

A NONCOMMUTATIVE ANALOGUE OF CLAUSEN'S VIEW ON THE IDÈLE CLASS GROUP

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Abstract Clausen a prédit que le groupe des classes d'idèles de Chevalley d'un corps de nombres F apparaît comme le premier K -groupe de la catégorie des F -espaces vectoriels localement compacts. Cela s'est avéré vrai, et se généralise même aux groupes K supérieurs dans un sens approprié. Nous remplaçons F par une \mathbb{Q} -algèbre semi-simple, et obtenons le groupe des classes d'idèles noncommutatif de Fröhlich de manière analogue, modulo les éléments de norme réduite une. Même dans le cas du corps de nombres, notre preuve est plus simple que celle existante, et repose sur le théorème de localisation pour des sous-catégories percolées. Enfin, en utilisant la théorie des corps de classes, nous interprétons la loi de réciprocité d'Hilbert (ainsi qu'une variante noncommutative) en termes de nos résultats.

Clausen predicted that Chevalley's idèle class group of a number field F appears as the first K -group of the category of locally compact F -vector spaces. This has turned out to be true and even generalizes to the higher K -groups in a suitable sense. We replace F by a semisimple \mathbb{Q} -algebra and obtain Fröhlich's noncommutative idèle class group in an analogous fashion, modulo the reduced norm one elements. Even in the number field case, our proof is simpler than the existing one and based on the localization theorem for percolating subcategories. Finally, using class field theory as input, we interpret Hilbert's reciprocity law (as well as a noncommutative variant) in terms of our results.

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1. Introduction

Let F be a number field and LCA_F the category of locally compact topological F -vector spaces, that is: Objects are topological F -vector spaces with a locally compact topology and morphisms are continuous F -linear maps. Clausen [Cla17] had predicted that

$$K_1(\mathrm{LCA}_F) = \text{Chevalley's idèle class group}$$

so that the first K -group gives the automorphic side of the idèle formulation of global class field theory in the number field situation. This is quite remarkable in that it requires no manual modifications in order to get the infinite places into the picture, which is frequently needed when using cohomological, cycle-theoretic or K -theoretic approaches in arithmetic applications.

This picture also finds the correct automorphic object for local class field theory or finite fields, every time just using LCA_F for the respective type of field. In this paper, we focus exclusively on the hardest part, the number field case. The original predictions were confirmed in [AB19], along with a determination of the entire K -theory spectrum of LCA_F . In this paper, we replace the number field F by an arbitrary finite-dimensional semisimple \mathbb{Q} -algebra A and LCA_A is analogously defined to consist of locally compact A -modules.

We find generalizations of the corresponding results in [AB19] and with simpler proofs. Let $K: \mathrm{Cat}_\infty^{\mathrm{ex}} \rightarrow \mathbf{A}$ be a localizing invariant in the sense of [BGT13] with values in a stable presentable ∞ -category \mathbf{A} which commutes with countable direct products of categories, for example, nonconnective algebraic K -theory.

Theorem A. *Let A be a finite-dimensional semisimple \mathbb{Q} -algebra. Then there is a fiber sequence*

$$K(A) \longrightarrow K(\mathrm{LCA}_{A,ab}) \longrightarrow K(\mathrm{LCA}_A),$$

where the middle category $\mathrm{LCA}_{A,ab}$ (the so-called “adelic blocks”) can be characterized in any of the following equivalent ways:

1. *It is the category of locally compact A -modules of type \mathbb{A} in the sense of Hoffmann–Spitzweck [HS07, Definition 2.1].*
2. *It is the full subcategory of objects in LCA_A which are simultaneously injective and projective.*
3. *It is a categorical restricted product,*

$$\mathrm{LCA}_{A,ab} \simeq \prod'_p \mathrm{proj}(A_p),$$

where the index p runs over all prime numbers as well as $p = \mathbb{R}$, with the meaning $A_{\mathbb{R}} = A \otimes_{\mathbb{Q}} \mathbb{R}$ and $A_p = A \otimes_{\mathbb{Q}} \mathbb{Q}_p$, respectively. A precise definition of the middle category is to employ $J_A^{(\infty)}$ from Definition 4.3 below.

The map $K(A) \rightarrow K(\mathrm{LCA}_{A,ab})$ is induced by the functor $- \otimes_A \mathbb{A}$, where \mathbb{A} is the ring of adèles of A .

The characterization (3) shows most clearly how the idea of the restricted product in Chevalley’s idèles is lifted to a categorical level by using locally compact modules. The characterization in (2) is new, has no previous counterpart in the literature and affirmatively settles [AB19, Conjecture 1]. The fiber sequence will be Theorem 6.2. The different characterizations of $\text{LCA}_{A,ab}$ follow from the definition in case of (1), from Theorem 4.15 for (2) and from Proposition 4.5 for (3).

To establish the fiber sequence in the above theorem, we follow the method of [BHR21] closely. Specifically, we perform two subsequent localizations of the category LCA_A . In the first step, we consider the quotient by the category $\text{LCA}_{A,com}$ of compact A -modules. In the second step, we further quotient by the (image of the) category $\text{LCA}_{A,ab}$ of adelic blocks. After performing these quotient constructions, one is left with the abelian category $\text{Mod } A / \text{mod } A$ and obtains the required fiber sequence. This will be done in §6.

The quotient $Q: \text{LCA}_A \rightarrow \text{LCA}_A / \text{LCA}_{A,com}$ is obtained as a localization of LCA_A at the following class of morphisms: all inflations (deflations) with cokernel (kernel) in $\text{LCA}_{A,com}$. It is not clear that this localization can be equipped with an exact structure such that the localization functor Q is exact, but it follows from [HR19b] that $\text{LCA}_A / \text{LCA}_{A,com}$ has a natural one-sided exact structure (in the sense of [BC13]) for which Q is exact. We will recall the necessary setup for one-sided exact categories and their place in the theory of localizations of exact categories in §5.

On the one hand, one-sided exact categories are a generalization of exact categories. On the other hand, a one-sided exact category \mathcal{E} can be completed 2-canonically to an exact category \mathcal{E}^{ex} , called the *exact hull* of \mathcal{E} . This completion yields an equivalence $\mathbf{D}_\infty^b(\mathcal{E}) \rightarrow \mathbf{D}_\infty^b(\mathcal{E}^{\text{ex}})$ of the derived ∞ -categories (see [HR19a]), and as such, the flexibility added by considering the one-sided exact setting comes at no cost to applications concerning localizing invariants.

For the next quotient, namely the quotient of $[\text{LCA}_A / \text{LCA}_{A,com}]^{\text{ex}}$ by the image of the category $\text{LCA}_{A,ab}$ of adelic blocks, we use the following attractive property of $[\text{LCA}_A / \text{LCA}_{A,com}]^{\text{ex}}$: All morphisms have cokernels and all cokernels are deflations. As is shown in §5, this property holds for $[\text{LCA}_A / \text{LCA}_{A,com}]^{\text{ex}}$, as it holds for LCA_A (since the latter is quasi-abelian), and is preserved by both the localization and the exact hull.

The primary application is using nonconnective K -theory as the localizing invariant K . It is easily shown to agree with ordinary Quillen K -theory (i.e., connective K -theory) for LCA_A . Following this path, we recover the noncommutative analogue of the idèle class group due to Fröhlich [Frö75]. That is, we rediscover a group which Fröhlich defined manually in 1975 and without any reference to K -theory, and every part of Fröhlich’s definition fits perfectly (including being a restricted product and the infinite places).

Theorem B. *Let A be a finite-dimensional semisimple \mathbb{Q} -algebra.*

1. *There is a natural isomorphism*

$$K_1(\text{LCA}_{A,ab}) \xrightarrow{\sim} \frac{J(A)}{J^1(A)},$$

where $J(A)$ is Fröhlich’s idèle class group and $J^1(A)$ the subgroup of reduced norm one elements.

2. There is a natural isomorphism

$$K_1(\mathrm{LCA}_A) \xrightarrow{\sim} \frac{J(A)}{J^1(A) \cdot \mathrm{im} A^\times},$$

where the units A^\times are diagonally mapped to the idèles.

This will be Theorem 6.10. Not only does this reestablish Clausen's prediction, but it also proves the natural noncommutative analogue.

We can also describe K_2 in a precise way. Unlike the previous results, this relies on tools from class field theory.

Theorem C. *Let A be a finite-dimensional semisimple \mathbb{Q} -algebra. Write $\zeta(A)$ for its center and $(-)^{\wedge}$ for profinite completion.*

1. There is a natural isomorphism

$$K_2(\mathrm{LCA}_{A,ab}) \cong \bigoplus_v K_2(\zeta(A)_v),$$

where v runs through the places of the number field $\zeta(A)$.

2. If A is commutative (i.e., a number field),

$$K_2(\mathrm{LCA}_A)^{\wedge} \cong \mu(A),$$

where $\mu(-)$ denotes the group of roots of unity.

3. If Conjecture 10.3 of Merkurjev–Suslin holds,

$$K_2(\mathrm{LCA}_A)^{\wedge} \cong \mu(\zeta(A)),$$

that is, no condition on commutativity as in (2) is needed.

The individual parts of the above theorem are shown in the last three sections of the paper. These results also allow us to phrase Moore's formulation of the Hilbert reciprocity law [Moo68],

$$K_2(A) \longrightarrow \bigoplus_{v \text{ noncomplex}} \mu(A_v) \longrightarrow \mu(A) \longrightarrow 0$$

in terms of the fiber sequence of our Theorem A. Let us stress, however, that our work does not give an independent proof of Hilbert reciprocity because the proof of Theorem C relies on class field theory itself. For this reason, it would be very interesting if one could prove Theorem C using different technology. Moreover, we conjecture the following.

Conjecture 1.1. *Suppose A is a finite-dimensional simple \mathbb{Q} -algebra with center F . Then*

$$K_2(\mathrm{LCA}_A) \cong K_2(\mathrm{LCA}_F).$$

This conjecture would follow from the aforementioned conjecture of Merkurjev–Suslin (see 10.3 for its statement).

2. Structure theory

Conventions. Algebras are associative and unital but not necessarily commutative. Ring homomorphisms preserve the unit. The notation $\text{mod}(R)$ resp. $\text{proj}(R)$ refers to the categories of finitely generated right R -modules, resp. finitely generated projective right modules. Unless otherwise stated, all modules will be right modules.

Suppose A is a finite-dimensional semisimple \mathbb{Q} -algebra.

Definition 2.1. The category LCA_A has

1. as objects locally compact right A -modules, where A is given the discrete topology,
2. as morphisms all continuous right A -module homomorphisms.

An exact structure is given by declaring closed injections inflations and open surjections deflations.

The proof that this describes a quasi-abelian category with its standard exact structure can be adapted verbatim from Hoffmann–Spitzweck [HS07], who pioneered this kind of consideration and studied $\text{LCA}_{\mathbb{Z}}$.

Remark 2.2. Alternatively, one can define LCA_A as follows. Regard the ring A as a category with one object with itself as its endomorphism ring. Then let

$$\text{LCA}_A := \text{Fun}(A, \text{LCA}_{\mathbb{Z}}),$$

that is, we consider the functor category to plain locally compact abelian groups. Equip the functor category with the pointwise exact structure from $\text{LCA}_{\mathbb{Z}}$. Kernels and cokernels are computed pointwise, and hence LCA_A is quasi-abelian because $\text{LCA}_{\mathbb{Z}}$ is.

Example 2.3. We stress that the condition to be a topological A -module is a nontrivial constraint even though A carries the discrete topology. For example, not every \mathbb{Q} -vector space, equipped with some locally compact topology on its additive group, is a locally compact \mathbb{Q} -module. Indeed, from the viewpoint of Remark 2.2, an object is given by some $M \in \text{LCA}_{\mathbb{Z}}$ along with a ring homomorphism $A \rightarrow \text{End}_{\text{LCA}_{\mathbb{Z}}}(M)$ and even if $A = \mathbb{Q}$, the induced endomorphisms need not be continuous on M .

Pontryagin duality induces an exact equivalence of exact categories

$$\text{LCA}_A^{op} \xrightarrow{\sim} \text{LCA}_{A^{op}}.$$

If A is commutative, this renders LCA_A an exact category with duality. We first set up some basic structure results about the objects of the category LCA_A . The main definition is the following.

Definition 2.4. We call a module $Q \in \text{LCA}_A$ a *quasi-adelic block* if it can be written as

$$Q \simeq V \oplus H \quad \text{with} \quad H = \bigcup_{n \geq 1} \frac{1}{n}C, \quad (2.1)$$

where V is a vector A -module and C a compact clopen \mathfrak{A} -submodule of H , where $\mathfrak{A} \subset A$ is any \mathbb{Z} -order. We call it an *adelic block* if additionally C can be chosen such that

$$\bigcap_{n \geq 1} nC = 0.$$

We write $\text{LCA}_{A,qab}$ for the full subcategory of quasi-adelic blocks, and $\text{LCA}_{A,ab}$ for the one of adelic blocks.

This definition generalizes [AB19, Definition 2.3].

Remark 2.5. If $\mathfrak{A}, \mathfrak{A}' \subset A$ are \mathbb{Z} -orders, then there exists some $N \geq 1$ such that $\mathbb{Z}[\frac{1}{N}] \cdot \mathfrak{A} = \mathbb{Z}[\frac{1}{N}] \cdot \mathfrak{A}'$, so there is no difference whether in the above definition we pick one order \mathfrak{A} once and for all or allow any order (the way the definition is stated).

Lemma 2.6. *Let $\mathfrak{A} \subset A$ be any order. Then the forgetful functors $\text{LCA}_A \rightarrow \text{LCA}_{\mathfrak{A}}$ and $\text{LCA}_A \rightarrow \text{LCA}_{\mathbb{Q}}$ are fully faithful and reflect exactness.*

Proof. Clear. □

Lemma 2.7. *We record the following observations:*

1. *Every discrete right A -module is a projective object in LCA_A .*
2. *Every compact right A -module is an injective object in LCA_A . Moreover, they are connected.*
3. *Every vector A -module is both injective and projective in LCA_A . Moreover, they are connected.*

Proof. (1) Since A is semisimple, every (right) A -module P is projective, so for any surjection $M \rightarrow P$ a splitting exists as an A -module homomorphism. This gives a splitting also in LCA_A as soon as it is continuous, but since P carries the discrete topology, this is automatically true. (2) If I is a compact right A -module, its Pontryagin dual I^\vee is a discrete right A^{op} -module. By (1) applied to I^\vee for the semisimple algebra A^{op} , it follows that I^\vee is a projective object in $\text{LCA}_{A^{op}}$, and since Pontryagin duality is an exact functor, it follows that $I = I^{\vee\vee}$ is injective in LCA_A . Finally, a compact LCA group is connected if and only if its dual is torsion-free, but all A -modules are \mathbb{Q} -vector spaces and thus torsion-free; see [AB19, Lemma 2.21]. (3) Pick any order $\mathfrak{A} \subset A$. Regard the vector module as a vector \mathfrak{A} -module. Then it is projective and injective by [Bra19, Proposition 8.1] in $\text{LCA}_{\mathfrak{A}}$. The relevant splitting maps thus exist in $\text{LCA}_{\mathfrak{A}}$, then apply Lemma 2.6. □

Definition 2.8. We call an object $M \in \text{LCA}_A$ *vector-free* if it does not have a nonzero vector A -module as a subobject (equivalently as a quotient or, equivalently, as a direct summand).

The equivalence of these characterizations follows from the fact that vector A -modules are both injective and projective by Lemma 2.7.

Lemma 2.9. *Any vector-free adelic block Q is a topological torsion LCA group. In particular, Q is totally disconnected.*

Proof. By definition, $Q = \bigcup_{n \geq 1} \frac{1}{n}C$ and $\bigcap_{n \geq 1} nC = 0$ for C a compact clopen \mathfrak{A} -submodule of Q . Thus, the intersection of all open \mathbb{Z} -submodules is also zero. It follows that Q is totally disconnected ([HR79, Theorems 7.8 and 7.3]). Combined with [HR79, Theorem 7.7], it follows that Q admits arbitrarily small compact clopen subgroups such that the quotients

$$Q/nC = \bigcup_{N \geq 1} \frac{1}{N}(C/nC)$$

are discrete torsion groups. The topology is the discrete one since nC is open in C since multiplication with \mathbb{Q}^\times acts as homeomorphisms. Thus, by [Arm81, Theorem 3.5] the claim follows. \square

Lemma 2.10. *Write $\text{LCA}_{A,dis}$ and $\text{LCA}_{A,com}$ for the full subcategories of LCA_A of discrete A -modules and of compact A -modules. The following hold:*

1. $\text{Hom}_{\text{LCA}_A}(\text{LCA}_{A,qab}, \text{LCA}_{A,dis}) = 0$;
2. $\text{Hom}_{\text{LCA}_A}(\text{LCA}_{A,com}, \text{LCA}_{A,ab}) = 0$.

Proof.

1. Let $h: Q \rightarrow D$ be a map with $Q \in \text{LCA}_{A,qab}$ and $D \in \text{LCA}_{A,dis}$. Let $Q \cong V \oplus H$ and $H = \bigcup_{n \geq 1} \frac{1}{n}C$ be as in Definition 2.4. By Lemma 2.7, $h(V)$ is connected and thus $h(V) = 0$ as D is discrete. Similarly, $h(C)$ is compact as C is compact and thus $h(C)$ is finite as D is discrete. It follows that $h(H)$ is a torsion group and an A -module, thus $h(H) = 0$.
2. Let $h: K \rightarrow W$ be a map with $K \in \text{LCA}_{A,com}$ and $W \in \text{LCA}_{A,ab}$. By Lemma 2.7, K is connected. By Definition 2.4 and Lemma 2.9, the result follows. \square

Proposition 2.11. *For all $M \in \text{LCA}_A$, there exists a split conflation*

$$Q \twoheadrightarrow M \twoheadrightarrow D,$$

unique up to isomorphism, where Q is a quasi-adelic A -module and D is a discrete A -module. In other words, $(\text{LCA}_{A,qab}, \text{LCA}_{A,dis})$ is a split torsion pair in LCA_A .

