequence[50]

$$1 \to \pi_1(\text{local holink}) \to \pi_1(\text{global holink}) \to \pi_1(\Sigma) \to 1$$

and then we would expect this fiber to be trivial at least when Σ is haspherical (by comparison to the aspherical case – as there is a map to the corresponding twisted assembly map over $B\pi_1(\Sigma)$).

All of this suggests that for $\mathrm{Cyl}(f)$, if the map is an approximate fibration,[51] and the base is aspherical, one might conjecture rigidity. A little thought shows that the reasoning discussed in Chapter 4 would prove the split injectivity. Which makes the conjecture of rigidity (despite its clear falsity due to Nil and UNil issues!) plausible. This is a special case of the twisted Borel conjecture in §5.1.

The connection between the neighborhoods and approximate fibrations is very close in the topological category. As we discussed earlier, there is no direct regular neighborhood theory. What there is, is the "teardrop neighborhood theorem" of Hughes *et al.* (2000). It says that there is a deleted neighborhood of $X - \Sigma$ that has a canonical approximate fibration structure over $\Sigma \times (0, 1)$. (Then Σ is glued in in the obvious way, and, when drawn appropriately, the open sets in the relevant topology look like teardrops.)

Crossing with $(0, 1)$ has the effect of taking a loop space, and one is thus led geometrically to the conjecture of existence and uniqueness of approximate fibration structures when the target is aspherical.

Remark 7.33 Note that, once we have decided that Σ is aspherical, then the Borel conjecture would give *its* rigidity, so $\mathbf{S}(X \operatorname{rel} \Sigma) \to \mathbf{S}(X)$ would be an isomorphism. Thus, we have two circumstances where we have vanishing of $\mathbf{S}\big(\mathrm{Cyl}(f), \operatorname{rel} \partial\big)$ – namely when the fiber is $\mathbb{CP}^{2k}$ or when the base is aspherical. When the fiber is $\mathbb{CP}^{2k}$ then any manifold M' homotopy-equivalent to M gives rise to a structure N' on N, so that M' approximately fibers over N'. In the situation of aspherical base, the base is itself rigid, so the result is that M' fibers over N itself.

Remark 7.34 Approximate fibrations arose naturally when we considered rigidity in the topological category. As we have emphasized many times in these notes, the topological setting is the natural one for rigidity.

Had one worked in the PL category one would have been led to consider block

[50] See Chapter 13 in Weinberger (1994).

[51] Actually, all one needs is that all "fibers" have the same π_1 (i.e. an approximate fibration with respect to 2-complexes). In that case, stratified surgery does not directly apply, but nevertheless controlled surgery would still give a suitable rigidity.

bundles instead. In that case, we would have trouble with decorations, which also reflects the way there are K-theory obstructions in Farrell's theorem.[52] While the whole Whitehead group enters in Farrell's theorem, in the problem of approximate fibering over the circle (and this is part of the Borel package) it is only the Nil part: writing, in the product situation, $\mathrm{Wh}(\mathbb{Z} \times \pi) = \mathrm{Wh}(\pi) \times \tilde{K}_0(\pi) \times \mathrm{Nil}_\pm(\pi)$. The $\tilde{K}_0(\pi)$ part arises from trying to put a boundary on the infinite cyclic cover. However, any such cover, with boundary or not, is the infinite cyclic cover of an approximate fibration over the circle (see Hughes and Ranicki, 1996, for a discussion of wrapping). The Whitehead part is entirely irrelevant to the problem. The h-cobordisms between different "almost fibers" can all be mapped to points in an approximate fibration.

Now let us turn to the possible structure of manifolds whose universal covers have finitely generated homology but have torsion in their fundamental group.

Conjecture 7.35 *For such a manifold to exist it is necessary and sufficient that* $\mathrm{HR}_i(\pi)$ *is zero for all but one dimension, and in that dimension it is isomorphic to* $\mathbb{Z}$*.*

Here HR is the "Rips homology" of the discrete metric space π (made into a metric using the word metric). It is defined as the limit of the locally finite homology of the nerve of the covering of π by balls of radius k, as $k \to \infty$ (see §4.8 or Block and Weinberger, 1997, or Roe, 2003). For groups of finite type, this is equivalent to Poincaré duality (see, for example, Brown, 1982).

The work we've already done on the Nielsen theorem indicates that there are groups satisfying the conditions of the conjecture where there is no $\underline{\mathrm{E}}\pi$-manifold. If one takes the counterexample to Nielsen and crosses it with S^3, one easily obtains a manifold whose universal cover is homotopy equivalent to the sphere but we cannot attribute this to, for example, an approximate fibering, as (we try to) in the finite type situation.