Proof. (This generalizes and simplifies [AB19, Theorem 2.7]) Let $\mathfrak{A} \subset A$ be any order. By the structure theorem of $\text{LCA}_{\mathfrak{A}}$ there exists a conflation

$$V \oplus C \twoheadrightarrow M \twoheadrightarrow D' \tag{2.2}$$

with V a vector \mathfrak{A} -module, C a compact clopen \mathfrak{A} -module in M and D' a discrete \mathfrak{A} -module. Define

$$Q := \bigcup_{n \geq 1} \frac{1}{n}(V \oplus C)$$

inside M . As Sequence 2.2 is exact in $\text{LCA}_{\mathfrak{A}}$ and D' discrete, it follows that $V \oplus C$ is open in M . As the action of $\mathbb{Q} \subset A$ is continuous, \mathbb{Q}^\times acts through homeomorphisms and therefore each $\frac{1}{n}(V \oplus C)$ is also open. Thus, Q is a union of open sets and therefore itself an open (and thus clopen) \mathfrak{A} -module of M . As $\mathbb{Q} \cdot \mathfrak{A} = A$, it follows that Q is even an A -submodule. Since the scalar action of A on M is continuous, the restriction to the submodule Q is also continuous. Thus, $Q \in \text{LCA}_A$. Moreover, we clearly have the direct sum decomposition

$$Q = V \oplus H \quad \text{with} \quad H = \bigcup_{n \geq 1} \frac{1}{n}C$$

in LCA_A since $V = \bigcup_{n \geq 1} \frac{1}{n}V$ is already an A -module and $V \cap \frac{1}{n}C = 0$ for all $n \geq 1$ because the intersection is a compact subset of a real vector space and thus trivial. As Q is a clopen A -submodule of M , it is clear that the quotient exists in LCA_A and carries the discrete topology. Thus, we have a conflation

$$Q \twoheadrightarrow M \twoheadrightarrow D.$$

By Lemma 2.7, the discrete A -module D is a projective object, so this conflation splits.

It now follows from Lemma 2.10 and the fact that every M fits into a (split) conflation $Q \twoheadrightarrow M \twoheadrightarrow D$ that $(\text{LCA}_{A,qab}, \text{LCA}_{A,dis})$ is a split torsion theory. \square

Lemma 2.12. *For all $Q \in \text{LCA}_{A,qab}$, there exists a split conflation*

$$K \twoheadrightarrow Q \twoheadrightarrow W,$$

unique up to isomorphism, where K is a compact A -module and W is an adelic block. In other words, $(\text{LCA}_{A,com}, \text{LCA}_{A,ab})$ is a split torsion theory in $\text{LCA}_{A,qab}$.

Proof. (This generalizes and simplifies [AB19, Proposition 2.23]) If Q has a vector module summand V , we can write $Q = Q' \oplus V$, where Q' is vector-free. If we prove our claim for Q' , we prove it in general because if we have $Q' \cong K' \oplus W'$, we get $Q \cong K' \oplus (W' \oplus V)$ and the second summand in brackets is an adelic block. Thus, we may assume without loss of generality that Q is vector-free. We shall first construct a conflation

$$K \twoheadrightarrow Q \twoheadrightarrow W, \tag{2.3}$$

where K is a compact A -module and W an adelic block. To this end, write

$$Q = H \quad \text{with} \quad H = \bigcup_{n \geq 1} \frac{1}{n}C$$

with C a compact clopen \mathfrak{A} -submodule of H as in Definition 2.4. Define

$$K := \bigcap_{n \geq 1} nC. \tag{2.4}$$

Since C is compact, each nC is compact (image under the multiplication by n map), and thus K is an intersection of closed sets and therefore closed. Moreover, since each nC is an \mathfrak{A} -module so is K . Finally, any $k \in K$ admits elements $c_N \in C$ such that $k = N \cdot c_N$ for all $N \in \mathbb{Z}_{\geq 1}$. Thus,

$$\frac{1}{n}k = \frac{1}{n}Nnc_{Nn} = Nc_{Nn}$$

holds for all N . This shows that $\frac{1}{n}k \in K$, and by $\mathbb{Q} \cdot \mathfrak{A} = A$ it follows that K is an A -submodule of H . Further, since the scalar action of A is continuous on Q , it remains so on K . Thus, $K \in \text{LCA}_A$ and since $K \subseteq C$, it is a compact A -module. The quotient $\pi: Q \twoheadrightarrow Q/K (= W)$ is an open map. We now show that W is an adelic block. As C is a clopen compact in Q , the image $\pi(C)$ is clopen compact in W . As π is surjective, we have

$$W = \bigcup_{n \geq 1} \frac{1}{n}\pi(C).$$

We need to show $\bigcap_{n \geq 1} n\pi(C) = 0$ in W . Suppose $x \in Q$ such that in the quotient W we have $q(x) = \bigcap_{n \geq 1} n\pi(C)$. Thus, for any $n \geq 1$ we find $x_n \in C$ such that

$$x = nx_n + k_n$$

with $k_n \in K$. Write $k_n = nh_n$ for some $h_n \in C$, which is possible by Equation (2.4). Hence, $x = n(x_n + h_n)$ with $x_n, h_n \in C$, so $x \in \bigcap_{n \geq 1} nC = K$, that is, x maps to zero in W . Thus, K is an adelic block. Since K is a compact A -module, it is injective by Lemma 2.7, so the conflation in Equation (2.3) splits. Finally, by Lemma 2.10, $\text{Hom}_{\text{LCA}_A}(\text{LCA}_{A,com}, \text{LCA}_{A,ab}) = 0$. The result follows. \square

Corollary 2.13. *Suppose $M \in \text{LCA}_A$. Then the canonical filtration of Hoffmann–Spitzweck [HS07, Proposition 2.2] lifts to a canonical filtration*

$$M_{\mathbb{S}^1} \twoheadrightarrow F_{\mathbb{Z}}M \twoheadrightarrow M$$

in LCA_A . Here $M_{\mathbb{S}^1}$ is a compact A -module, $M_{\mathbb{A}} := F_{\mathbb{Z}}M/M_{\mathbb{S}^1}$ is an adelic block and $M_{\mathbb{Z}} := M/F_{\mathbb{Z}}M$ is a discrete vector space. Each step of the filtration splits and thus yields a (noncanonical) direct sum decomposition

$$M = K \oplus X \oplus D \tag{2.5}$$

such that X is an adelic block, K a compact A -module and D a discrete A -module.

Let us also recall that the Hoffmann–Spitzweck filtration is functorial: Any continuous morphism respects the filtration [HS07, Proposition 2.2].

Proof. (This generalizes and simplifies [AB19, Theorem 3.10]) Combining Proposition 2.11 and Lemma 2.12 gives the second claim, Equation (2.5). Now, combine this with [HS07, Proposition 2.2] and observe that the topological properties of K, X, D pin down how they have to occur in the canonical filtration. The rough idea is as follows: Let M^0 be the connected component of M . This is canonically determined. Using

$$M = K \oplus X \oplus D,$$

we see that it can only stem from $K \oplus V$, where V is the vector module summand in X , since D is discrete and adelic blocks aside from the vector module are totally disconnected (Lemma 2.9). Thus, $K \oplus V$ is canonically determined. The Pontryagin dual is $K^\vee \oplus V^\vee$, where V^\vee is connected and K^\vee discrete. So, again the connected component uniquely pins down a subobject and thus allows us to canonically characterize V as a quotient of M^0 . This settles $M_{\mathbb{S}^1}$. □

Remark 2.14. It is not true that the entire category $\text{LCA}_{A,qab}$ is a direct sum of $\text{LCA}_{A,com}$ and $\text{LCA}_{A,ab}$ as there can be nonzero morphisms from adelic blocks to compact A -modules. To see this, consider the dense image morphism

$$\mathbb{Q} \longrightarrow \mathbb{Q}_p$$

in $\text{LCA}_{\mathbb{Q}}$ and take its Pontryagin dual, noting that $\mathbb{Q}_p^\vee \simeq \mathbb{Q}_p$ is self-dual, but the dual of \mathbb{Q} is a compact \mathbb{Q} -vector space.

Lemma 2.15. *For an object $M \in \text{LCA}_A$ its discrete piece $M_{\mathbb{Z}}$ in the Hoffmann–Spitzweck filtration is a finitely generated A -module if and only if $\text{Hom}_{\text{LCA}_A}(M, A_{\mathbb{R}})$ is a finite-dimensional real vector space.*

As the output object $A_{\mathbb{R}}$ has a \mathbb{R} -vector space structure, this Hom-group canonically comes with an enriched structure as an \mathbb{R} -vector space itself.

Proof. By Corollary 2.13, we have

$$\text{Hom}(M, A_{\mathbb{R}}) = \text{Hom}(K, A_{\mathbb{R}}) \oplus \text{Hom}(X, A_{\mathbb{R}}) \oplus \text{Hom}(D, A_{\mathbb{R}})$$

and since morphisms respect the Hoffmann–Spitzweck filtration [HS07, Proposition 2.2], the first summand is always zero, the second always finite-dimensional (since the vector-free summand of the adelic block maps trivially to $A_{\mathbb{R}}$ because the compact C in $H = \bigcup \frac{1}{n}C$ as in Equation (2.1) must map to zero in $A_{\mathbb{R}}$), so we deduce that $\text{Hom}(M, A_{\mathbb{R}})$ is finite-dimensional if and only if $\text{Hom}(D, A_{\mathbb{R}})$ is. We have $\text{Hom}(A, A_{\mathbb{R}}) \cong \mathbb{R}^n$ for $n := \dim_{\mathbb{Q}}(A)$ since A is a free rank one module over itself, so an A -module homomorphism is given by assigning an arbitrary value, which, since A is discrete, any such map is automatically continuous. Since A is semisimple, any indecomposable $D \subseteq A$ correspondingly has $\text{Hom}(A, A_{\mathbb{R}})$ a finite-dimensional \mathbb{R} -vector space, and moreover any discrete A -module is a (possibly infinite) direct sum of indecomposables, $D = \bigoplus_{i \in I} D_i$. Thus, $\text{Hom}(\bigoplus_i D_i, A_{\mathbb{R}}) = \prod_i \text{Hom}(D_i, A_{\mathbb{R}})$ is a finite-dimensional \mathbb{R} -vector space if and only if I is a finite set, that is, if and only if D is finitely generated. □

Next, we prove a “Serre subcategory”-type statement about whether the discrete piece in the Hoffmann–Spitzweck filtration is finitely generated as an A -module.

Lemma 2.16. *Suppose $X' \twoheadrightarrow X \twoheadrightarrow X''$ is a conflation in LCA_A . Then the middle object X in its decomposition of Corollary 2.13 has finitely generated discrete A -module part $X_{\mathbb{Z}}$, if and only if $X'_{\mathbb{Z}}$ and $X''_{\mathbb{Z}}$ are finitely generated A -modules.*

Proof. (This generalizes and isolates the essence of [AB19, Propositions 2.14, 2.17 and Theorem 2.19]) Since $A_{\mathbb{R}}$ is an injective object by Lemma 2.7, the sequence

$$\mathrm{Hom}(X'', A_{\mathbb{R}}) \twoheadrightarrow \mathrm{Hom}(X, A_{\mathbb{R}}) \twoheadrightarrow \mathrm{Hom}(X', A_{\mathbb{R}})$$

is exact and now the claim follows from Lemma 2.15 and the fact that finitely generated real vector spaces are a Serre subcategory in the category of all real vector spaces. \square

Example 2.17. The above lemma is only true for “finite generation”, but not in a stronger form which would test being nonzero. The adèle sequence $A \twoheadrightarrow \mathbb{A}_A \twoheadrightarrow \mathbb{A}_A/A$, for example, has $(\mathbb{A}_A)_{\mathbb{Z}} = 0$, yet $A_{\mathbb{Z}} = A$ is nonzero.

Lemma 2.18. *We also obtain the following facts.*

1. *If $M \in \mathrm{LCA}_A$ and $\mathrm{Hom}(M, A) = 0$, then $M \in \mathrm{LCA}_{A, qab}$.*
2. *If $M \twoheadrightarrow M''$ is a deflation and M quasi-adelic, then M'' is quasi-adelic.*
3. *The subcategories $\mathrm{LCA}_{A, qab}$ and $\mathrm{LCA}_{A, ab}$ of LCA_A both are extension-closed.*
4. *If $M' \twoheadrightarrow M$ is an inflation with M' discrete and M quasi-adelic, then M' is finitely generated.*

Proof. Let $M \in \mathrm{LCA}_A$ be arbitrary. Then we may decompose M as a direct sum as done in Corollary 2.13, giving

$$\mathrm{Hom}(K \oplus X \oplus D, A) = \mathrm{Hom}(D, A)$$

because all morphisms from K or X to the discrete module A must be zero by preservation of the Hoffmann–Spitzweck filtration. As A is semisimple, any $D \neq 0$ is a projective A -module, that is, it occurs as a direct summand of some direct sum $A^{\oplus \kappa}$. It follows that $D \neq 0$ is equivalent to $\mathrm{Hom}(D, A) \neq 0$.

1. Assume that M is quasi-adelic. By Lemma 2.10, $\mathrm{Hom}(M, A) = 0$ as A is discrete. Conversely, assume that $\mathrm{Hom}(M, A) = 0$. As above, $\mathrm{Hom}(M, A) = \mathrm{Hom}(D, A) = 0$ implies that $D = 0$ and thus M is quasi-adelic.
2. Let $M' \twoheadrightarrow M \twoheadrightarrow M''$ be a conflation in LCA_A . Applying $\mathrm{Hom}(-, A)$, we obtain the left-exact sequence

$$0 \rightarrow \mathrm{Hom}(M'', A) \rightarrow \mathrm{Hom}(M, A) \rightarrow \mathrm{Hom}(M', A).$$

If M is quasi-adelic, $\mathrm{Hom}(M, A) = 0$ by (1) and thus $\mathrm{Hom}(M'', A) = 0$ as well. Again, by (1), M'' is quasi-adelic.

3. Analogously to (2), one finds that if $M', M'' \in \mathrm{LCA}_{A, qab}$, then $M \in \mathrm{LCA}_{A, qab}$ and thus $\mathrm{LCA}_{A, qab} \subseteq \mathrm{LCA}_A$ is extension-closed. If $M', M'' \in \mathrm{LCA}_{A, ab}$, then $M \in \mathrm{LCA}_{A, qab}$. Hence, $M \cong K \oplus X$, furthermore, by Lemma 2.10, $\mathrm{Hom}(K, M'') = 0$ and thus K is a direct summand of M' . As $M' \in \mathrm{LCA}_{A, ab}$, $K = 0$ and thus $M \cong X \in \mathrm{LCA}_{A, ab}$.
4. This follows from Lemma 2.16 by observing that $M_{\mathbb{Z}} = 0$ if M is quasi-adelic. \square

3. Recollections on the idèle class group

Let $\mathfrak{A} \subset A$ be an arbitrary \mathbb{Z} -order. We shall use standard notation, inspired by the choices in [Frö75, §2], [BF01, §2.7]. For any prime number p define

$$\mathfrak{A}_p := \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \text{and} \quad A_p := A \otimes_{\mathbb{Q}} \mathbb{Q}_p,$$

where \mathbb{Z}_p denotes the p -adic integers and \mathbb{Q}_p its field of fractions, the p -adic numbers.

Definition 3.1. We write $\mathcal{P}\ell_F$ for the set of places of a number field F . For simplicity, we also write $\mathcal{P}\ell$ in place of $\mathcal{P}\ell_{\mathbb{Q}}$. We write $\mathcal{P}\ell_{\text{fin}, F}$ for the finite places. We denote the infinite place of \mathbb{Q} , often denoted by ∞ in the literature, by \mathbb{R} . This increases the compatibility to literature, where the real completion is denoted by $A_{\mathbb{R}}$, in the style of a base change, rather than the less customary A_{∞} (as, for example, in [BF01]).

Recall that

$$\mathfrak{A}_{\mathbb{R}} = A_{\mathbb{R}} = A \otimes_{\mathbb{Q}} \mathbb{R}.$$

As A is a finite-dimensional semisimple \mathbb{Q} -algebra, it follows that A_p (for p a prime number) is a finite-dimensional semisimple \mathbb{Q}_p -algebra and \mathfrak{A}_p is a \mathbb{Z}_p -order in it (this is [Rei03, (11.1), (11.2) and (11.5)]). We also define

$$\widehat{\mathfrak{A}} = \mathfrak{A} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \prod_{p \neq \mathbb{R}} \mathfrak{A}_p \quad \text{and} \quad \widehat{A} = \widehat{\mathfrak{A}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Here, $\widehat{\mathbb{Z}}$ is the profinite completion of the integers. The alternative characterization follows from the decomposition into topological p -parts, $\widehat{\mathbb{Z}} = \prod \mathbb{Z}_p$.

Let us recall the concept of idèles. The classical commutative story is, for example, reviewed in [Har20, §12.3]. For the noncommutative generalization, we follow Fröhlich’s paper [Frö75, §2]. The general *idèle group* is defined to be

$$J(A) := \left\{ (a_p)_p \in \prod_{p \in \mathcal{P}\ell} A_p^{\times} \mid a_p \in \mathfrak{A}_p^{\times} \text{ for all but finitely many places } p \right\}. \tag{3.1}$$

At the infinite place $p = \mathbb{R}$ this means $\mathfrak{A}_{\mathbb{R}} = A_{\mathbb{R}}$, so there the condition is void.

It turns out that this group actually does not depend on the choice of an order \mathfrak{A} because if $\mathfrak{A}' \subset A$ is another \mathbb{Z} -order, we have $\mathfrak{A}'_p = (\mathfrak{A}')_p$ for all but finitely many places p . But then in view of Equation (3.1), the group $J(A)$ does not change. The group

$$J^1(A) := \{ (a_p)_p \in J(A) \mid \text{nr}_{A_p}(a_p) = 1 \} \tag{3.2}$$

is the subgroup of *reduced norm one idèles*. We denote the center of a ring A by $\zeta(A)$.

We write

$$\prod'_v K_n(A_v) := \left\{ (\alpha_v)_v \in \prod_{v \in \mathcal{P}\ell_F} K_n(A_v) \mid \begin{array}{l} \alpha_v \in \text{im } K_n(\mathfrak{A}_v) \\ \text{for all but finitely many } v \end{array} \right\} \tag{3.3}$$

for $\mathfrak{A} \subset A$ any order and $\mathfrak{A}_v := \mathfrak{A} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v}$. The group defined in Equation 3.3 is independent of the choice of the order $\mathfrak{A} \subset A$. Here and everywhere below, we use the

notation \prod'_v with the above meaning and not in the sense of a restricted product of topological groups.

Example 3.2. Suppose $A = F$ is a number field and $n = 1$. Then each $K_1(F_v) = F_v^\times$ may also be regarded as a locally compact topological group with respect to the metric on F_v and $\mathcal{O}_{F,v}^\times \subseteq F_v^\times$ is a compact clopen subgroup. Then the restricted product $\prod'_v K_1(F_v)$ in the sense of topological groups has the same underlying abstract group as the group in Equation (3.3). This justifies using the notation \prod'_v also for K_n with $n \neq 1$.

4. Adelic blocks

4.1. Description as a 2-colimit

Let $\mathfrak{A} \subset A$ be a maximal order. Such always exists [Rei03, Corollary 10.4], but it is not unique.

As \mathfrak{A} is maximal, it follows that each \mathbb{Z}_p -order $\mathfrak{A}_p \subset A_p$ is also maximal (this is [Rei03, Corollary 11.6]). We shall need the following basic fact.

Lemma 4.1. *Every finitely generated \mathfrak{A} -module M (for $\mathfrak{A} \subset A$ a maximal order) has the shape*

$$M \cong M_{tor} \oplus P,$$

where M_{tor} is the submodule of \mathbb{Z} -torsion elements, and P a finitely generated projective \mathfrak{A} -module. The same is true for maximal orders $\mathfrak{A}_p \subseteq A_p$.

Proof. This is standard but apparently not spelled out in the standard reference [Rei03]. Since any morphism sends a \mathbb{Z} -torsion element to a \mathbb{Z} -torsion element, this in particular holds for the \mathfrak{A} -module structure. Thus, the \mathbb{Z} -torsion elements M_{tor} form a right \mathfrak{A} -submodule of M . As M is finitely generated (even over \mathbb{Z}), it follows that M_{tor} is finite as a set. We get a conflation $M_{tor} \hookrightarrow M \rightarrow M/M_{tor}$. Thus, we are done if we can show that every \mathbb{Z} -torsion-free \mathfrak{A} -module is projective. So, suppose M is torsion-free. We follow [Rei03, Theorem 10.6]. Tensoring with \mathbb{Q} gives the embedding $M \hookrightarrow M \otimes_{\mathbb{Z}} \mathbb{Q}$, the latter is an A -module, thus has some free resolution, giving a surjection

$$A^{\oplus n} \twoheadrightarrow M \otimes_{\mathbb{Z}} \mathbb{Q}$$

for some integer n . As A is semisimple, this sequence must split, giving $M \subseteq M \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq A^{\oplus n}$. As M is finitely generated, there exists some $N \geq 1$ such that $M \subseteq \frac{1}{N} \mathfrak{A}^{\oplus n}$ (just using that $\mathfrak{A} \cdot \mathbb{Q} = A$). Since \mathfrak{A} is a maximal order, it is hereditary ([Rei03, Theorem 21.4]), and thus its submodule M is projective. \square

For any exact categories $(C_i)_{i \in I}$ and index set I one may define a product exact category $\prod_{i \in I} C_i$. Its objects are arrays $X = (X_i)_{i \in I}$ with $X_i \in C_i$ and morphisms $X \rightarrow X'$ are termwise morphisms $X_i \rightarrow X'_i$ in C_i . Conflations are termwise conflations.

Lemma 4.2. *If each exact category C_i is split exact, so is $\prod_{i \in I} C_i$.*

Proof. Given a conflation $(X'_i) \hookrightarrow (X_i) \twoheadrightarrow (X''_i)$ in the product category, we have conflations $X'_i \hookrightarrow X_i \xrightarrow{q_i} X''_i$ in \mathbf{C}_i , and if s_i is a splitting of q_i , the morphism $(q_i)_{i \in I}$ defines a splitting in the product category. \square

Definition 4.3. Suppose $\mathfrak{A} \subset A$ is a maximal order. Let S be a finite set of prime numbers. Define

$$J_A^{(S)} := \text{proj}(A_{\mathbb{R}}) \times \prod_{p \notin S} \text{proj}(\mathfrak{A}_p) \times \prod_{p \in S} \text{proj}(A_p).$$

For any inclusion of finite sets $S \subseteq S'$ of primes, there is an exact functor $J^{(S)} \rightarrow J^{(S')}$ induced from $(-) \mapsto (-) \otimes_{\mathfrak{A}_p} A_p$ termwise for all primes $p \in S' \setminus S$ and the identity functor for the remaining p . Define

$$J_A^{(\infty)} := \text{colim}_S J_A^{(S)}. \tag{4.1}$$

This definition is actually independent of the choice of the maximal order, see Remark 4.11.

Remark 4.4 (2-Colimit). The colimit in Equation (4.1) is taken of a diagram, indexed by a partially ordered set, taking values in the 2-category of exact categories. One might call this a 2-colimit, but it is also common to merely call this a colimit.

Proposition 4.5. *Suppose $\mathfrak{A} \subset A$ is a maximal order. Then there is an exact equivalence of exact categories*

$$\text{LCA}_{A,ab} \xrightarrow{\sim} J_A^{(\infty)}$$

and in particular the definition of $J_A^{(\infty)}$ is independent of the choice of the order $\mathfrak{A} \subset A$. The functor sends X to the array $(X_p)_p$ of topological p -torsion parts,¹ resp. the vector module part, both on objects as well as morphisms.

Note that the individual categories $J_A^{(S)}$ do not even possess an enrichment in A -modules (or even \mathbb{Q} -vector spaces). This only exists in the colimit.

4.2. Proof of Proposition 4.5

We sketch the strategy. We first ‘fatten up’ the category $\text{LCA}_{A,ab}$ to a category M_0 with extra data and a forgetful functor

$$M_0 \xrightarrow{\sim} \text{LCA}_{A,ab}.$$

Then we consider what looks like a subcategory $M_1 \subseteq M_0$ at first but turns out to be an equivalence as well: $M_1 \xrightarrow{\sim} M_0$. Next, on M_1 one can explicitly write down a functor to $J_A^{(\infty)}$ and show that it is an equivalence. We obtain a chain of exact equivalences

$$J_A^{(\infty)} \xleftarrow{\sim} M_1 \xrightarrow{\sim} M_0 \xrightarrow{\sim} \text{LCA}_{A,ab}.$$

Finally, we check that these functors transform Pontryagin duality into linear duality.

¹See Definition 4.7.

4.2.1. Step 1. We define an exact category M_0 first. Its objects are pairs

$$(X, C),$$

where X is a vector-free adelic block and $C \subset X$ is a compact clopen \mathfrak{A} -submodule as in Equation (2.1), that is,

$$\bigcup_n \frac{1}{n}C = X \quad \text{and} \quad \bigcap_n \frac{1}{n}C = 0. \tag{4.2}$$

Morphisms $(X, C) \rightarrow (X', C')$ are just morphisms of adelic blocks $X \rightarrow X'$ in $LCA_{A, ab}$. Conflations are such if the adelic blocks form a conflation. In particular, neither for morphisms nor conflations, there is any dependency on C . There is an obvious functor

$$\Psi_0 : M_0 \longrightarrow LCA_{A, ab}, \quad (X, C) \longmapsto X$$

forgetting the choice of C . This is clearly fully faithful and exact. It is also essentially surjective since each adelic block possesses a choice of C by definition. Thus, the above functor Ψ_0 is an exact equivalence of exact categories,

Definition/Proposition 4.6. For a prime number p , define the *topological p -torsion part*

$$C_p := \left\{ x \in C \mid \lim_{n \rightarrow \infty} p^n x = 0 \right\}. \tag{4.3}$$

This is a closed \mathfrak{A} -submodule of C . Moreover, it carries a canonical right \mathfrak{A}_p -module structure through

$$x \cdot \alpha = \lim_{n \rightarrow \infty} (x \cdot \alpha_n), \tag{4.4}$$

where (α_n) is any sequence with $\alpha_n \in \mathfrak{A}$ converging to α in the p -adic topology of \mathfrak{A}_p .

Proof. For a general LCA group, the topological p -torsion part need not be a closed subgroup, but we know that C is totally disconnected and can invoke [Arm81, Lemma 3.8], settling that C_p is closed in C . The limit condition is preserved under the \mathfrak{A} -module action. The \mathfrak{A}_p -module stems from $\mathfrak{A}_p = \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and \mathbb{Z} being dense in \mathbb{Z}_p in the p -adic topology. The existence of the limit is ensured since in the p -adic topology $|\alpha_n - \alpha|$ getting smaller means $\alpha_n - \alpha \in p^m \mathbb{Z}_p$ for $m \rightarrow \infty$ as $n \rightarrow \infty$, and therefore $(\alpha_n - \alpha)x \rightarrow 0$ thanks to the topological p -torsion condition in Equation (4.3). \square

Definition 4.7. For a prime number p , we also define

$$X_p := \left\{ x \in X \mid \lim_{n \rightarrow \infty} p^n x = 0 \right\}.$$

This is a closed A -submodule of X . It carries a canonical right A_p -module structure, again defined by Equation (4.4).

This is shown analogously. Again, [Arm81, Lemma 3.8] settles that $X_p \subseteq X$ is closed. The rest can be deduced in the same way, but by a simple exercise we also have the description

$$X_p = \bigcup_n \frac{1}{n}C_p \quad \text{and} \quad \bigcap_n \frac{1}{n}C_p = 0 \tag{4.5}$$

analogous to Equation (4.2), so one can alternatively reduce this to the previous considerations about C_p . Equation (4.5) also shows that each X_p is a vector-free adelic block.

Lemma 4.8. *C_p is a finitely generated projective \mathfrak{A}_p -module, X_p is a finitely generated projective A_p -module and*

$$X_p = C_p \otimes_{\mathfrak{A}_p} A_p \tag{4.6}$$

holds.

Proof. From $C_p \subseteq C$ we get $C_p/pC_p \subseteq C/pC$. However, $p \in A^\times$ is a unit, so multiplication by p is a homeomorphism of X to itself, so it maps the open C to an open pC . Thus, C/pC is compact (as C is compact) and discrete (as pC is open) and therefore finite. It follows that C_p/pC_p is finite. Now, conclude that C_p is finitely generated by the topological Nakayama lemma for compact modules; see [L90, Chapter 5, §1] (instead of C_p being finitely generated, this takes compactness as input). Next, $X_p = C_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ follows directly from Equation (4.5). This yields Equation (4.6) and also implies the finite generation property for X_p . As A_p is semisimple, it is clear that X_p is projective. Next, by Lemma 4.1 we learn that C_p is projective since by $C_p \subseteq X_p$ it must be \mathbb{Z} -torsion-free. \square

Lemma 4.9. *There is an isomorphism of topological \mathfrak{A} -modules, $\prod_p C_p \xrightarrow{\sim} C$, induced from the respective inclusions $C_p \subseteq C$.*

Proof. [Arm81, Proposition 3.10]. \square

4.2.2. Step 2. Next, we define a category M_1 . Its objects are pairs (X, C) as before, but this time morphisms $(X, C) \rightarrow (X', C')$ are morphisms of adelic blocks

$$f: X \longrightarrow X' \quad \text{such that} \quad f(C_p) \subseteq C'_p$$

holds for all but finitely many prime numbers p . This is clearly closed under composition. There is an obvious faithful functor

$$\Psi_1: M_1 \longrightarrow M_0,$$

as we have just imposed an additional condition on morphisms. We claim that the functor is also full: Let $(X, C) \rightarrow (X', C')$ be any morphism in M_0 , say coming from $f: X \rightarrow X'$. As C is compact, the set-theoretic image $f(C) \subseteq X'$ is compact. Further, $X' = \bigcup_n \frac{1}{n}C'$ is an open cover, so since $f(C)$ is compact, a finite subcover suffices. Thus, there exists some $n \geq 1$ such that $f(C) \subseteq \frac{1}{n}C'$. In particular, $f(C_p) \subseteq \frac{1}{n}C'_p$. However, C'_p is an \mathbb{Z}_p -module (as it is a \mathfrak{A}_p -module by the module structure of Definition 4.6). Hence, for all primes p not dividing n , $\frac{1}{n} \in \mathbb{Z}_p^\times$ is a unit, so $\frac{1}{n}C'_p = C'_p$. In other words: Except for finitely many

primes we have $f(C_p) \subseteq C'_p$, as required. Since Ψ_1 is an equivalence of categories, we can transport the exact structure from M_0 to M_1 : It just amounts to kernel-cokernel sequences

$$(X', C') \longrightarrow (X, C) \longrightarrow (X'', C'')$$

being a conflation if and only if $X' \hookrightarrow X \twoheadrightarrow X''$ is a conflation of adelic blocks.

Lemma 4.10. *If $f: (X, C) \rightarrow (X', C')$ is a morphism in M_1 , we have $f(X_p) \subseteq X'_p$. Moreover, $f|_{X_p}$ is a morphism of A_p -modules.*

Proof. We had seen a little above the lemma that $f(C) \subseteq \frac{1}{n}C'$. Suppose $x \in C_p$. Then the element $f(x)$ is also topological p -torsion because f is continuous and therefore $\lim_{n \rightarrow \infty} p^n f(x) = f(\lim_{n \rightarrow \infty} p^n x) = 0$. If ℓ denotes a distinct prime, let pr_ℓ denote the projection $C' \rightarrow C'_p$ coming from applying Lemma 4.9 to C' . It follows that $pr_\ell f(x)$ is also topological p -torsion (note that $\frac{1}{n}C'$ is isomorphic to C'). Hence, it is topological I -torsion in for the ideal $I = (p, \ell) = (1)$. Thus, $pr_\ell f(x) = 0$. As this works for all primes $\ell \neq p$, we deduce that $f(x) \in \frac{1}{n}C'_p$. Now, for a general $x \in X_p$ some multiple mx lies in C_p , so our claim follows. Note that $f|_{X_p}$ being an A_p -module homomorphism just comes from the compatible way how the module structure is defined in Definition 4.7. \square

4.2.3. Step 3. Now, we define a functor

$$\Psi_2: M_1 \longrightarrow \operatorname{colim}_S J_A^{(S)}.$$

On objects, send (X, C) to the array $(C_p)_p$. By Lemma 4.8 each C_p is a finitely generated projective \mathfrak{A}_p -module, so we can take $S = \emptyset$. For any morphism

$$(X, C) \longrightarrow (X', C')$$

in M_1 , we know that for all but finitely many primes p we have $f(C_p) \subseteq C'_p$ by the very definition of M_1 . Take S to be these primes. Then for all primes $p \notin S$ we have $f|_{C_p}: C_p \rightarrow C'_p$ as a morphism of \mathfrak{A}_p -modules by Lemma 4.10. For the primes $p \in S$, the same two lemmata say that $f|_{X_p}$ is a morphism of finitely generated projective A_p -modules. These constructions are compatible with the transition morphisms of the 2-colimit by Equation (4.6). It is clear that this describes a faithful functor. It is also full: Suppose we are given a morphism in $J_A^{(S)}$ for a finite set S . For $p \notin S$, every morphism of finitely generated \mathfrak{A}_p -modules is automatically continuous. Thus, it defines a map $C_p \rightarrow C'_p$ for each such p and then induces one to the product

$$\prod_{p \notin S} C_p \rightarrow \prod_{p \notin S} C'_p. \tag{4.7}$$

For the $p \in S$, one can do this similarly: Since the source module is finitely generated, its image is finitely generated, thus lies in $\frac{1}{p^N}C'_p$ for a suitable N and just use this instead of C'_p on the right in Equation (4.7) for the $p \in S$. By Lemma 4.9 and since $X = \bigcup \frac{1}{n}C$ (and similarly for X'), this defines a morphism $X \rightarrow X'$.

Finally, we claim that Ψ_2 is essentially surjective: Let Y be an object in $J_A^{(S)}$. Define for all primes p ,

$$\widehat{Y}_p := Y \otimes_{\mathfrak{A}_p} A_p,$$

and note that this is well defined on the 2-colimit (it does not change under the transition maps). Each \widehat{Y}_p is a finitely generated projective A_p -module, having a natural topology as a finite-dimensional \mathbb{Q}_p -vector space. Hence, we may choose some finitely generated torsion-free \mathfrak{A}_p -submodule \widehat{C}_p such that $\mathbb{Q} \cdot \widehat{C}_p = \widehat{Y}_p$. Then each \widehat{Y}_p is locally compact and \widehat{C}_p a compact clopen group. Now, take the restricted product $\prod'(\widehat{Y}_p, \widehat{C}_p)$, [Arm81, p. 6, P.14].² This is a locally compact abelian group and a \mathfrak{A} -module through the inclusions $\mathfrak{A} \subset \mathfrak{A}_p$. Its topological p -torsion component is precisely \widehat{Y}_p .

Thus, Ψ_2 is an equivalence of categories. It also preserves the exact structures since by the previous considerations an inflation in M_1 just corresponds to an injection, and conversely all these are indeed automatically closed immersions (and analogously for deflations).

This finishes the proof of Proposition 4.5.

Remark 4.11. Suppose we pick a different maximal order $\mathfrak{A}' \subset A$ in Definition 4.3. Then the definition does not actually change because the local maximal orders $\mathfrak{A}'_p \subset A_p$ will be isomorphic, $\mathfrak{A}_p \simeq \mathfrak{A}'_p$, [AG60, Proposition 3.5]. If A_p is a division algebra, the maximal order is unique.

4.3. Duality

There is an exact functor to the opposite category

$$(-)_A^* : J_A^{(S)} \longrightarrow J_{A^{op}}^{(S),op}, \tag{4.8}$$

defined on the factor categories by

$$X_p \longmapsto \begin{cases} \text{Hom}_{A_{\mathbb{R}}}(X_{\mathbb{R}}, A_{\mathbb{R}}) & \text{for } p = \mathbb{R} \\ \text{Hom}_{\mathfrak{A}_p}(X_p, \mathfrak{A}_p) & \text{for } p \notin S \\ \text{Hom}_{A_p}(X_p, A_p) & \text{for } p \in S. \end{cases} \tag{4.9}$$

These Hom-modules use up the right module structures and thus are then left $A_{\mathbb{R}}$ (resp. \mathfrak{A}_p, A_p) modules. Or, as we shall phrase it, right modules over the opposite ring. Note that \mathfrak{A}^{op} is a maximal order in A^{op} (if there was a bigger one, its opposite would be bigger than \mathfrak{A}). Thus, all our considerations above regarding $J_A^{(S)}$ also apply to $J_{A^{op}}^{(S)}$. Since the input modules are projective, these functors are all exact and moreover there is a natural equivalence of functors

$$\text{id} \longrightarrow (-)_A^* \circ [(-)_{A^{op}}^*]^{op}.$$

In particular, $(-)_A^*$ is an exact equivalence of exact categories.

²Instead of restricted product, it is called a *local direct product* in the cited reference.

Proposition 4.12. *The functors in Equation (4.8) induce an exact equivalence of categories in the 2-colimit*

$$J_A^{(\infty)} \xrightarrow{\sim} J_{A^{op}}^{(\infty),op}$$

and under the exact equivalence of Proposition 4.5 this identifies with Pontryagin duality on adelic blocks $LCA_{A,ab} \xrightarrow{\sim} LCA_{A^{op},ab}^{op}$.

Proof. For the first claim, we only need to check that the duality functor respects the transition morphisms in the 2-colimit diagram. We only need to check this for those p where the transition map is not the identity, that is, where X_p is a finitely generated \mathfrak{A}_p -module. However, the compatibility then amounts to the harmless isomorphism

$$A_p \otimes_{\mathfrak{A}_p} \text{Hom}_{\mathfrak{A}_p}(X_p, \mathfrak{A}_p) \cong \text{Hom}_{A_p}(X_p \otimes_{\mathfrak{A}_p} A_p, A_p), \tag{4.10}$$

where we see the relevant transition maps on either side, respectively. Equation (4.10) is an isomorphism because it is a central localization by $\mathbb{Z}_p \setminus \{0\}$.