Of course, there are situations where $\underline{\mathrm{E}}\pi$ geometrically exists. For example, this is the case when π is a uniform lattice in a connected Lie group G. In that case, the Borel package gives one a nice characterization:

Theorem 7.36 *If M is a manifold whose fundamental group is that of a uniform lattice in G and whose universal cover has finitely generated homology, then there is a π-equivariant* UV^1 *approximate fibration $\tilde{M} \to G/K$ iff appropriate* Nil *and* UNil *obstructions vanish. Of course, if the fundamental group is, in addition, torsion free, then conclusion holds unconditionally.*

The need to handle such obstructions is easy to see. Suppose we take one of

52 That the version of Farrell's theorem for approximate fibrations only involves Nil was observed by Ferry in the late 1970s.

Cappell's manifolds $M = \mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}$ terms that is not in itself a connected sum. Suppose that there were a D_∞ equivariant approximate fibration f of the universal cover to $\mathbb{R}$ (with the usual action where one involution is $x \to -x$ and the other is $x \to 1 - x$). Then $f^{-1}(1/3, 2/3)$ would descend into M, and would be a copy of $\mathcal{S}^{4k} \times (0,1)$ – as there is a unique manifold element in $S^p\big(\mathcal{S}^{4k} \times (0,1)\big)$. Taking a slice, M is thus decomposed into a connected sum. Similarly, it is not hard to realize Nil obstructions on some manifolds with fundamental group $\mathbb{Z} \times F$, where F is a finite group, and then there would be no $\mathbb{Z} \times F$ equivariant map of the universal cover to $\mathbb{R}$ (with trivial F-action, and $\mathbb{Z}$ acting by translation).

The proof of the above theorem is now identical to the discussion of the torsion free case above.

7.6 Manifolds with Excessive Symmetry[53]

In this section we will describe some theorems in Riemannian geometry that have a philosophical relation to the topological issues explored in the previous few chapters. We will not do more than give the slightest hints of the arguments.

Recall again Borel's theorem:

Theorem 7.37 *If M is aspherical and $\pi_1 M$ is centerless, then any group G that acts on M injects into* $\mathrm{Out}(\pi_1 M)$.

If M is compact locally symmetric (with no virtual hyperbolic surface factors) then Mostow rigidity implies that $\mathrm{Isom}(M) = \mathrm{Out}(\pi_1 M)$. Then we can assert that, for action on M, there is a semiconjugacy (i.e., an equivariant map that's not necessarily a homeomorphism) homotopic to the identity

$$H\colon M \to M$$

to an action by isometries. In this sense, the locally symmetric metric is maximally symmetric among all metrics.

Of course, there are other metrics that are equally symmetric. $\mathrm{Isom}(M)$ is a finite group, and any metric equivariant with respect to this group has the same symmetry. Is there any way we can be more demanding and perhaps characterize the locally symmetric metric?

Another point to note is this. Instead of just considering *metrics on M*, we can also consider manifolds N with $\pi_1 N \cong \pi_1 M$. If the G-action on N is trivial

[53] Benson Farb has criticized the title of this section and suggested "Manifolds that only have the slightest bit more symmetry than they need to have" as an alternative.

in Out(π), then one can factor the natural map $N \to M$ through N/G.[54] In particular, if the homology class of N is nontrivial in $H_n(B\pi \cong M; \mathbb{Q})$, i.e. if N is essential, then G is finite.[55] If the map were degree 1, we would have Borel's injectivity for this larger class of manifolds.

This connects to our higher-signature localization discussion, since the fixed sets would have no choice but to be lower dimension for a nontrivial action, but such wouldn't be able to have this relevant higher signature. It's not shabby getting this far just from homology considerations without using any high technology.

In short, we see that there is a rather larger class of manifolds for which Isom(M) provides a bound on their symmetry.

We will warm up with the following result:

Theorem 7.38 (Farb and Weinberger, 2005) *If $N \to M$ is a map of nonzero degree, M an irreducible locally symmetric manifold, and* Iso(N') $\cong$ Iso(M') *for every finite sheeted cover, then:*

(1) *if M is arithmetic, then N is isometric to M;*[56]
(2) *if M is not arithmetic, then there are such N not homotopy-equivalent to M (above dimension* 3, *for trivial reasons) and there are N diffeomorphic to M that are not locally symmetric.*

The key to the theorem comes from the existence and uniqueness theorems for harmonic maps in a given homotopy class when the target is compact and non-positively curved. The difference between the arithmetic and non-arithmetic cases is because of a theorem of Margulis: *Every non-arithmetic lattice is included in a maximal lattice that includes all of the lattices that are commensurable to it*. This makes part of (2) very simple. We can take N to be $M\#$ several $(S^2 \times S^{n-2})$.