The second fact is more surprising since Pontryagin duality is based on $\text{Hom}_{LCA}(-, \mathbb{T})$ instead. To check this, we proceed by the following reductions: For each prime p , the equivalence identifies the full subcategory $\text{proj}(A_p) \subset J_A^{(\infty)}$ with the full subcategory of topological p -torsion adelic blocks. As we already have an equivalence of categories, restrict it to $\text{proj}(A_p)$ and its essential image. Being an equivalence of categories, it suffices to work with $\text{proj}(A_p)$ and compare its natural duality (Equation (4.9)) with the duality transported from the topological p -torsion adelic blocks (i.e., Pontryagin duality) along this equivalence to $\text{proj}(A_p)$. If these are compatible for all primes p and $p = \mathbb{R}$, then compatibility follows for the 2-colimit. So, first, fix a prime p . Since A_p is a projective generator of $\text{proj}(A_p)$, it suffices to prove compatibility on this object. We have

$$\text{Hom}_{\mathbb{Q}_p}(A_p, \mathbb{Q}_p) \cong \text{Hom}_{A_p}(A_p, A_p) = A_p^*$$

by tensoring with A_p . Now, compose any φ in the left group with the composition of maps

$$\mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \subset \mathbb{Q}/\mathbb{Z} \xrightarrow{e} \mathbb{T} \quad e(x) := \exp(2\pi ix), \tag{4.11}$$

inducing a map $\text{Hom}_{\mathbb{Q}_p}(A_p, \mathbb{Q}_p) \rightarrow \text{Hom}_{LCA}(A_p, \mathbb{T}) = A_p^\vee$. The map e is continuous as $\mathbb{Q}_p/\mathbb{Z}_p$ carries the discrete topology. As a composition of continuous maps, the output is indeed a continuous character. For the inverse map, one needs to check that any character

$$\psi: A_p \longrightarrow \mathbb{T}$$

actually takes values in the subgroup $\mathbb{Q}_p/\mathbb{Z}_p \subset \mathbb{Q}/\mathbb{Z}$ (under the same maps as in Equation (4.11)). Every element in A_p is topological p -torsion, so a continuous character sends it to a topological p -torsion element in the circle, so $\psi(A_p) \subseteq \mathbb{T}_p$, and the latter group is just the p -primary roots of unity by [Arm81, Lemma 2.6], that is, precisely the image of $\mathbb{Q}_p/\mathbb{Z}_p$ under the map e in Equation (4.11). A further computation checks that the left A_p -module structures are compatible under this isomorphism. The argument for $p = \mathbb{R}$ is analogous but more straightforward. \square

Proposition 4.13. *The category $\text{LCA}_{A,ab}$ is extension-closed in LCA_A . As such, it is a fully exact subcategory. Equipped with this exact structure, it is a split quasi-abelian category.*

Proof. (Step 1) First, we show that $\text{LCA}_{A,ab}$ is extension-closed. Suppose

$$X' \hookrightarrow M \twoheadrightarrow X''$$

is a conflation in LCA_A with X', X'' adelic blocks. By Corollary 2.13, we may write $M = K \oplus X \oplus D$ and by the canonical filtration, applied to either map, we must have $K = D = 0$ (for example, K must map to zero in X'' by the filtration, so K is a subobject of X' , but compacts can only map trivially to adelic blocks, so $K = 0$). Like any extension-closed full subcategory of an exact category, this equips $\text{LCA}_{A,ab}$ with an exact structure and then renders $\text{LCA}_{A,ab}$ fully exact in LCA_A .

(Step 2) We show that each $J_A^{(S)}$ has kernels. This just uses that kernels (in the abelian category of all modules) in each of the cases \mathfrak{A}_p, A_p or $A_{\mathbb{R}}$ are finitely generated and since all these rings are hereditary ([Rei03, Theorem 21.4]) must again also be projective. As all $J_A^{(S)}$ have kernels, so has the 2-colimit $J_A^{(\infty)}$. This also applies to A^{op} , so $J_{A^{op}}^{(\infty)}$ also has all kernels. Under the equivalence of Proposition 4.12 this implies that $J_A^{(\infty)}$ has all cokernels. Finally, invoke Proposition 4.5. Having all kernels and cokernels, we deduce that the category is quasi-abelian.

(Step 3) It remains to show that $\text{LCA}_{A,ab}$ is split exact. We use Proposition 4.5 again. Each category $J_A^{(S)}$ is split exact by Lemma 4.2. Any exact sequence in the 2-colimit $J_A^{(\infty)}$ stems from $J_A^{(S)}$ for S big enough and induces a splitting in the 2-colimit. \square

Example 4.14. The category $\text{LCA}_{A,ab}$ is not an abelian category. Suppose $A = \mathbb{Q}$. There is an injective continuous morphism

$$r : \mathbb{A} \longrightarrow \mathbb{A}$$

given by multiplication with the adèle $(2, 3, 5, \dots, 1)$, that is, the multiplication with p on \mathbb{Q}_p and the identity on \mathbb{R} . That this is continuous follows, for example, from the adèles being a topological ring or by using $J_A^{(\infty)}$ instead and the correspondence of Proposition 4.5. If $\text{LCA}_{A,ab}$ were an abelian category, the morphism r would have the analysis

$$r : \mathbb{A} \twoheadrightarrow \text{coim}(r) \hookrightarrow \mathbb{A}.$$

However, the categorical coimage in LCA_A agrees with the set-theoretic image, and thus the latter would have to be closed in \mathbb{A} . As the category is split exact by Proposition 4.13, this would force \mathbb{A} to be a nontrivial direct summand of itself. While such a thing can happen in categories, it cannot happen in $J_A^{(\infty)}$ by a rank consideration for the topological p -torsion parts.

Theorem 4.15. *Every adelic block Y is an injective and projective object in LCA_A . No other objects are simultaneously injective and projective.*

Proof. We show that any $Y \in \text{LCA}_{A,ab}$ is injective. To this end, it suffices to show that $\text{Ext}^1(M, Y) = 0$ holds for all $M \in \text{LCA}_A$. By Corollary 2.13, we have $M \simeq K \oplus X \oplus D$ with

K a compact A -module, X adelic and D discrete. We have $\text{Ext}^1(D, Y) = 0$ since D is projective by Lemma 2.7, and $\text{Ext}^1(X, Y) = \text{Ext}^1_{\text{LCA}_{A, ab}}(X, Y) = 0$ since the subcategory of adelic blocks is split exact by Proposition 4.13. Thus, we only need to show that $\text{Ext}^1(K, Y) = 0$. Suppose $\phi \in \text{Ext}^1(K, Y)$ is arbitrary. It is represented by a conflation

$$Y \xrightarrow{\alpha} Q \twoheadrightarrow K. \tag{4.12}$$

By Lemma 2.18 (2), as Y and K are quasi-adelic, so is Q , and then $Q \simeq X \oplus C$ with X adelic (and possibly different from the earlier use of X) and C compact. We now show that the splitting for X in this direct sum produces a splitting of α . To this end, we form the pullback of the inflation with $C \twoheadrightarrow Q$, giving the commutative diagram

$$\begin{array}{ccccc} Y \cap C & \twoheadrightarrow & C & \twoheadrightarrow & C/(Y \cap C) \\ \downarrow & & \downarrow & & \downarrow \tau \\ Y & \xrightarrow{\alpha} & Q & \twoheadrightarrow & K. \end{array}$$

The downward arrows are admissible (clear for left and middle, and the right is between compact A -modules, which form an abelian category). As LCA_A is quasi-abelian, we can apply the snake lemma in the style of [Büh10, Corollary 8.13]. We get an exact sequence

$$\ker \tau \twoheadrightarrow Y/(Y \cap C) \xrightarrow{\tilde{\alpha}} X \twoheadrightarrow \text{coker } \tau, \tag{4.13}$$

where $\tilde{\alpha}$ is induced from α . However, we must have $Y \cap C = 0$ since this is a compact A -module, which cannot be nonzero inside an adelic block since that would violate the Hoffmann–Spitzweck filtration (Corollary 2.13). As Y is adelic and $\ker \tau$ compact, we must have $\ker \tau = 0$, and then $\text{coker } \tau \cong K/C$. Thus, Equation (4.13) simplifies to the conflation

$$Y \xrightarrow{\tilde{\alpha}} X \twoheadrightarrow K/C,$$

showing that K/C is simultaneously a compact A -module (as K and C are) and adelic as X and Y are. Thus, $K/C = 0$, showing that the map $\tilde{\alpha} : Y \twoheadrightarrow X$ was an isomorphism to start with, and $K \cong C$. The diagram of the snake lemma thus yields the commutative diagram

$$\begin{array}{ccc} & & Q \\ & \nearrow \alpha & \downarrow \text{quot} \\ Y & & X \\ & \searrow \tilde{\alpha} & \end{array}$$

showing that α (coming from our input conflation) is a splitting for the direct sum splitting $Q \cong X \oplus C$. It follows that $\phi = 0$. Thus, $\text{Ext}^1(K, Y) = 0$. It follows that Y is injective. Finally, by Pontryagin double dualization $Y \cong (Y^\vee)^\vee$, and since Y^\vee is an adelic block in $\text{LCA}_{A^{op}}$ and thus an injective object by the above part of the proof, $(Y^\vee)^\vee$ is projective.

For the converse: Suppose a discrete A -module D is injective. For every element $d \in D$ there is the map of discrete right A -modules $A \rightarrow D, a \mapsto da$. Lift this map along the inflation in $A \twoheadrightarrow \mathbb{A}_A \twoheadrightarrow \mathbb{A}_A/A$ using that D is injective. But by the Hoffmann–Spitzweck

filtration, any morphism $\mathbb{A}_A \rightarrow D$ from the adèles of A to D must be zero, so $d = 0$ by commutativity. Hence, $D = 0$. By Pontryagin duality, no compact module can be projective. Corollary 2.13 now implies the claim. \square

5. One-sided exact categories and quotients

Our approach to the calculation of the K -theory spectrum is similar to the one in [BHR21], that is, we construct two subsequent quotients of the category LCA_A . First, we take the quotient by the category $\text{LCA}_{A, \text{com}}$ of compact A -modules; secondly, we take a further quotient by the category $\text{LCA}_{A, \text{ab}}$ of adelic blocks. These quotients are taken using the framework in [HR19a, HR19b], that means we show that they are inflation-percolating subcategories (see Definition 5.5).

To better understand the first quotient, the following property will be useful: We say that a (one-sided) exact category has *admissible cokernels* (see Definition 5.9) if every morphism has a cokernel and this cokernel is admissible, that is, the cokernel is a deflation. We show in this section that the property of having admissible cokernels is stable under both quotients of inflation-exact categories and taking exact hulls of such categories. These observations allow us to bypass some technical difficulties in the computation of the K -theory spectra of LCA_A in the next section.

5.1. Quotients of (one-sided) exact categories by percolating subcategories

A *conflation category* \mathcal{E} is an additive category \mathcal{E} together with a chosen class of kernel-cokernel pairs (closed under isomorphisms), called *conflations*. We refer to the kernel part of a conflation as an *inflation* (depicted by \rightarrow) and the cokernel-part as a *deflation* (depicted by \twoheadrightarrow). An additive functor $F: \mathcal{C} \rightarrow \mathcal{D}$ of conflation categories is called *exact* (or *conflation-exact*) if it maps conflations to conflations.

Definition 5.1. A conflation category \mathcal{E} is called an *inflation-exact category* if the following axioms are satisfied:

L0 For each $X \in \mathcal{E}$, the map $0 \rightarrow X$ is an inflation.

L1 Inflations are closed under composition.

L2 Pushouts along inflations exist, and inflations are stable under pushouts.

The notion of a *deflation-exact category* is defined dually; the dual axioms are called **R0–R2**.

In addition to the axioms listed above, we formulate a (weakly idempotent complete) version of Quillen's obscure axiom.

Definition 5.2. Let \mathcal{E} be a conflation category. We define the following axiom:

L3⁺ If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are maps such that $g \circ f$ is an inflation, then f is an inflation.

The dual axiom is called **R3⁺**.

Remark 5.3.

1. What we call axiom **L0** is stronger than the corresponding axiom in [BC13]. The axiom as presented here is necessary for all split kernel-cokernel pairs to be conflations.
2. A Quillen exact category is a two-sided exact category in the sense that it is a conflation category that is both inflation-exact and deflation-exact (see [Kel90, Appendix A]).
3. An inflation-exact category satisfies axiom **L3**⁺ if and only if the snake lemma holds (see [HR20a, Theorem 1.2]).

The theory of one-sided exact categories parallels the theory of exact categories, and many notions from exact categories can be transferred easily to the one-sided exact setting. For example, the notions of *admissible morphisms* or *strict morphisms* (morphisms that admit a deflation-inflation factorization) carries over verbatim from [Büh10]. Similarly, one can define *acyclic* or *exact* sequences. Given an inflation-exact category \mathcal{E} , the bounded derived category $\mathbf{D}^b(\mathcal{E})$ is defined as the Verdier localization $\mathbf{K}^b(\mathcal{E})/\mathbf{Ac}^b(\mathcal{E})$, where $\mathbf{K}^b(\mathcal{E})$ is the bounded homotopy category of cochain complexes and $\mathbf{Ac}^b(\mathcal{E})$ is the triangulated subcategory (not necessarily closed under isomorphisms) of $\mathbf{K}^b(\mathcal{E})$ consisting of bounded acyclic sequences (see [BC13, Section 7]). In a similar vein, the stable ∞ -category $\mathbf{D}_\infty^b(\mathcal{E})$ is defined as well (see [HR19a, Section Section 8]).

Let \mathcal{E} be an inflation-exact category. The *exact hull* \mathcal{E}^{ex} of \mathcal{E} is the extension-closure of $i(\mathcal{E}) \subseteq \mathbf{D}^b(\mathcal{E})$, where $i: \mathcal{E} \rightarrow \mathbf{D}^b(\mathcal{E})$ is the natural embedding (see [HR19a] or [Ros, Proposition I.7.5]). Following [HR19a], the exact hull is endowed with the structure of an exact category via the triangle structure of $\mathbf{D}^b(\mathcal{E})$. Moreover, both the 2-natural embedding $j: \mathcal{E} \rightarrow \mathcal{E}^{\text{ex}}$ and the embedding $\mathcal{E}^{\text{ex}} \rightarrow \mathbf{D}^b(\mathcal{E})$ lift to triangle equivalences $\mathbf{D}^b(\mathcal{E}) \simeq \mathbf{D}^b(\mathcal{E}^{\text{ex}})$. In fact, j lifts to an equivalence $\mathbf{D}_\infty^b(\mathcal{E}) \simeq \mathbf{D}_\infty^b(\mathcal{E}^{\text{ex}})$ on the ∞ -derived categories (these are stable ∞ -categories in the sense of [Lur17]; see [HR19a] and the references therein).

We will use the following lemma (see [HKRW21, Lemma 2.26]).

Lemma 5.4. *Let \mathcal{E} be an inflation-exact category. Let $f: X \rightarrow Y$ be a map in \mathcal{E} . The map f is an epimorphism (monomorphism) in \mathcal{E} if and only if $j(f)$ is an epimorphism (monomorphism) in \mathcal{E}^{ex} .*

We recall the following two definitions:

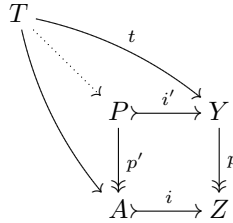
Definition 5.5. Let \mathcal{E} be a conflation category. A full additive subcategory $\mathcal{A} \subseteq \mathcal{E}$ is called an *inflation-percolating subcategory* if the following axioms are satisfied:

P1 \mathcal{A} is a *Serre subcategory*, meaning:

If $A' \twoheadrightarrow A \rightarrow A''$ is a conflation in \mathcal{E} , then $A \in \mathcal{A}$ if and only if $A', A'' \in \mathcal{A}$.

P2 Any morphism $f: A \rightarrow X$ with $A \in \mathcal{A}$ factors as $A \xrightarrow{g} A' \xrightarrow{h} X$, where $A' \in \mathcal{A}$.

P3 For any composition $T \xrightarrow{t} Y \xrightarrow{p} Z$ which factors through some object $B \in \mathcal{A}$, there exists a commutative diagram



with $A \in \mathcal{A}$ and such that the square $PYAZ$ is a pullback square.

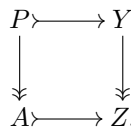
P4 For all maps $X \xrightarrow{f} Y$ that factor through \mathcal{A} and for any deflation $Y \xrightarrow{p} A$ with $A \in \mathcal{A}$ such that $p \circ f = 0$, the induced map $X \rightarrow \ker(p)$ factors through \mathcal{A} .

Similarly, $\mathcal{A} \subseteq \mathcal{E}$ is called a *strictly inflation-percolating* (or *admissibly inflation-percolating*) if the following three axioms are satisfied:

A1 \mathcal{A} is a *Serre subcategory*.

A2 Any morphism $f: A \rightarrow X$ with $A \in \mathcal{A}$ is admissible (with image in \mathcal{A}), that is, factors as $A \rightarrow A' \rightarrow X$ with $A' \in \mathcal{A}$.

A3 Any cospan $A \rightarrow Z \leftarrow Y$ with $A \in \mathcal{A}$ can be completed to a pullback square of the form:



Remark 5.6.

1. Axiom **P3** does not assume any relation between the objects A and B .
2. In many applications, a percolating subcategory satisfies a slightly stronger version of axiom **P2**. We say that $\mathcal{A} \subseteq \mathcal{E}$ satisfies *strong axiom P2* if g can be chosen as an epimorphism in axiom **P2**.
3. Note that axiom **A2** implies strong axiom **P2**.
4. A strictly inflation-percolating subcategory of an inflation-exact category is automatically an abelian subcategory (see [HR19b, Proposition 6.4]).
5. If \mathcal{E} is an exact category and $\mathcal{A} \subseteq \mathcal{E}$ is a full additive subcategory, then \mathcal{A} automatically satisfies axiom **A3** (see [Büh10, Proposition 2.15]).

Definition 5.7. Let \mathcal{E} be a conflation category, and let $\mathcal{A} \subseteq \mathcal{E}$ be a full additive subcategory. An \mathcal{A} -inflation is an inflation with cokernel in \mathcal{A} , similarly, an \mathcal{A} -deflation is a deflation with kernel in \mathcal{A} . A *weak \mathcal{A} -isomorphism* is a finite composition of \mathcal{A} -inflations and \mathcal{A} -deflations. The set of weak isomorphisms is denoted by $S_{\mathcal{A}}$.

The following theorem summarizes the main results of [HR19a, HR19b] in a convenient form.

Theorem 5.8. *Let \mathcal{E} be an inflation-exact category, and let $\mathcal{A} \subseteq \mathcal{E}$ be an inflation-percolating subcategory. Write $Q: \mathcal{E} \rightarrow \mathcal{E}[S_{\mathcal{A}}^{-1}]$ for the localization functor with respect to the set of weak isomorphisms $S_{\mathcal{A}}$. The following hold:*

1. *The set $S_{\mathcal{A}}$ is a left multiplicative system.*
2. *The smallest conflation structure on $\mathcal{E}[S_{\mathcal{A}}^{-1}]$ for which the functor $Q: \mathcal{E} \rightarrow \mathcal{E}[S_{\mathcal{A}}^{-1}]$ is conflation-exact is an inflation-exact structure.*
3. *The functor Q satisfies the 2-universal property of a quotient in the category of inflation-exact categories. This motivates the notation $\mathcal{E}/\mathcal{A} := \mathcal{E}[S_{\mathcal{A}}^{-1}]$.*
4. *The localization sequence $\mathcal{A} \rightarrow \mathcal{E} \xrightarrow{Q} \mathcal{E}/\mathcal{A}$ lifts to a Verdier localization sequence*

$$\mathbf{D}_{\mathcal{A}}^b(\mathcal{E}) \rightarrow \mathbf{D}^b(\mathcal{E}) \rightarrow \mathbf{D}^b(\mathcal{E}/\mathcal{A}).$$

Here, $\mathbf{D}_{\mathcal{A}}^b(\mathcal{E})$ is the thick triangulated subcategory of $\mathbf{D}^b(\mathcal{E})$ generated by \mathcal{A} .

5. *If the natural functor $\mathbf{D}^b(\mathcal{A}) \rightarrow \mathbf{D}_{\mathcal{A}}^b(\mathcal{E})$ is an equivalence, we obtain an exact sequence*

$$\mathbf{D}_{\infty}^b(\mathcal{A}) \rightarrow \mathbf{D}_{\infty}^b(\mathcal{E}) \rightarrow \mathbf{D}_{\infty}^b(\mathcal{E}/\mathcal{A})$$

in the sense of [BGT13].

6. *For any morphism f in \mathcal{E} , we have that $Q(f) = 0$ if and only if f factors through \mathcal{A} .*

Moreover, if $\mathcal{B} \subseteq \mathcal{E}$ is a strictly inflation-percolating subcategory, then $\mathcal{B} \subseteq \mathcal{E}$ is inflation-percolating (in particular all of the above holds). In addition, the following properties hold:

1. *Every map $f \in S_{\mathcal{B}}$ is strict (this explains the terminology).*
2. *The set $S_{\mathcal{B}}$ is saturated, that is, if $Q(f)$ is an isomorphism, then $f \in S_{\mathcal{B}}$.*

5.2. Inflation-exact categories with admissible cokernels

We will use the above results for an inflation-exact category with admissible cokernels (see Definition 5.9). We use this structure to avoid later technicalities.

Definition 5.9. Let \mathcal{E} be an inflation-exact category. We say that \mathcal{E} has *admissible cokernels* if every morphism has a cokernel which is a deflation.

The following description is [HKRW21, Proposition 4.10].

Proposition 5.10. *The following are equivalent for an inflation-exact category \mathcal{E} :*

1. *\mathcal{E} has admissible kernels,*
2. *every map in \mathcal{E} has an epi-inflation factorization, that is, every map f factors as $f = i \circ k$, where k is a kernel and i is an inflation.*

Remark 5.11. Following [BGO71, BB04], a category is called *coregular* if (1) every morphism f has factors as $f = k \circ p$, where p is an epimorphism and k is a kernel map and (2) kernels are stable under pushouts. It is shown in [HKRW21, Proposition 4.12] that

an additive coregular category is an inflation-exact category with admissible cokernels; here, the conflations are given by all kernel-cokernel pairs.

Remark 5.12.

1. One readily verifies that an inflation-exact category \mathcal{E} having admissible cokernels satisfies axiom $\mathbf{L3}^+$. In particular, the snake lemma holds in \mathcal{E} by Remark 5.3.
2. Every quasi-abelian category \mathcal{E} is inflation-exact and has admissible cokernels. In fact, a conflation category \mathcal{E} is a quasi-abelian category if and only if it is both an inflation-exact category having admissible cokernels and a deflation-exact category having admissible kernels.

Having admissible cokernels is preserved under localizations at inflation-percolating subcategories as well as under taking the exact hull (see the duals of [HKRW21, Theorem 5.11 and Proposition 6.5]).

Proposition 5.13. *Let \mathcal{E} be an inflation-exact category, and let $\mathcal{A} \subseteq \mathcal{E}$ be an inflation-percolating subcategory. If \mathcal{E} has admissible cokernels, then so does \mathcal{E}/\mathcal{A} .*

Proposition 5.14. *Let \mathcal{E} be an inflation-exact category. If \mathcal{E} has admissible cokernels, then so does \mathcal{E}^{ex} .*

The following is an analogue of [BHR21, Proposition 2.7].

Proposition 5.15. *Let \mathcal{E} be an inflation-exact category, and let $\mathcal{A} \subseteq \mathcal{E}$ be a full additive subcategory satisfying strong axiom **P2**. If \mathcal{E} has admissible cokernels, then $\mathcal{A} \subseteq \mathcal{E}^{\text{ex}}$ satisfies strong axiom **P2**.*

Proof. Let $f: A \rightarrow X$ be a map in \mathcal{E}^{ex} with $A \in \mathcal{A}$. By [HR19a, Corollary 7.5], there is a conflation $X \xrightarrow{i} Y \xrightarrow{p} Z$ in \mathcal{E}^{ex} such that $Y, Z \in \mathcal{E}$. By strong axiom P2, the composition $i \circ f$ factors as $A \xrightarrow{e} A' \rightarrow Y$ in \mathcal{E} with $A' \in \mathcal{A}$ and e epic. By Lemma 5.4, the latter yields an epi-inflation factorization of $i \circ f$ in \mathcal{E}^{ex} .

By Proposition 5.14, f admits an epi-inflation factorization $A \xrightarrow{e'} T \xrightarrow{k} X$ in \mathcal{E}^{ex} . By axiom L1, the composition $i \circ k$ is an inflation in \mathcal{E}^{ex} . Hence, we obtain the epi-inflation factorization $A \xrightarrow{e'} T \xrightarrow{i \circ k} X$ of $i \circ f$. As epi-inflation factorizations are unique up to isomorphism, we conclude that $T \cong A' \in \mathcal{A}$. This shows strong axiom P2. □

6. K -theory computations

We rely on the nomenclature of *localizing invariants* from [BGT13, Definition 8.1]. Following their notation, $\text{Cat}_{\infty}^{\text{ex}}$ is the ∞ -category of small stable ∞ -categories. An invariant is *weakly localizing* if one drops the requirement to commute with filtering colimits.

Theorem 6.1. *Suppose $\mathbb{K}: \text{Cat}_{\infty}^{\text{ex}} \rightarrow \mathbf{A}$ is a localizing invariant with values in a stable presentable ∞ -category \mathbf{A} . If \mathbb{K} commutes with countable products, then there is a natural equivalence*

$$\mathbb{K}(\text{LCA}_{A,ab}) \xrightarrow{\sim} \text{hocolim}_S \left(\prod_{p \notin S} \mathbb{K}(\mathfrak{A}_p) \times \prod_{p \in S} \mathbb{K}(A_p) \right),$$

where S runs through all finite subsets of $\mathcal{P}\ell$, partially ordered by inclusion. If A is commutative, all categories involved are naturally categories with dualities (as detailed in §4.3) and if \mathbb{K} is an invariant incorporating duality, the equivalence still holds.

Proof. This follows directly from the equivalence of exact categories in Proposition 4.5, which implies the equivalence of the attached stable ∞ -categories of bounded complexes (modulo acyclic complexes). Regarding duality, combine this with Proposition 4.12. \square

Nonconnective K -theory is an example of a localizing invariant commuting with infinite products by [KW19]. Since all our input categories stem from exact categories, one could get by with the classical result of [Car95].

Weakly localizing invariants decidedly do not suffice for the above because of the colimit in Definition 4.3 (so Topological Cyclic Homology (TC) or negative cyclic homology cannot be taken for \mathbb{K}).

Theorem 6.2. *Suppose $\mathbb{K}: \text{Cat}_\infty^{\text{ex}} \rightarrow \mathbf{A}$ is a weakly localizing invariant with values in a stable presentable ∞ -category \mathbf{A} . There is a natural fiber sequence in \mathbf{A} ,*

$$\mathbb{K}(A) \rightarrow \mathbb{K}(\text{LCA}_{A,ab}) \rightarrow \mathbb{K}(\text{LCA}_A).$$

This result does not require \mathbb{K} to commute with countable products.

The proof of Theorem 6.2 follows Section 4 of [BHR21] closely.

The following is the analogue of [BHR21, Proposition 4.1]. Recall that $\text{LCA}_{A,com}$ is the full subcategory of LCA_A consisting of all compact A -modules.

Proposition 6.3. *The category $\text{LCA}_{A,com}$ is a strictly inflation-percolating subcategory of LCA_A and the localization functor*

$$Q_c: \text{LCA}_A \rightarrow \text{LCA}_A / \text{LCA}_{A,com} (=:\mathcal{E})$$

induces an exact sequence of stable ∞ -categories

$$D_\infty^b(\text{LCA}_{A,com}) \rightarrow D_\infty^b(\text{LCA}_A) \rightarrow D_\infty^b(\mathcal{E}).$$

As the category $\text{LCA}_{A,com}$ admits all coproducts, there is an equivalence $\mathbb{K}(D_\infty^b(\text{LCA}_A)) \simeq \mathbb{K}(\mathcal{E})$.

Moreover, as LCA_A is a quasi-abelian category, it has admissible cokernels, and thus, \mathcal{E} has admissible cokernels as well.

Proof. That $\text{LCA}_{A,com} \subseteq \text{LCA}_A$ is a strictly inflation-percolating subcategory is shown in [HR19b, Proposition 8.34]. As $\text{LCA}_{A,com}$ contains enough injectives for LCA_A , $D^b(\text{LCA}_{A,com}) \rightarrow D_{\text{LCA}_{A,com}}^b(\text{LCA}_A)$ lifts to a triangle equivalence. The remainder of the statement follows from Theorem 5.8. The last statement follows from Proposition 5.13. \square

In light of Corollary 2.13, we now wish to further annihilate the adelic blocks from \mathcal{E} . To that end, write \mathcal{V} for the full additive subcategory of \mathcal{E} generated by the adelic

blocks. Unfortunately, it is not clear whether $\mathcal{V} \subseteq \mathcal{E}$ is an inflation-percolating subcategory. However, as $\mathbb{K}(\mathcal{E}^{\text{ex}}) \simeq \mathbb{K}(\mathcal{E})$, for our purposes, we content ourselves with showing that \mathcal{V} is an inflation-percolating subcategory of the exact hull \mathcal{E}^{ex} . We first consider the following analogue of [BHR21, Lemma 4.2].

Lemma 6.4.

- 1. Let V be an adelic block, the localization functor Q_c induces a natural equivalence

$$Q_c: \text{Hom}_{\text{LCA}_A}(-, V) \rightarrow \text{Hom}_{\mathcal{E}}(Q_c(-), Q_c(V)).$$

In particular, it follows that V is injective in \mathcal{E} .

- 2. The category \mathcal{V} is equivalent to the category $\text{LCA}_{A,ab}$.

Proof.

- 1. Let $f \in \text{Hom}_{\text{LCA}_A}(X, V)$ be a map such that $Q_c(f) = 0$. By the quotient/localization theory of percolating subcategories, \mathcal{E} is the localization $\text{LCA}_A[S_{\text{LCA}_{A,com}}^{-1}]$. Here, $S_{\text{LCA}_{A,com}}$ is the collection of admissible morphisms with kernel and cokernel belonging to $\text{LCA}_{A,com}$ and $S_{\text{LCA}_{A,com}}$ is a left multiplicative system. As such, $Q_c(f) = 0$ implies that there exists a $t: V \rightarrow Y$ in $S_{\text{LCA}_{A,com}}$ such that $t \circ f = 0$. It follows that f factors through $\ker(t)$. On the other hand, $\ker(t) \rightarrow V$ must be zero by Lemma 2.10 as $\ker(t) \in \text{LCA}_{A,com}$. This shows that $f = 0$ and thus Q_c is injective.

To show that Q_c is surjective, let $g \in \text{Hom}_{\mathcal{E}}(X, V)$ be represented by the roof $X \xrightarrow{f} Y \xleftarrow{s} V$ with $s \in S_{\text{LCA}_{A,com}}$. By definition s is admissible and $\ker(s) \in \text{LCA}_{A,com}$. As V has no nontrivial compact submodules, s is an inflation. As V is injective by Theorem 4.15, s splits. Let $t: Y \rightarrow V$ be a splitting, that is, $t \circ s = 1_V$, then $Q_c(t \circ f) = g$. This shows the desired bijection.

As V is injective in LCA_A , $\text{Hom}_{\text{LCA}_A}(-, V)$ is an exact functor. Hence, $\text{Hom}_{\mathcal{E}}(-, V)$ is an exact functor and thus V is injective in \mathcal{E} .

- 2. This follows immediately from the above natural equivalence. □

The main difference with [BHR21] is that \mathcal{V} cannot be a strictly inflation-percolating subcategory of \mathcal{E} . Indeed, by Example 4.14, $\text{LCA}_{A,ab}$ is not abelian and thus Remark 5.6 implies that \mathcal{V} cannot be strictly percolating. Nonetheless, we obtain the following analogue of [BHR21, Proposition 4.3].

Proposition 6.5. *The subcategory $\mathcal{V} \subseteq \mathcal{E}^{\text{ex}}$ is an inflation-percolating subcategory.*

Proof. We first show strong axiom P2. By Proposition 5.14, the category \mathcal{E}^{ex} has admissible cokernels. By Proposition 5.15, it now suffices to show that $\mathcal{V} \subseteq \mathcal{E}$ satisfies strong axiom P2. To that end, let $f: V \rightarrow X$ be a map in \mathcal{E} with $V \in \mathcal{V}$. As \mathcal{E} has admissible cokernels, f admits an epi-inflation factorization $V \xrightarrow{g} T \xrightarrow{h} X$ in \mathcal{E} . As $\mathcal{E} = \text{LCA}_A[S_{\text{LCA}_{A,com}}^{-1}]$ is a localization and $S_{\text{LCA}_{A,com}}$ is a left multiplicative system, the map g can be represented as a roof $V \xrightarrow{g'} T' \xleftarrow{s} T$ with $s \in S_{\text{LCA}_{A,com}}$. Note that g' is an epimorphism in \mathcal{E} and thus its cokernel is zero in \mathcal{E} . As the localization functor Q_c

commutes with cokernels, the cokernel of g' belongs to $\text{LCA}_{A,com}$. Hence, as LCA_A has admissible cokernels, we obtain the sequence in LCA_A :

$$V \rightarrow \text{im}(g') \twoheadrightarrow T' \twoheadrightarrow C.$$

Here, $C \in \text{LCA}_{A,com}$, $V \rightarrow \text{im}(g')$ is epic and the composition $V \rightarrow \text{im}(g') \twoheadrightarrow T'$ is g' . It follows from Corollary 2.13 that epimorphic quotient of adelic blocks are quasi-adelic blocks, and thus $\text{im}(g') \in \text{LCA}_{A,qab}$. By Lemma 2.12, $\text{im}(g')$ is isomorphic to an adelic block in \mathcal{E} as compacts are annihilated. It follows that $\text{im}(g') \in \mathcal{V}$. This shows that $\mathcal{V} \subseteq \mathcal{E}$ satisfies strong axiom P2 as Q_c maps epics to epics.

We now claim that every object of \mathcal{V} is injective in \mathcal{E}^{ex} . Let $V \in \mathcal{V}$. By Lemma 6.4, V is injective in \mathcal{E} . As \mathcal{E}^{ex} is simply the extension-closure of \mathcal{E} in \mathcal{E}^{ex} , one readily verifies that V remains injective in \mathcal{E}^{ex} . It follows that $\mathcal{V} \subseteq \mathcal{E}^{\text{ex}}$ is extension-closed.

Now, let $X \xrightarrow{i} V \xrightarrow{p} Z$ be a conflation in \mathcal{E}^{ex} with $V \in \mathcal{V}$. By strong axiom P2, $Z \cong \text{im}(p) \in \mathcal{V}$. It follows from Proposition 4.13 that $X \in \mathcal{V}$. This shows axiom P1.

As \mathcal{E}^{ex} is exact, axiom P3 is automatic. Axiom P4 follows from (the dual of) [HR19b, Proposition 4.11] and Proposition 4.13. This concludes the proof. \square

Corollary 6.6. *The localization functor $Q_{\mathcal{V}}: \mathcal{E}^{\text{ex}} \rightarrow \mathcal{F}(=\mathcal{E}^{\text{ex}}/\mathcal{V})$ induces a fiber sequence*

$$\mathbb{K}(\mathcal{V}) \rightarrow \mathbb{K}(\mathcal{E}^{\text{ex}}) \rightarrow \mathbb{K}(\mathcal{F}).$$

Proof. By Lemma 6.4, $\mathcal{V} \subseteq \mathcal{E}^{\text{ex}}$ contains enough injectives. The result now follows from Theorem 5.8. \square

Proposition 6.7. *The composition functor $\text{LCA}_A \rightarrow \mathcal{E} \rightarrow \mathcal{E}^{\text{ex}} \rightarrow \mathcal{F}$ is 2-universal with respect to conflation-exact functors $F: \text{LCA}_A \rightarrow \mathcal{C}$ with \mathcal{C} exact such that $F(\text{LCA}_{A,ab}) = 0$.*

Proof. This follows by simply gluing together all 2-universal properties. \square

Our next goal is to show that \mathcal{F} is equivalent to the category $\text{Mod}(A)/\text{mod}(A)$ (see [BHR21, Proposition 4.9]). Consider the localization functor $Q_A: \text{Mod}(A) \rightarrow \text{Mod}(A)/\text{mod}(A)$. We write $D: \text{LCA}_A \rightarrow \text{LCA}_{A,dis} \simeq \text{Mod}(A)$ for the functor mapping an object to its torsion-free part with respect to the torsion theory in Proposition 2.11. Note that D is not conflation-exact (the standard adèle sequence $A \twoheadrightarrow \mathbb{A} \twoheadrightarrow \mathbb{A}/A$ is mapped to $A \rightarrow 0 \rightarrow 0$). However, we need not to change much to remedy this (see also [BHR21, Proposition 4.7]).