For the arithmetic case, one makes use of many conjugates of π in the group $G(\mathbb{Q})$ which produce many extra isometries that finite covers have. Ultimately this transfers the whole action of G from the universal cover of M to that of N (by taking limits).

Indeed, a little work makes the same conclusion follow from the hypothesis that M and N have the same dimension and fundamental group. The map of universal covers is smooth, and, given the action of G, every point of M is

[54] This uses the finiteness of Out(π), the existence of a lattice Γ in G such that $\pi \triangleleft \Gamma \to$ Out(π), all of which follows from Mostow rigidity, and the fact that G/K is an $\underline{E}\Gamma$.

[55] And we can be more quantitative: depending on how divisible $[N]$ is in $H_n(B\pi)$, we can bound $\#G$.

[56] Actually homothetic to M; one can rescale the metric!

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a regular value, so the map is a finite sheeted cover, and the conclusion then follows.

One can push these ideas further in a number of ways:

(1) One doesn't need that all of the isometries of all of the finite covers extend. One just needs that there are many that do.
(2) One can, in the spirit of the previous paragraph, remove essentiality[57] types of conditions (like asphericity) and end up with statements about fibering, i.e. that the harmonic map that one produces has no singularities, and that the domain manifold Riemannianly fibers over the target.

It turns out that the correct setting for these results is the following (that follows from the same ingredients together with some Lie theory):

Theorem 7.39 (Farb and Weinberger, 2008) *Suppose M is a compact aspherical Riemannian manifold whose fundamental group has no normal abelian subgroup and is not virtually a product of manifolds. Then if* $[\mathrm{Isom}(\tilde{M})\colon \pi]$ *is infinite, then π is a uniform lattice in a semisimple group G, and M is isometric to $K\backslash G/\pi$.*

If M is not assumed aspherical, then one must assume that $\mathrm{Isom}(\tilde{M})$ is not a compact extension of π, and one obtains that a finite cover of M is a Riemannian fiber bundle over $K\backslash G/\pi$.

The condition about compact extension is to avoid situations like the following. Suppose π has, for example, a dense representation in a compact group H, then the quotient manifold under the diagonal action $(K\backslash G \times H)$ has a large isometry group for its universal cover (i.e. containing π with infinite index), even if one gives $K\backslash G$ a highly non-symmetric metric, that is merely π-invariant.

Normal abelian subgroups truly are the enemy. A three-dimensional solvmanifold is abstractly a torus bundle over a circle. One can consider families of flat structures on the torus, parameterized over the circle, with a small compatibility condition (and don't even all have the same volume) and obtain metrics that have excessive symmetry and do not come from locally symmetric metrics or fiber over anything.

It seems very reasonable to try to improve the condition "$[\mathrm{Isom}(\tilde{M})\colon \pi]$ is infinite" to something more quantitative. This is most salient in the "no normal abelian subgroup" situation. The example on the Conner–Raymond conjecture gives an example of an obstacle involved in removing this condition and replacing the condition in the conclusion by something that takes the abelian subgroup explicitly into account.

[57] A manifold is called essential when it represents a nontrivial cycle in the group homology of its fundamental group.

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Conjecture 7.40 ("Magic number" conjecture) *For each group π that has no normal abelian subgroups, there is a number $C(\pi)$ such that any Riemannian aspherical manifold with fundamental group π such that $[\mathrm{Isom}(\tilde{M})\colon \pi] > C(\pi)$ is isometric to a locally symmetric Riemannian manifold.*

This conjecture is true when π is a lattice, or a word hyperbolic group. It seems that a counterexample would have to be rather exotic.

Our discussion also makes the following seem possible (although, I confess, unlikely[58]).

Conjecture 7.41 *An aspherical manifold M has Lipschitz Riemannian metrics for which $[\mathrm{Isom}(\tilde{M})\colon \pi_1 M]$ is arbitrarily large iff $\pi_1 M$ has a normal abelian subgroup.*

7.7 Notes

The paper of Borel on symmetry of aspherical manifolds (that we began our discussion with in §7.1) was unpublished for many years, but appeared in his collected works. The theorem itself was published by Conner and Raymond much earlier. I recommend the book by Lee and Raymond (2010) for many results about Lie group actions on aspherical manifolds, which builds on and reviews the excellent work of Conner and Raymond.

The Nielsen problem can be viewed as a variant of the Borel conjecture: if one believes that a homotopy equivalence gives rise to a canonical homeomorphism in the homotopy class, then one would have been led to the Nielsen problem. In the classical setting of surfaces, Nielsen proved it for cyclic groups. As mentioned in the text, the general case was first proved by Kerckhoff (1983).

Borel's theorem, of course, only applies to closed aspherical manifolds: after all, there are many finite group actions on Euclidean space and $\mathrm{Out}(\pi) = e$! However, it is natural to ask about locally symmetric manifolds of finite volume, i.e. quotients by non-uniform lattices. I had been interested in this question for many years, and the result that Borel's theorem holds for these was proved by G. Avramidi (2013). However, as he points out, the proof leaves open many questions: for example, given an arbitrary action of a finite group A on $K\backslash G/\Gamma$, is the dimension of the fixed set the same as in the classical action? Even for A a p-group this is open.

The results of Borel underscore a problem for understanding G-manifolds. For ordinary closed manifolds, we make a lot of use of the comparison of M

[58] And therefore this problem's demise will be a measure of how much we have to learn.

to $B\pi_1(M)$ (at least in the torsion-free case and to related things when π has torsion). The fact that for nonabelian compact Lie groups, and more generally, there is no very nice model for $K(\pi_1^G(M), 1)$ where $\pi_1^G(M)$ is the map from the orbit category of G, $\mathrm{Orb}(G) \to \mathrm{Gpd}$, the category of groupoids (since fixed sets can be disconnected or empty) makes it much harder to understand equivariant structure sets even when we have a theoretical analysis via a surgery sequence.

In §7.2 the connection of the Wall conjecture to the Borel conjecture requires Ranicki's total surgery obstruction (Ranicki, 1979b). This in turn is related to the idea that "coherent Poincaré transversality" gives a reduction of the Spivak fibration (Levitt and Ranicki, 1987). Actually, the (integral form of the) Novikov conjecture therefore gives the reduction of the Spivak fibration, but there is a surgery obstruction in principle – but surjectivity of the assembly map would give this, which is clearly part of the Borel conjecture. When Bryant *et al.* (1993) came out, it became clear (it seems to me) that it was more reasonable to ask for homology manifolds instead of manifolds. Davis (2000) suggested that one ask the question about aspherical R-homology manifolds.

This was disproved by Fowler (2009) in his thesis, which we have followed in spirit in the text.

Thanks to intersection homology there are other settings where one can ask for Poincaré duality. It would be interesting to find groups that naturally act on IH-acyclic "Witt spaces" or something similar. Then the Wall conjecture would somehow resolve the space (although perhaps in a non-local way).

Moving to §7.3, hyperbolic groups were first studied by I. Rips, who showed that they act properly discontinuously and cocompactly on a finite-dimensional contractible complex (the Rips construction). This result was published and then much elaborated by Gromov (1987), who explained their stability properties, their boundaries, isoperimetric inequalities, rationality of their word zeta functions, etc. They have become a much studied class of groups and are central to geometric group theory. I recommend Coornaert *et al.* (1990), Ghys and de la Harpe (1990), and Alonso *et al.* (1991) as good references (although there are a number of others).

As hyperbolic groups are generalizations of the idea of the fundamental group of a closed hyperbolic (or negatively curved) manifold, relatively hyperbolic groups are a generalization of hyperbolic manifolds with cusps. (These are never hyperbolic in dimension greater than 2, because they contain nontrivial abelian subgroups.) The basic paper is Farb (1998) which establishes many of their properties. We use the fact that relative hyperbolization (see Davis *et al.*, 2001) can be made hyperbolic relative to the boundary (Belegradek, 2007). That the pieces glue together to form a hyperbolic group is based on "combination theorems." The original combination theorem was Thurston's uniformization of

Haken 3-manifolds. For hyperbolic and relatively hyperbolic groups, Bestvina and Feighn (1992) and Dahamani (2003) provide analogues. (Drutu and Sapir, 2005 provide the quasi-convexity necessary for applying these theorems.)

The Conner–Raymond conjecture grew out of their work on injective torus actions (all actions on aspherical manifolds are injective). Their understanding of group actions on aspherical manifolds led to many examples, including, for example, the first examples of closed manifolds that have no symmetry (Conner *et al.*, 1972).

As discussed earlier, pseudo-isotopy theory is a deep subject connecting higher algebraic K-theory to groups of homeomorphisms and diffeomorphisms of manifolds. Besides the paper by Farrell and Hsiang (1978a) already referred to in the text, I recommend Cohen (1987) and Weiss and Williams (2001) as useful surveys, although Cohen's (1987) description of the then "recent" results was a bit optimistic.