Proposition 6.8. *The functor $Q_A \circ D: \text{LCA}_A \rightarrow \text{Mod}(A)/\text{mod}(A)$ is conflation-exact.*

Proof. Let $X \twoheadrightarrow Y \twoheadrightarrow Z$ be a conflation in LCA_A . By Proposition 2.11, we obtain the following commutative diagram:

$$\begin{array}{ccccc} W_X \twoheadrightarrow & X & \twoheadrightarrow & D_X & \\ \downarrow \text{dotted} & \downarrow & & \downarrow g & \\ W_Y \twoheadrightarrow & Y & \twoheadrightarrow & D_Y & \end{array}$$

Note that the left downwards arrow is an inflation by [Büh10, Proposition 7.6]. Clearly, the map g is strict. Applying the short snake lemma ([Büh10, Corollary 8.13]), we obtain an exact sequence

$$\ker(g) \xrightarrow{i} W_Y/W_X \rightarrow Z \rightarrow \operatorname{coker}(g).$$

By Lemma 2.18, $W_Y/W_X \in \operatorname{LCA}_{A,qab}$ and hence $\operatorname{coker}(i) \in \operatorname{LCA}_{A,qab}$. Note that the conflation $\operatorname{coker}(i) \rightarrow Z \rightarrow \operatorname{coker}(g)$ thus is the torsion sequence of Z (from Proposition 2.11). Thus, $\operatorname{coker}(g) \cong D_Z$.

Lastly, it follows from Lemma 2.18 that $\ker(g)$ is finitely generated and discrete. Hence, we find a conflation $Q_A(D_X) \rightarrow Q_A(D_Y) \rightarrow Q_A(D_Z)$ as required. \square

Corollary 6.9. *The functor $Q_A \circ D: \operatorname{LCA}_A \rightarrow \operatorname{Mod}(A)/\operatorname{mod}(A)$ induces an equivalence $\mathcal{F} \rightarrow \operatorname{Mod}(A)/\operatorname{mod}(A)$ of abelian categories.*

Proof. The proof is identical to [BHR21, Construction 4.8 and Proposition 4.9]. \square

We are now in a position to prove Theorem 6.2.

Proof of Theorem 6.2. Consider the following natural commutative diagram

$$\begin{array}{ccccc} \operatorname{mod}(A) & \longrightarrow & \operatorname{Mod}(A) & \longrightarrow & \operatorname{Mod}(A)/\operatorname{mod}(A) \\ \downarrow & & \downarrow & & \downarrow \simeq \\ \mathcal{V} & \longrightarrow & \mathcal{E}^{\operatorname{ex}} & \longrightarrow & \mathcal{F}. \end{array}$$

whose rows are localization sequences. Note that this diagram lifts to a diagram on the bounded derived ∞ -categories such that the rows are exact. As $\mathcal{V} \simeq \operatorname{LCA}_{A,ab}$ by Lemma 6.4, we obtain the following bi-Cartesian square of stable ∞ -categories:

$$\begin{array}{ccc} D_\infty^b(\operatorname{mod}(A)) & \longrightarrow & D_\infty^b(\operatorname{Mod}(A)) \\ \downarrow & & \downarrow \\ D_\infty^b(\operatorname{LCA}_{A,ab}) & \longrightarrow & D_\infty^b(\mathcal{E}^{\operatorname{ex}}). \end{array}$$

Again, using the Eilenberg swindle, every object of $D_\infty^b(\operatorname{Mod}(A))$ gets trivialized under a localizing invariant $\mathbb{K}: \operatorname{Cat}_\infty^{\operatorname{Ex}} \rightarrow \mathbf{A}$. Hence, for each such \mathbb{K} , there is a fiber sequence $\mathbb{K}(\operatorname{mod}(A)) \rightarrow \mathbb{K}(\operatorname{LCA}_{A,ab}) \rightarrow \mathbb{K}(\mathcal{E}^{\operatorname{ex}})$. By Proposition 6.3, $D_\infty^b(\mathcal{E}^{\operatorname{ex}}) \simeq D_\infty^b(\operatorname{LCA}_A)$. The result follows. \square

All of the previous results give a complete computation of the K -theory spectrum of LCA_A . In the special case of the first homotopy group $\pi_1 K$, we recover a lot of well-known classical invariants:

Theorem 6.10. *Let A be a finite-dimensional semisimple \mathbb{Q} -algebra, and let K denote Quillen’s algebraic K -theory.*

1. There is a natural isomorphism

$$K_1(\text{LCA}_{A,ab}) \xrightarrow{\sim} \frac{J(A)}{J^1(A)},$$

where $J(A)$ is Fröhlich’s idèle class group and J^1 the subgroup of reduced norm one elements.

2. There is a natural isomorphism

$$K_1(\text{LCA}_A) \xrightarrow{\sim} \frac{J(A)}{J^1(A) \cdot \text{im } A^\times},$$

where the units A^\times are diagonally mapped to the idèles.

Remark 6.11. The reduced norm one subgroup $J^1(A)$ always contains the commutator subgroup of $J(A)$, explaining why both quotients on the right side are abelian.

Remark 6.12. The theorem generalizes the main result of [AB19]: If F is a number field, $J(F)$ simplifies to the usual idèles, the reduced norm one subgroup $J^1(F)$ is trivial, so the second claim simplifies to $K_1(\text{LCA}_F) \cong J(F)/F^\times$, which is Chevalley’s idèle class group and the automorphic side of global class field theory in dimension one. This statement was the original prediction made by Clausen in [Cla17].

In higher degrees, we obtain the following, also matching the picture from the number field case.

Theorem 6.13. Let A be a finite-dimensional semisimple \mathbb{Q} -algebra. Let $\mathfrak{A} \subset A$ be any \mathbb{Z} -order.

$$K_n(\text{LCA}_{A,ab}) \cong \left\{ (\alpha_P)_P \in \prod_{p \in \mathcal{P}^\ell} K_n(A_p) \mid \begin{array}{l} \alpha_P \in \text{im } K_n(\mathfrak{A}_p) \text{ for all but finitely} \\ \text{many } p \in \mathcal{P}^{\ell_{\text{fin}}} \end{array} \right\},$$

where $\mathfrak{A}_p := \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. The right-hand side is independent of the choice of the order \mathfrak{A} (see 3.3, where we had introduced the above notation for restricted products of K -groups).

Proof of both theorems. (Step 1) Use Theorem 6.2 and Theorem 6.1 for \mathbb{K} being nonconnective K -theory. This is a localizing invariant (in fact the universal one by [BGT13]) and commutes with products by [KW19]. However, A and each A_p are regular, so their nonconnective K -theory agrees with Quillen’s K -theory (By [BGT13, Remark 9.33] there is an equivalence of the 0-connective cover of connective versus nonconnective K -theory, and [Sch06, Remarks 3 and 7] show that all negative degree homotopy groups of nonconnective K -theory vanish and that it further is an equivalence on π_0 . Thus, it is an equivalence on the whole). We thus get the exact sequence of abelian groups

$$\cdots \longrightarrow K_1(A) \longrightarrow \prod' K_1(A_p) \longrightarrow K_1(\text{LCA}_A) \longrightarrow K_0(A) \longrightarrow \cdots, \tag{6.1}$$

where p runs over all places of \mathbb{Q} .

This finishes the proof of Theorem 6.13, so henceforth assume $n = 1$. Then the group in Equation (3.3) is also called $JK_1(A)$ (e.g., in [Wil77, p. 2, end of page] or [CR87, (49.16), p. 224]), so we have recovered a classical invariant with our K -theory computations:

$$K_1(\text{LCA}_{A,ab}) \cong JK_1(A). \tag{6.2}$$

In order to relate this to $J(A)$, we follow classical ideas of Wilson [Wil77]: There is a fundamental commutative diagram

$$\begin{array}{ccc}
 J(A) & \longrightarrow & JK_1(A) \\
 & \searrow \text{nr} & \downarrow \cong \\
 & & J(\zeta(A)),
 \end{array}
 \tag{6.3}$$

which we describe now. The group $J(A)$ is Fröhlich’s idèle class group of Equation (3.1). The horizontal arrow is induced from the map $A_p^\times \rightarrow K_1(A_p)$. It is surjective for each p by [CR87, Theorem 40.31]. The downward arrow is induced for every place p by the reduced norm map

$$\text{nr}: K_1(A_p) \xrightarrow{\sim} \zeta(A_p)^\times$$

taking values in the units of the center $\zeta(A_p)$. That these are isomorphisms is shown for central simple algebras in [CR87, Theorem 45.3] whose center is a finite field extension of \mathbb{Q}_p resp. \mathbb{R} (i.e., is a local field). However, then the general case follows by the Artin–Wedderburn decomposition and Morita invariance of K -theory on the left side and on the right side since $\zeta(R) \xrightarrow{\sim} \zeta(M_n(R))$ by constant diagonal matrices. The commutativity of Diagram 6.3 implies that both arrows originating from $J(A)$ have the same kernel (which for the diagonal arrow is the reduced norm one idèles of Equation (3.2)),

$$J(A)/J^1(A) \xrightarrow{\sim} JK_1(A).$$

Alongside Equation (6.2), we have proven the first claim.

(Step 2) Since A and each A_p are semisimple, one has a canonical isomorphism $K_0(A) \cong \bigoplus_W \mathbb{Z}$, where W is the set of isomorphism classes of simple objects in $\text{proj}(A)$ (and analogously for A_p). Thus, the kernel $K_0(A) \rightarrow \prod' K_0(A_p)$ which would occur at the right end of Sequence 6.1 is zero (it suffices to check what happens to each simple module generator of $K_0(A)$ and since the underlying functor is just tensoring $P \mapsto P \otimes_A A_p$, none goes to zero). Thus, we obtain

$$K_1(\text{LCA}_A) \cong JK_1(A)/\text{im } K_1(A),$$

and we need to understand how $K_1(A)$ is mapped to $JK_1(A)$ here. To this end, return to Diagram 6.3 and observe that this map agrees with the horizontal arrow. It follows that the map is indeed just the diagonal embedding. Our second claim follows. \square

7. The K -theory of the category of adelic blocks

In our main theorem (Theorem A), we describe the K -theory spectrum of LCA_A via a fiber sequence $K(A) \rightarrow K(\text{LCA}_{A,ab}) \rightarrow K(\text{LCA}_A)$. In this sequence, we use the equivalence $\text{LCA}_{A,ab} \simeq J_A^{(\infty)}$ to construct a fiber sequence

$$\bigoplus_{p \in \mathcal{P}^{\ell}_{\text{fin}}} K(\text{tor} \mathfrak{A}_p) \rightarrow \prod_{p \in \mathcal{P}^{\ell}} K(\text{mod} \mathfrak{A}_p) \rightarrow K(\text{LCA}_{A,ab})$$

describing the K -theory spectrum of $K(\text{LCA}_{A,ab})$. Here, \mathfrak{A}_p (for $p \in \mathcal{P}^{\ell}_{\mathbb{Q}}$) is as introduced in §3 and $\text{tor} \mathfrak{A}_p$ stands for the subcategory of $\text{mod} \mathfrak{A}_p$ consisting of the \mathbb{Z}_p -torsion modules.

Thus, choose a maximal order $\mathfrak{A} \subset A$. For any finite set of prime numbers $S \subset \mathcal{P}^{\ell}_{\text{fin}}$, consider the following category:

$$\mathbb{L}_A^{(S)} := \text{mod}(A_{\mathbb{R}}) \times \prod_{p \notin S} \text{mod}(\mathfrak{A}_p) \times \prod_{p \in S} \text{mod}(A_p).$$

The category $\mathbb{L}_A^{(S)}$ is a product of abelian categories and, as such, is an abelian category itself. For any inclusion of finite sets $S \subseteq S'$ of primes there is an exact functor $\mathbb{U}^{(S)} \rightarrow \mathbb{U}^{(S')}$ induced by $(-) \mapsto (-) \otimes_{\mathfrak{A}_p} A_p$ termwise for all primes $p \in S' \setminus S$, and the identity functor for the remaining p . Define

$$\mathbb{L}_A^{(\infty)} := \text{colim}_S \mathbb{L}_A^{(S)}.$$

We write \mathbb{T} for the category for the full subcategory of $\mathbb{L}_A^{(\emptyset)}$ consisting of those objects which are finite direct sums of torsion \mathfrak{A}_p -modules. Put differently, as sequences indexed by the set of places, the objects are almost everywhere zero and contain a torsion \mathfrak{A}_p -module for finitely many primes p , or thus:

$$\mathbb{T} := \bigoplus_{p \in \mathcal{P}^{\ell}_{\text{fin}}} \text{tor}(\mathfrak{A}_p) = \text{colim}_S \prod_{p \in S} \text{tor}(\mathfrak{A}_p),$$

where S ranges over the finite subsets of $\mathcal{P}^{\ell}_{\text{fin},\mathbb{Q}}$. We have the following lemma.

Lemma 7.1. *Let S be a finite set of primes.*

1. $\mathbb{L}_A^{(S)} \simeq \mathbb{L}_A^{(S)} / \bigoplus_{p \in S} \text{tor}(\mathfrak{A}_p)$.
2. *The category $\mathbb{L}_A^{(S)}$ has enough projectives.*
3. *The essential image of the embedding $J_A^{(S)} \rightarrow \mathbb{L}_A^{(S)}$ consists of all projective objects in $\mathbb{L}_A^{(S)}$.*
4. *The global dimension of $\mathbb{L}_A^{(S)}$ is one.*

Proof. As $A_p = \mathfrak{A}_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, for any prime p , we have $\text{mod} A_p \simeq \text{mod} \mathfrak{A}_p / \text{tor} \mathfrak{A}_p$. The first statement follows from this observation. For the other statements, note that both the projective objects and projective resolutions in $\mathbb{L}_A^{(S)}$ can be determined componentwise. The statements follow easily from this. □

From this, we find the following properties of $\mathbb{L}_A^{(\infty)}$.

Proposition 7.2. *With notation as above, we have:*

1. $\mathbf{L}_A^{(\infty)} \simeq \mathbf{L}_A^{(\emptyset)} / \oplus_p \text{tor}(\mathfrak{A}_p)$. In particular, $\mathbf{L}_A^{(\infty)}$ is abelian.
2. The category $\mathbf{L}_A^{(\infty)}$ has enough projectives.
3. The essential image of the embedding $\mathbf{J}_A^{(\infty)} \rightarrow \mathbf{L}_A^{(\infty)}$ consists of all projective objects in $\mathbf{L}_A^{(\infty)}$.
4. The global dimension of $\mathbf{L}_A^{(\infty)}$ is one.

Proof. It follows from Lemma 7.1 that, up to equivalence, for each $p \in S$, we may replace the component $\text{mod}(A_p)$ in $\mathbf{L}_A^{(S)}$ by $\text{mod}(\mathfrak{A}_p) / \text{tor}(\mathfrak{A}_p)$. In this way, each of the transition maps $\mathbf{L}_A^{(S)} \rightarrow \mathbf{L}_A^{(S')}$ is a localization (at the Serre subcategory $\oplus_{p \in S' \setminus S} \text{tor}(\mathfrak{A}_p)$). The first statement now follows easily.

In the rest of the proof, we identify $\mathbf{L}_A^{(\infty)}$ with $\mathbf{L}_A^{(\emptyset)} / \oplus_p \text{tor}(\mathfrak{A}_p)$, meaning that the objects of $\mathbf{L}_A^{(\infty)}$ are the objects of $\mathbf{L}_A^{(\emptyset)}$.

To see that $\mathbf{L}_A^{(\infty)}$ has enough projectives, it suffices to see that the objects $P = (P_p)_p \in \mathbf{L}_A^{(\infty)}$ for which every $P_p \in \text{mod}(\mathfrak{A}_p)$ is projective, are projective in $\mathbf{L}_A^{(\infty)}$. This is straightforward.

Now, let $P = (P_p)_p \in \mathbf{L}_A^{(\infty)}$ be any projective object. As $\mathbf{L}_A^{(\emptyset)}$ has enough projectives, we may consider an epimorphism $\alpha: Q \rightarrow P$ in $\mathbf{L}_A^{(\emptyset)}$ with Q projective in $\mathbf{L}_A^{(\emptyset)}$ (thus, $Q \in \mathbf{J}_A^{(\emptyset)}$). As P is projective in $\mathbf{L}_A^{(\infty)}$, there is a section $\gamma^{-1} \circ \beta: P \rightarrow Q$ in $\mathbf{L}_A^{(\infty)}$. Hence, there is a finite set of primes S such that $\gamma^{-1} \circ \beta \in \text{Hom}_{\mathbf{L}_A^{(S)}}(P, Q)$ and $\alpha \circ \gamma^{-1} \circ \beta = 1_P$.

Hence, P is projective in $\mathbf{L}_A^{(S)}$ and so $P \in \mathbf{J}_A^{(S)}$ by Lemma 7.1. This shows that $P \in \mathbf{J}_A^{(\infty)}$, as required.

Finally, as projective resolutions can be taken componentwise, we see that the global dimension of $\mathbf{L}_A^{(\infty)}$ is at most one. As not all objects are projective, for example, an object $X = (X_p)_p$, where each $X_p \in \text{mod}(\mathfrak{A}_p)$ is nonprojective, the global dimension is not zero. □

Remark 7.3. As $\mathbf{J}_A^{(\infty)}$ is equivalent to the category of projectives of an abelian category of global dimension at most one, every morphism in $\mathbf{J}_A^{(\infty)}$ has a kernel and all kernels are split monomorphisms. This gives an alternative to steps 2 and 3 of the proof of Proposition 4.13.

In the next proposition, we write $\mathbf{D}_\mathbb{T}^b(\mathbf{L}_A^{(\infty)})$ for the full subcategory of $\mathbf{D}^b(\mathbf{L}_A^{(\emptyset)})$ consisting of those objects whose cohomology groups lie in $\mathbb{T} = \oplus_{p \in \mathcal{P}l_{\text{fin}}} \text{tor}(\mathfrak{A}_p)$.

Proposition 7.4. *The natural triangle functor $\mathbf{D}^b(\mathbb{T}) \rightarrow \mathbf{D}_\mathbb{T}^b(\mathbf{L}_A^{(\emptyset)})$ is an equivalence.*

Proof. We show that the conditions of [KS06, Theorem 13.2.8] are satisfied, namely that for each monomorphism $f: X \rightarrow Y$ in $\mathbf{L}_A^{(\emptyset)}$ with $X \in \mathbb{T}$, there is a morphism $Y \rightarrow Y'$ with $Y' \in \mathbb{T}$ such that the composition $X \rightarrow Y \rightarrow Y'$ is a monomorphism. Thus, let $f: X \rightarrow Y$ be a monomorphism in $\mathbf{L}_A^{(\emptyset)}$ with $X \in \mathbb{T}$. Recall that each object of $L = (L_p)_{p \in \mathcal{P}l}$ of $\mathbf{L}_A^{(\infty)}$ is a string of objects, where each $L_p \in \text{mod } \mathfrak{A}_p$. As $X \in \mathbb{T}$, there is a finite set $S \subset \mathcal{P}l_{\text{fin}}$ such that $X_p \in \text{tor } \mathfrak{A}_p$ for $p \in S$ and $X_p = 0$ for $p \notin S$.