The material on products in §7.4 should be well known, but doesn't seem to be. That products are likely not isomorphisms when the controlled symmetric signature is not a unit (in a relevant) ring is obvious in retrospect and also gives rise to failure of equivariant transversality – as discussed at the beginning of Chapter 6, first appearing in the form of lack of stability of equivariant classifying spaces – and some forms of transversality for homology manifolds (if one asks for bundle neighborhoods).

That bounded over $\mathbb{R}$ boils down to the proper theory is because tame ends of manifolds can be "wrapped up" and have an automatic periodicity. The relevant geometry is part of Siebenmann (1970a), which gave an alternative approach to Farrell's thesis for the problem of fibering over the circle.[59]

The Browder–Livesay theory is an elegant one wherein this particular non-simply connected problem is boiled down to a simply connected problem. The quadratic form $\langle u, Tv\rangle$, where $\langle\,,\,\rangle$ denotes cup product and T denotes the involution on the 2-fold cover, plays a major role. The orientation reversing nature of T interchanges the usual ± symmetry. In more modern L-theory this is called a change of "antistructure" and there is now a much more general theory of Browder–Livesay groups, associated to quadratic extensions of rings. Given our discussion, it should not surprise the reader that Cappell and Shaneson relied on such a Browder–Livesay theory in their calculations leading to the existence of nonlinear similarity.

The discussion on fibering in §7.5 is surely folklore and I am not sure who noticed what, and when. Ferry had told me decades ago that approximately

[59] Farrell's original approach did not place the algebraic obstruction all at once in Wh: it lived in several pieces. The connection between the two approaches is given by the formula for the Whitehead group of a twisted extension $\mathbb{Z}[\pi \rtimes \mathbb{Z}]$.

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fibering over the circle is much less obstructed than fibering, i.e. that it's controlled by the Nil part of the fibering obstruction, and that the homotopy fiber does not need to be a finite complex. When considering the twisted analogue of the Borel conjecture, one quickly realizes its connection to block bundles, except that it doesn't get the decorations right. So, for "K-flat" groups, one gets general block fibering theorems (like those relevant to our question about spaces with finitely generated homology in their universal covers). Farrell and Jones (1989) point this out, and I had pointed out such things based on thinking about approximately fibered neighborhoods and possible stratified rigidity in Weinberger (1994). See Farrell *et al.* (2018) for recent results about approximately fibering compact manifolds over aspherical ones.

That one can prove results about approximate fibrations over ANR homology manifolds was the struggle in Bryant *et al.* (2007).

The teardrop neighborhood theorem of Hughes *et al.* (2000) is a variation on the periodic structure that can be given to a tame end, referred to in the notes in S6.11.

In §7.6, the proof of the main theorem characterizing Riemannian manifolds with excessive symmetry is a combination of the Myers–Steenrod theorem, which tells us that the isometry group of any Riemannian manifold is naturally a Lie group, the theory of harmonic maps (in order to build canonical maps to model spaces), and the Conner conjecture (a theorem of Oliver, 1976a), which asserts that the quotient of a finite-dimensional contractible space under a compact group action is contractible. This enables one to get information about isotropy groups and use homological algebra.

The use of harmonic maps to rigidify homotopy theory and make maps automatically equivariant arose earlier in work of Schoen and Yau (1979b). They also play a role in Frankel's proof of a conjecture of Kazhdan, of which Farb and Weinberger (2008) give an alternative proof.

These techniques are somewhat extended to noncompact manifolds in Farb and Weinberger (2010), except that the issues in general are much more complicated. As a result, attention is concentrated on moduli space (of curves). We give a new proof of some theorems of Ahlfors by showing that no complete Finsler metric on moduli space with finite co-volume has even a single point in its universal cover at which it is symmetric, i.e. possesses an involution with an isolated fixed point. We also obtained that, for any such metric, the symmetry of moduli space is never excessive. Avramidi (2014) strongly improved on this by showing that there are no unexpected isometries at all in any finite volume metric on any finite cover of moduli space.

On the other hand, he also gave a very simple construction of complete infinite volume metrics on moduli space (and non-uniform locally symmetric spaces) that do have excessive symmetry, so the finite co-volume conditions in Farb and Weinberger (2010) were necessary.

The fibration in the non-aspherical situation was significantly extended by van Limbeek (2014). Melnick (2009) has extended some of these results to Lorentzian manifolds.