Let Y' be the direct summand of Y given by $Y'_p = Y_p$ when $p \in S$ and $Y'_p = 0$ otherwise. The composition $X \rightarrow Y \rightarrow Y'$ is still a monomorphism.

It now follows from Lemma 4.1 that $Y' = Y'_{\text{proj}} \oplus Y'_{\text{tor}}$ with Y'_{proj} projective and Y'_{tor} torsion. As $\text{Hom}(X, Y'_{\text{proj}}) = 0$, the composition $X \rightarrow Y \rightarrow Y' \rightarrow Y'_{\text{tor}}$ is a monomorphism with codomain in \mathbb{T} . The statement now follows from [KS06, Theorem 13.2.8]. \square

The following theorem gives the required fiber sequence. It is an easy consequence of the results in this section.

Theorem 7.5. *Let $\mathbb{K}: \text{Cat}_\infty^{\text{ex}} \rightarrow \mathbf{A}$ be a localizing invariant with values in a stable presentable ∞ -category \mathbf{A} . If \mathbb{K} commutes with countable products, then there is a fiber sequence*

$$\bigoplus_{p \in \mathcal{P}^{\ell_{\text{fin}}}} \mathbb{K}(\text{tor } \mathfrak{A}_p) \rightarrow \prod_{p \in \mathcal{P}^{\ell}} \mathbb{K}(\text{mod } \mathfrak{A}_p) \rightarrow \mathbb{K}(\text{LCA}_{A,ab}).$$

Proof. The localization sequence $\bigoplus_p \text{tor}(\mathfrak{A}_p) \rightarrow \prod_p \text{mod}(\mathfrak{A}_p) \rightarrow \mathbf{L}_A^{(\infty)}$ of abelian categories, given by Proposition 7.2, yields an exact sequence $\mathbf{D}_\infty^b(\bigoplus_p \text{tor}(\mathfrak{A}_p)) \rightarrow \mathbf{D}_\infty^b(\prod_p \text{mod}(\mathfrak{A}_p)) \rightarrow \mathbf{D}_\infty^b(\mathbf{L}_A^{(\infty)})$ of stable ∞ -categories (in the sense of [Lur17]); this uses that the natural functor $\mathbf{D}^b(\bigoplus_p \text{tor}(\mathfrak{A}_p)) \rightarrow \mathbf{D}_\mathbb{T}^b(\mathbf{L}_A^{(\infty)})$ is a triangle equivalence; see [BGT13, Corollary 5.11] or [NS18]. Furthermore, the natural embedding $\mathbf{J}_A^{(\infty)} \rightarrow \mathbf{L}_A^{(\infty)}$ induces an equivalence $\mathbf{D}_\infty^b(\mathbf{J}_A^{(\infty)}) \rightarrow \mathbf{D}_\infty^b(\mathbf{L}_A^{(\infty)})$. Indeed, since $\mathbf{J}_A^{(\infty)}$ is the category of projectives of $\mathbf{L}_A^{(\infty)}$, and the latter has enough projectives (see Proposition 7.2 for both claims), the equivalence follows from [HR20b, Remark 3.15]. This leads to an exact sequence $\mathbf{D}_\infty^b(\bigoplus_p \text{tor}(\mathfrak{A}_p)) \rightarrow \mathbf{D}_\infty^b(\prod_p \text{mod}(\mathfrak{A}_p)) \rightarrow \mathbf{D}_\infty^b(\mathbf{J}_A^{(\infty)})$ of stable ∞ -categories. Applying a localizing invariant as in the statement of the theorem gives the required sequence. Finally, use Proposition 4.5. \square

Remark 7.6. For a weakly localizing invariant \mathbb{K} which need not commute with countable products, there is a fiber sequence $\bigoplus_{p \in \mathcal{P}^{\ell_{\text{fin}}}} \mathbb{K}(\text{tor } \mathfrak{A}_p) \rightarrow \mathbb{K}(\prod_{p \in \mathcal{P}^{\ell}} \text{mod } \mathfrak{A}_p) \rightarrow \mathbb{K}(\text{LCA}_{A,ab})$.

8. A corollary regarding the second K -group

Suppose G is an abelian group. We denote its profinite completion by

$$G^\wedge := \varprojlim_N G/N,$$

where the limit is taken over all finite-index subgroups $N \subseteq G$, partially ordered by inclusion. This notation is not to be confused with the Pontryagin dual G^\vee .

We shall only need G^\wedge as an abelian group and ignore that one could consider it as a profinite topological group.

One can formulate the above in a more categorical fashion: The forgetful functor from profinite groups to groups has profinite completion as its left adjoint.

In particular, the unit of this adjunction provides a natural map $G \rightarrow G^\wedge$ to the profinite completion. Despite the name ‘completion’, this map need not be injective.

Example 8.1.

1. If G is a finite group, the map $G \rightarrow G^\wedge$ is an isomorphism.
2. We have $\mathbb{Z}^\wedge \cong \prod_p \mathbb{Z}_p$, where \mathbb{Z}_p denotes the p -adic integers.
3. As \mathbb{Q} has no nonzero finite quotient groups, we have $\mathbb{Q}^\wedge = 0$. Similarly, we have $\mathbb{R}^\wedge = 0$, and $(\mathbb{Q}/\mathbb{Z})^\wedge = 0$.

Construction 8.2. Let A be a simple finite-dimensional \mathbb{Q} -algebra with center F . Below, we construct an exact sequence

$$K_2(A) \longrightarrow \prod'_v K_2(A_v) \longrightarrow K_2(\text{LCA}_A) \longrightarrow 0, \tag{8.1}$$

where v runs over the set $\mathcal{P}\ell_F$. The middle term is to be understood in the meaning of Equation (3.3).

The construction only relies on Theorem A, applied with nonconnective K -theory as the input invariant, and uses no class field theory.

Proof. Apply Theorem A to A using nonconnective K -theory as the localizing invariant. This outputs a fiber sequence of spectra and its long exact sequence of homotopy groups reads as follows

$$\cdots \rightarrow K_2(A) \rightarrow \prod'_p K_2(A_p) \rightarrow K_2(\text{LCA}_A) \xrightarrow{\alpha} K_1(A) \xrightarrow{\beta} \prod'_p K_1(A_p) \rightarrow \cdots, \tag{8.2}$$

where p runs through $\mathcal{P}\ell_{\mathbb{Q}}$ and where we have labeled various arrows for later reference. Here, we use that for $n \geq 1$ nonconnective K -groups $K_n(-)$ agree with ordinary Quillen K -groups ([Sch06, Remarks 3 and 7]). For any finite place $p \in \mathcal{P}\ell_{\mathbb{Q}}$, we have

$$A_p = A \otimes_{\mathbb{Q}} \mathbb{Q}_p = \bigoplus_{v|p} A_v, \tag{8.3}$$

where v runs through the finitely many places v of F lying over p . This also holds analogously for $p = \mathbb{R}$. Hence, $K_i(A_p) \cong \bigoplus_{v|p} K_i(A_v)$. Finally, the morphism β is injective (it suffices to check that the map to the localization at a single prime p of our choice is injective. In detail: Apply Remark 8.3 to each simple summand of A , regarded over its individual center). Thus, we must have $\alpha = 0$ and we can truncate the exact sequence at this point. We arrive at Equation (8.1). □

Remark 8.3. Suppose A is a central simple F -algebra. Then A_v is a central simple F_v -algebra. The reduced norm respects any base change, so

$$\begin{array}{ccc} K_i(A) & \longrightarrow & K_i(A_v) \\ \text{nr} \downarrow & & \downarrow \text{nr} \\ K_i(F) & \longrightarrow & K_i(F_v) \end{array}$$

commutes for $i = 0, 1, 2$ (and just for these i because the reduced norm only exists in this range). Suppose $i = 1$. The reduced norm $K_1(A) \rightarrow K_1(F)$ is injective ([CR90, Theorem 7.51] or [CR87, Theorem 45.3]). The bottom horizontal arrow is just $F^\times \rightarrow F_v^\times$ and thus

also injective as this is just the metric completion at v . It follows that $K_1(A) \rightarrow K_1(A_v)$ is injective. If v is a finite place, the reduced norm $K_1(A_v) \rightarrow K_1(F_v)$ is also an isomorphism.

9. Hilbert reciprocity law (classical)

In this section, we shall freely use the Hilbert symbol and its properties. A useful survey reference is [Gra03, Chapter II, §7], and proofs of the key properties can be found in [FV02, Chapter IV, §5.1].

Let F be a number field. We write $m := \#\mu(F)$ for the number of its roots of unity. Similarly, write $m_v := \#(\mu(F_v)^\wedge)$ for any place $v \in \mathcal{P}_F$. Thus, $m_v \geq 1$ is a finite number in any case. See Remark 9.8 for an explanation why this holds and an alternative, perhaps more common, description of the value m_v .

The Hilbert reciprocity law states that for any $\alpha, \beta \in F^\times$ we have the identity

$$\prod_{v \in \mathcal{P}_F} h_v(\alpha, \beta)^{\frac{m_v}{m}} = 1, \tag{9.1}$$

where

$$h_v: F_v^\times \times F_v^\times \longrightarrow \mu(F_v) \tag{9.2}$$

is the *Hilbert symbol* for the local field F_v , the completion of F at the place v . All but finitely many factors in the product are 1.

It does not make a difference whether the product over $v \in \mathcal{P}_F$ includes the complex places or not because for these the Hilbert symbol is trivial by construction.³

The Hilbert reciprocity law is a consequence of the stronger Artin reciprocity law (see, for example, [Sha00]) but is still a result with many important applications on its own. For example, it immediately implies Gauss’ original quadratic reciprocity law.

Example 9.1. Suppose $F = \mathbb{Q}$. Then for all primes $p \neq 2$ the Hilbert symbol

$$h_p: \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \longrightarrow \mu(\mathbb{Q}_p) \cong \mathbb{F}_p^\times$$

agrees with the tame symbol. We have $\frac{m_p}{m} = \frac{p-1}{2}$ and therefore only the image of h_p in the quotient group $\mathbb{F}_p^\times / \mathbb{F}_p^{\times 2} \cong \{\pm 1\}$ matters for the reciprocity law. This makes it possible to express h_p in terms of Legendre symbols, leading to the classical formulation of quadratic reciprocity. Only for $p = 2, \infty$, the Hilbert symbol differs from the tame symbol: For $p = 2$ the residue field \mathbb{F}_2^\times is trivial and so is the tame symbol, for $p = \infty$ the definition of the tame symbol collapses as there is no valuation ideal. For more, see [Har20, Remark 14.12].

Moore [Moo68] later found a strengthening of the Hilbert reciprocity law. We recall his way of phrasing it. As a first step, one uses that the Hilbert symbol satisfies the Steinberg relation

$$h_v(\alpha, 1 - \alpha) = 1 \quad \text{for all} \quad \alpha \in F \setminus \{0, 1\}$$

³We recall the construction of the Hilbert symbol in Equation (9.4) below. The Hilbert symbol concerns the action of the Artin symbol on radical extensions of the local field. As \mathbb{C} is algebraically closed, no nontrivial extensions exist, so the action is necessarily by the identity map. For the same reason, the action at real places is necessarily 2-torsion.

and therefore, using Milnor’s formula, namely

$$K_2(F) \cong (F^\times \otimes_{\mathbb{Z}} F^\times) / (\text{Steinberg relation}), \tag{9.3}$$

can be reinterpreted as a map

$$h_v : K_2(F_v) \longrightarrow \mu(F_v)$$

replacing Equation (9.2). Then Moore proves the following.

Theorem 9.2 (Moore sequence). *Let F be a number field. Then the sequence*

$$K_2(F) \longrightarrow \bigoplus_{v \text{ noncomplex}} \mu(F_v) \xrightarrow{\cdot \frac{m_v}{m}} \mu(F) \longrightarrow 0$$

is exact, where the first arrow is the Hilbert symbol, and the second is taking the $\frac{m_v}{m}$ -th power.

The statement that this is a complex agrees with Hilbert’s reciprocity law. The exactness is Moore’s refinement. A nice presentation and proof of Moore’s theorem can be found in [CW72].

The new perspective brought by Moore’s formulation directly suggests numerous follow-up questions. Is there a natural way to extend Moore’s sequence to the left? Does the sequence have a counterpart for K_n with $n \neq 2$?

Below, we present a novel viewpoint which might be considered an answer to these questions. Building on top of this angle, we later go on to replace the number field by a possibly noncommutative simple algebra.

Write $\mathcal{O}_v := \mathcal{O}_{F,v}$ as a shorthand for the valuation ring and $\kappa(v)$ for its residue field.

Lemma 9.3. *For any place $v \in \mathcal{P}l_F$, let $h_v : K_2(F_v) \rightarrow \mu(F_v)$ be the Hilbert symbol.*

1. *For any complex place v , the group $K_2(F_v)$ is divisible.*
2. *For noncomplex places v , we have*
 - (a) *the maps $h_v : K_2(F_v) \rightarrow \mu(F_v)$ are epimorphisms with uniquely divisible kernels,*
 - (b) *for all finite places, the maps $h_v : K_2(\mathcal{O}_v) \rightarrow \mu(F_v)[p^\infty]$ are epimorphisms with uniquely divisible kernels, and*
 - (c) *for all but finitely many places, we have $h_v(\mathcal{O}_v) = 0$.*

Proof. We start by considering a complex place v . In this case, $F_v \cong \mathbb{C}$ is algebraically closed and by $K_2(\mathbb{C})/lK_2(\mathbb{C}) \cong H^2(\mathbb{C}, \mu_l^{\otimes 2}) = 0$ for all $l \geq 1$ (or a direct computation using Milnor K -groups, $\{a, b\} = l\{a, \sqrt[l]{b}\}$) one sees that $K_2(\mathbb{C})$ is a divisible group.

Consider now the case where v is noncomplex. Using local class field theory, Moore [Moo68] has shown the exactness of the sequence

$$0 \longrightarrow m_v K_2(F_v) \longrightarrow K_2(F_v) \xrightarrow{h_v} \mu(F_v) \longrightarrow 0,$$

where h_v is the Hilbert symbol. This proceeds as follows: Firstly, one uses local class field theory to construct the Hilbert symbol, using the explicit presentation of K_2 through Equation (9.3),

$$h_v(a \otimes b) := \frac{\text{Art}(a)^{m_v \sqrt{b}}}{m_v \sqrt{b}} \in \mu(F_v) \quad \text{for } a, b \in F_v^\times, \tag{9.4}$$

where $m_v \sqrt{b}$ is any m_v -th root of $b \in F_v^\times$ and $\text{Art}: F_v^\times \rightarrow \text{Gal}(F_v^{\text{ab}}/F)$ denotes the local Artin map. Secondly, one verifies that h_v respects the Steinberg relation, so it factors over $K_2(F_v)$ and then that h_v is zero on $m_v K_2(F_v)$, which is pretty immediate. Moore then goes on to prove that $m_v K_2(F_v)$ is divisible, which Merkurjev [Mer83] strengthened to being uniquely divisible. A complete textbook proof of all these results with full details can be found in [FV02, Ch. IX, §4].

Suppose that v is a finite place. The torsion group $\mu(F_v)$ splits into its different ℓ -primary torsion parts,

$$\mu(F_v) \cong \mu(F_v)[p^\infty] \oplus \mu(F_v)[\text{prime-to-}p^\infty].$$

The group $\mu(F_v)[p^\infty]$ denotes the p -primary part, where p is the prime in \mathbb{Z} lying below the finite place v . Its complement, the prime to p part, is known to agree with $\mu(\kappa(v))$ through the (canonical) Teichmüller lift.⁴ The quotient sequence of abelian categories

$$\text{mod}_{m_v}(\mathcal{O}_v) \longrightarrow \text{mod}(\mathcal{O}_v) \longrightarrow \text{mod}(F_v)$$

induces a long exact sequence in K -theory (by the localization theorem; $\text{mod}_{m_v}(\mathcal{O}_v)$ denotes finitely generated modules supported on the prime m_v). We get

$$\cdots \longrightarrow K_2(\mathcal{O}_v) \longrightarrow K_2(F_v) \xrightarrow{\partial} K_1(\kappa(v)) \longrightarrow \cdots,$$

where ∂ is the tame symbol (see [Sri08, Lemma 9.12]), and $K_1(\kappa(v)) \cong \mu(F_v)[\text{prime-to-}p^\infty]$. One can replace “ \cdots ” on either side by zero and still get an exact sequence. On the right, this follows because ∂ is split by (using Milnor K -group notation)

$$x \mapsto \{\pi, \omega(x)\},$$

where $\omega: \kappa(v)^\times \rightarrow F_v^\times$ is the Teichmüller lift and π any (fixed once and for all) uniformizer. On the left because K_2 of a finite field vanishes.

As the tame symbol factors as $K_2(F_v) \xrightarrow{h_v} \mu(F_v) \rightarrow \mu(F_v)[\text{prime-to-}p^\infty]$, where the last map is the canonical projection, we find the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & m_v K_2(F_v) & \longrightarrow & K_2(F_v) & \xrightarrow{h_v} & \mu(F_v) & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & K_2(\mathcal{O}_v) & \longrightarrow & K_2(F_v) & \xrightarrow{\partial} & \mu(F_v)[\text{prime-to-}p^\infty] & \longrightarrow & 0 \end{array}$$

⁴See [FV02, Chapter I, §7.1-7.3] for a detailed construction of the Teichmüller representatives. One can also define a map $\omega: \kappa(v)^\times \rightarrow \mu(F_v)[\text{prime-to-}p^\infty]$ by

$$\omega(x) := \lim_{n \rightarrow \infty} \tilde{x}^{q^n},$$

where $q := \#\kappa(v)$ and \tilde{x} is any preimage of x in \mathcal{O}_v . Once one shows that the limit exists and is independent of the choice of \tilde{x} , it is clear that ω is a group homomorphism and a section to $\mathcal{O}_v^\times \rightarrow \kappa(v)^\times$.

with exact rows. The left downward arrow is a split monomorphism since $m_v K_2(F_v)$ is divisible (and hence injective). The snake lemma now gives a (split) exact sequence

$$0 \rightarrow m_v K_2(F_v) \rightarrow K_2(\mathcal{O}_v) \xrightarrow{h'_v} \mu(F_v)[p^\infty] \rightarrow 0,$$

where h'_v the Hilbert symbol, with the domain restricted to $K_2(\mathcal{O}_v) \subseteq K_2(F_v)$ and the codomain restricted to $\mu(F_v)[p^\infty] \subseteq \mu(F_v)$. We note that F_v only has nontrivial p -primary roots of unity if F_v/\mathbb{Q}_p is ramified. This only happens at those primes where F/\mathbb{Q} is ramified, and thus finitely many. Hence, $\mu(F_v)[p^\infty] = 0$ for all but finitely many places v . □

Construction 9.4. We now consider a map $T: K_2(\text{LCA}_{F,ab}) \rightarrow \bigoplus_{v \text{ noncomplex}} \mu(F_v)$ as the composition of the maps $K_2(\text{LCA}_{F,ab}) \rightarrow \prod'_v K_2(F_v) \rightarrow \bigoplus_{v \text{ noncomplex}} \mu(F_v)$, given as follows.

1. Applying Theorem A with nonconnective K -theory as the localizing invariant, we find an isomorphism $K_2(\text{LCA}_{F,ab}) \xrightarrow{\cong} \prod'_v K_2(F_v)$. The isomorphism is induced by the equivalence $\text{LCA}_{F,ab} \xrightarrow{\cong} J_F^{(\infty)}$ from Proposition 4.5.
2. For each place v , we consider the Hilbert symbol $h_v: K_2(F_v) \rightarrow \mu(F_v)$. This then induces a map $\prod_v h_v: \prod_v K_2(F_v) \rightarrow \prod_v \mu(F_v)$, which restricts to a map

$$\prod'_v K_2(F_v) \rightarrow \bigoplus_{v \text{ noncomplex}} \mu(F_v)$$

by the last claim of Lemma 9.3.

We remind the reader that

$$\prod'_v K_2(F_v) = \left\{ (\alpha_v)_v \in \prod_{v \in \mathcal{P}l_F} K_2(F_v) \mid \begin{array}{l} \alpha_v \in K_2(\mathcal{O}_v) \text{ for all but} \\ \text{finitely many finite places } v \end{array} \right\}.$$

Theorem 9.5. *Let F be a number field.*

1. *The diagram*

$$\begin{array}{ccccccc} K_2(F) & \longrightarrow & K_2(\text{LCA}_{F,ab}) & \longrightarrow & K_2(\text{LCA}_F) & \longrightarrow & 0 \\ & & & \downarrow T & & \downarrow \bar{T} & \\ & & & & & & \\ K_2(F) & \longrightarrow & \bigoplus_{v \text{ noncomplex}} \mu(F_v) & \longrightarrow & \mu(F) & \longrightarrow & 0 \end{array} \tag{9.5}$$

commutes, where the bottom row is the Moore sequence (Theorem 9.2) and the top row comes from Theorem A. The arrow T comes from the Hilbert symbol (see Construction 9.4), and arrow \bar{T} is the induced map on the quotient.

2. *Both T and \bar{T} are surjective, and their kernels consist of the divisible elements of their respective domains.*
3. *Both rows in Diagram 9.5 are exact.*

4. Taking the profinite completion of the entire diagram, all downward arrows become isomorphisms.
5. $K_2(\text{LCA}_F)^\wedge \cong \mu(F)$.

As the top row directly comes from the results in this paper, statement (4) could be summarized as the following slogan: *Upon profinite completion, Moore’s sequence can be identified with an excerpt of the long exact sequence coming from the fibration in Theorem A.*

We note that our proof of Theorem A, and thus the fibration sequence referenced in the slogan, did not rely on any use of class field theory.

Proof. The top row in Diagram 9.5 is exact by Construction 8.2. The exactness of the bottom row is Moore’s theorem, [CW72]. This shows that statement (3) holds.

For the commutativity of the left square, we observe that the top-right branch is given by

$$\begin{array}{ccccccc}
 K_2(F) & \longrightarrow & K_2(\text{LCA}_{F,ab}) & \xrightarrow{\cong} & \prod'_v K_2(F_v) & \longrightarrow & \bigoplus_v \text{noncomplex} \mu(F_v) \\
 \alpha \otimes \beta & \longmapsto & & \longrightarrow & (\alpha \otimes \beta)_v & \longmapsto & (h_v(\alpha \otimes \beta))_v.
 \end{array}$$

The commutativity of the left square is now immediate from the definition of the first arrow in Moore’s sequence. The commutativity of the right square holds by construction. This shows statement (1).

It follows from Lemma 9.3 that T is an epimorphism. The commutativity of the rightmost square then shows that the composition $K_2(\text{LCA}_{F,ab}) \rightarrow K_2(\text{LCA}_F) \xrightarrow{\bar{T}} \mu(F)$ is an epimorphism. We conclude that \bar{T} is an epimorphism as well.

It follows from Lemma 9.3 that $\ker T$ is a uniquely divisible subgroup of $K_2(\text{LCA}_{F,ab})$. As the codomain of T is torsion, the $\ker T$ needs to contain all divisible elements of $K_2(\text{LCA}_{F,ab})$. This shows that $\ker T$ consists of all divisible elements of $K_2(\text{LCA}_{F,ab})$.

It follows from the snake lemma (or from a straightforward diagram chase) that the canonical map $\ker T \rightarrow \ker \bar{T}$ is an epimorphism. As $\ker T$ is a divisible group so is $\ker \bar{T}$. As $\mu(F)$ is torsion, we know that all divisible elements of $K_2(\text{LCA}_F)$ lie in $\ker \bar{T}$. This establishes statement (2).

We now prove statement (4). Consider the exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker T & \longrightarrow & K_2(\text{LCA}_{F,ab}) & \xrightarrow{T} & \bigoplus_v \text{noncomplex} \mu(F_v) \longrightarrow 0, \text{ and} \\
 0 & \longrightarrow & \ker \bar{T} & \longrightarrow & K_2(\text{LCA}_F) & \longrightarrow & \mu(F) \longrightarrow 0.
 \end{array}$$

As both $\ker T$ and $\ker \bar{T}$ are divisible groups (and hence injective), these sequences are split exact. Since taking profinite completions is an additive functor, these stay (split) exact after profinite completion. Using that $(\ker T)^\wedge = 0 = (\ker \bar{T})^\wedge$, as profinite completions of divisible groups are zero, we find that T^\wedge and \bar{T}^\wedge are isomorphisms, as required.

Finally, statement (5) follows from (4), together with the observation that $\mu(F) \cong \mu(F)^\wedge$ since $\mu(F)$ is a finite group. □

We also deduce the following characterization:

Corollary 9.6. *Up to isomorphisms, the Moore sequence arises from Construction 8.2 by quotienting out the subgroups of divisible elements in the right two terms.*

$$\begin{array}{ccccccc}
 K_2(F) & \longrightarrow & K_2(\text{LCA}_{F,ab})_{/div} & \longrightarrow & K_2(\text{LCA}_F)_{/div} & \longrightarrow & 0 \\
 \parallel & & \downarrow \cong & & \downarrow \cong & & \\
 K_2(F) & \longrightarrow & \bigoplus_{v \text{ noncomplex}} \mu(F_v) & \longrightarrow & \mu(F) & \longrightarrow & 0.
 \end{array}$$

This shows that, up to isomorphism, the Moore sequence stems from Theorem A. Ingredients from class field theory only enter in the identification of the K_2 -groups with groups of roots of unity. For the middle term, this only uses tools from local class field theory but showing $K_2(\text{LCA}_F)_{/div} \cong \mu(F)$ relied on using global class field theory, too.

Remark 9.7 (Wild kernel, [Hut04]). The kernel of the leftmost horizontal maps in the above diagram is understood. Historically, this kernel is known as the *wild kernel* $WK_2(F)$, as it is the joint kernel under all Hilbert symbols. The nomenclature stems from the fact that the joint kernel of all tame symbols, which agrees with $K_2(\mathcal{O}_F)$, is known as the *tame kernel*. It is clear that the subgroup of divisible elements $K_2(F)_{div} \subseteq K_2(F)$ must be contained in $WK_2(F)$. By Hutchinson [Hut01, Corollary 4.5], following ideas of Tate, the quotient $WK_2(F)/K_2(F)_{div}$ is either 0 or $\mathbb{Z}/2$, depending on the number field F . Both cases occur and it is understood how one can distinguish these cases.

Remark 9.8. We complement the definition given for m_v at the beginning of §9 with a different viewpoint. For each finite or real place $\mu(F_v)$ is finite, so the profinite completion $\mu(F_v) \rightarrow \mu(F_v)^\wedge$ is an isomorphism and therefore $m_v = \#\mu(F_v)$. Only for a complex place $\mu(F_v) \cong \mathbb{Q}/\mathbb{Z}$ is infinite, and we have $(\mathbb{Q}/\mathbb{Z})^\wedge = 0$. Hence, instead of using the profinite completion in our definition of m_v , we could also agree on the convention that $m_v := 0$ for complex places, and $m_v = \#\mu(F_v)$ for all noncomplex places.

10. Hilbert reciprocity law (noncommutative version)

The considerations of the previous section suggest a kind of noncommutative Moore sequence and thus a noncommutative Hilbert reciprocity law. As we had already seen in Construction 8.2, the top row of

$$\begin{array}{ccccccc}
 K_2(F) & \longrightarrow & K_2(\text{LCA}_{F,ab}) & \longrightarrow & K_2(\text{LCA}_F) & \longrightarrow & 0 \\
 \parallel & & \downarrow & & \downarrow & & \\
 K_2(F) & \longrightarrow & \bigoplus_{v \text{ noncomplex}} \mu(F_v) & \longrightarrow & \mu(F) & \longrightarrow & 0
 \end{array}$$

is also available in the noncommutative setting. Let us explore this. Let A be a simple finite-dimensional \mathbb{Q} -algebra; let F be its center. Then for every place v , A_v is a central simple F_v -algebra.

We could hope that there is a nice concept of noncommutative Hilbert symbols

$$h_v : K_2(A_v) \longrightarrow (\dots)$$

with values in a yet unknown group (\dots) so that the h_v satisfy an interesting reciprocity law which would reduce to the ordinary Hilbert reciprocity law if $A = F$. Indeed, it is not so difficult to identify what this might be. As a first step, imitating the commutative situation, we could expect that the kernel of h_v should be the subgroup of divisible elements in $K_2(A_v)$. Unless *vis a real place such that A_v does not split over F_v* , the reduced norm

$$K_2(A_v) \xrightarrow{\text{nr}} K_2(F_v)$$

is an isomorphism (in general, it is not: if A are the rational Hamilton quaternions, $A \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$ so that $K_2(\mathbb{H}) = 0$, yet $K_2(\mathbb{R}) \cong \{\pm 1\}$, so -1 cannot possibly lie in the image of the reduced norm).

However, in all situations where the reduced norm is an isomorphism, the subgroup of divisible elements in $K_2(A_v)$ gets identified with the divisible elements in $K_2(F_v)$. Thus,

$$K_2(A_v)/K_2(A_v)_{\text{div}} \cong \mu(F_v),$$

where the isomorphism is given by the Hilbert symbol of the local field F_v . Thus, the noncommutative Hilbert symbol reasonably would be taken to be

$$h_v^{\text{nc}} : K_2(A_v) \longrightarrow \mu(F_v) \tag{10.1}$$

$$h_v^{\text{nc}} := h_v \circ \text{nr}_{A_v}$$

and is merely the reduced norm, followed by the Hilbert symbol of the center. Unlike the situation for number fields, this map h_v^{nc} can fail to be surjective also for noncomplex places but only in the rather exceptional case where v is a real place over which A does not split.

One immediately deduces a noncommutative Hilbert reciprocity law with ease, given the structure of the above definition.

Proposition 10.1. *Let A be a finite-dimensional simple \mathbb{Q} -algebra with center F . Then for any $\alpha \in K_2(A)$ we have*

$$\prod_{v \in \mathcal{P}_{\ell_F}} h_v^{\text{nc}}(\alpha)^{\frac{m_v}{m}} = 1,$$

where h_v^{nc} denotes the noncommutative Hilbert symbol of Equation (10.1). If $A = F$ the statement specializes to the classical Hilbert reciprocity law of Equation (9.1).

Proof. The proof is tautological by construction of h_v^{nc} . We compute

$$\prod_{v \in \mathcal{P}\ell_F} h_v^{\text{nc}}(\alpha)^{\frac{m_v}{m}} = \prod_{v \in \mathcal{P}\ell_F} h_v(\text{nr}_{A_v}(\alpha \otimes A_v))^{\frac{m_v}{m}},$$

where $\text{nr}_{A_v} : K_2(A_v) \rightarrow K_2(F_v)$ is the reduced norm and $\alpha \otimes A_v$ refers to the image of $\alpha \in K_2(A)$ in $K_2(A_v)$. By Remark 8.3, we have $\text{nr}_A(\alpha) \otimes F_v = \text{nr}_{A_v}(\alpha \otimes A_v)$, so the right term equals $h_v(\text{nr}_A(\alpha))^{\frac{m_v}{m}}$ and thus our claim reduces to the classical Hilbert reciprocity law, applied to the element $\text{nr}_A(\alpha) \in K_2(F)$. If $A = F$, the reduced norm is the identity map. □

Clearly, the above proposition is not very interesting because it reduces everything to the commutative case. But we can push the analogy further. We would expect that Moore’s sequence holds analogously in the noncommutative setting and should also reduce to the commutative Moore sequence, except perhaps with some modifications when the reduced norm fails to be surjective. This suggests the following.

Conjecture 10.2. *Suppose A is a finite-dimensional simple \mathbb{Q} -algebra with center F . Then*

$$K_2(\text{LCA}_A) \cong K_2(\text{LCA}_F).$$

Curiously, when trying to prove this, we were naturally led to an old conjecture of Merkurjev and Suslin, which is still open.

Conjecture 10.3 (Merkurjev–Suslin, 1982). *Suppose A is a finite-dimensional central simple F -algebra, where F is a local or global field (say of characteristic zero). Then there is an exact sequence*

$$0 \longrightarrow K_2(A) \xrightarrow{\text{nr}} K_2(F) \longrightarrow \bigoplus_v \mathbb{Z}/2\mathbb{Z} \longrightarrow 0, \tag{10.2}$$

where v runs over all real places of F for which the algebra A_v is nonsplit. (If F is a p -adic or complex local field, then the set of such v is necessarily empty.)

This conjecture is stated in [MS82, Remark 17.5]. The conjecture is fairly parallel to what is proven to happen for K_1 in view of the Hasse–Schilling–Maass theorem [CR87, Theorem 45.3].

Remark 10.4. We should comment on the status of this conjecture:

1. (Merkurjev–Suslin) If A has square-free index, the conjecture is true. This is [MS82, Theorem 17.4].
2. (Merkurjev–Suslin, Kahn–Levine) If A has square-free index, the injectivity of $K_2(A) \xrightarrow{\text{nr}} K_2(F)$ is actually true for all fields of characteristic zero. This was proven independently by Merkurjev–Suslin [MS10, Theorem 2.3] and Kahn–Levine [KL10, Corollary 2]. The latter exhibit an exact sequence

$$0 \longrightarrow K_2(A) \xrightarrow{\text{nr}} K_2(F) \longrightarrow H^4(F, \mathbb{Z}/d(3)) \longrightarrow H^2(F(X), \mathbb{Z}/d(3)),$$

where X is the Severi–Brauer variety of A and d the index. Thus, if F is a field of cohomological dimension ≤ 3 , the reduced norm on K_2 is an isomorphism for square-free d .

- Sometimes, one finds a mistaken claim in the literature attributing the injectivity of the reduced norm on K_2 , that is, $SK_2(A) = 0$, over number fields to the paper [BR84]. However, this stems from a confusion and this paper does not even claim to have shown this.

Assuming the Merkurjev–Suslin conjecture, we obtain a nice analogue of Moore’s sequence in the noncommutative setting which for $A = F$ reduces to the original Moore sequence. In here, we write T^{nc} for the map as in Construction 9.4, replacing the Hilbert symbol h_v by h_v^{nc} . Specifically, T^{nc} is the composition

$$K_2(\text{LCA}_{A,ab}) \xrightarrow{\cong} \prod'_v K_2(A_v) \xrightarrow{h_v^{\text{nc}}} \bigoplus_{v \text{ noncomplex}} \mu(F_v).$$

Theorem 10.5. *Let A be a simple finite-dimensional \mathbb{Q} -algebra. Write F for its center. Assume Sequence 10.2 is exact for A and all completions A_v at the places of F (e.g., if A has square-free index).*

- The diagram

$$\begin{array}{ccccccc} K_2(A) & \longrightarrow & K_2(\text{LCA}_{A,ab}) & \longrightarrow & K_2(\text{LCA}_A) & \longrightarrow & 0 \\ \downarrow & & \downarrow T^{\text{nc}} & & \downarrow \bar{T}^{\text{nc}} & & \\ K_2(F) & \longrightarrow & \bigoplus_{v \text{ noncomplex}} \mu(F_v) & \longrightarrow & \mu(F) & \longrightarrow & 0 \end{array} \tag{10.3}$$

commutes, where the bottom row is the Moore sequence (Theorem 9.2) and the top row comes from Theorem A. The arrow T^{nc} arrow arises from the reduced norm, followed by the Hilbert symbol (defined above), and arrow \bar{T}^{nc} is the induced map on the quotient.

- The map T^{nc} is an epimorphism after inverting 2. The kernel $\ker T^{\text{nc}}$ agrees with the subgroup of divisible elements in $K_2(\text{LCA}_{A,ab})$.
- Both rows in Diagram 10.3 are exact.
- Taking the profinite completion of the entire diagram, the left two downward arrows become isomorphisms after inverting 2, and the right downward arrow becomes an isomorphism on the nose.
- $K_2(\text{LCA}_A)^\wedge \cong \mu(F)$.

We repeat that, if Conjecture 10.3 is true, the exactness of Sequence 10.2 holds unconditionally. In this case, the proof will also show the validity of Conjecture 10.2.

Proof. We first use Construction 8.2 twice: first for A itself and then for its center F . This yields the exact rows in the following diagram

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
K_2(A) & \longrightarrow & \prod' K_2(A_v) & \longrightarrow & K_2(\text{LCA}_A) & \longrightarrow & 0 \\
\text{nr} \downarrow & & \text{nr} \downarrow & & \downarrow & & \\
K_2(F) & \longrightarrow & \prod' K_2(F_v) & \longrightarrow & K_2(\text{LCA}_F) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\bigoplus_w \mathbb{Z}/2 & \longrightarrow & \bigoplus_w \mathbb{Z}/2 & & & &
\end{array}$$

and we map the top row to the bottom row using the reduced norm on K_2 . By the commutative square in Remark 8.3, the top-left square commutes and the right square commutes by the exactness of the rows. Next, as already indicated in the above diagram, we use Sequence 10.2 to obtain the exact two columns on the left. As both cokernels match, we deduce that the reduced norm induces an isomorphism

$$K_2(\text{LCA}_A) \xrightarrow[\text{nr}]{\cong} K_2(\text{LCA}_F),$$

that is, we obtain Conjecture 10.2. All remaining claims now follow from Theorem 9.5. \square

If one could show that $K_2(\text{LCA}_A) \cong K_2(\text{LCA}_F)$ by a different method, this might be a promising start to reverse the logic in order to prove Conjecture 10.3. However, one might also need some knowledge of the boundary map $K_3(\text{LCA}_A) \rightarrow K_2(A)$.

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