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INVARIANT SCALAR-FLAT KÄHLER METRICS ON LINE BUNDLES OVER GENERALIZED FLAG VARIETIES

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Abstract. Let G be a simply-connected semisimple compact Lie group, X a simply-connected compact Kähler manifold homogeneous under G, and L a negative holomorphic line bundle over X. We prove that all G-invariant Kähler metrics on the total space of L arise from the Calabi ansatz. Using this, we show that there exists a unique G-invariant scalar-flat Kähler metric in each G-invariant Kähler class of L. The G-invariant scalar-flat Kähler metrics are automatically asymptotically conical.

1. Introduction

During the past few decades, many works arise on the explicit construction of scalar-flat Kähler metrics on noncompact manifolds, usually with certain symmetry conditions. The prototypical example goes back to a seminal work by E. Calabi [10], who constructed complete Ricci-flat metrics on the canonical bundle of $\mathbb{CP}^n$. In complex dimension two, Calabi's construction produces the Eguchi-Hanson instanton [17] on the total space of $\mathcal{O}(-2)$. As a generalization of Eguchi-Hanson, LeBrun [25] constructed a family of ALE scalar-flat Kähler metrics in the total space of $\mathcal{O}(-n)$ with $U(2)$ symmetry. Some other works arise to construct families of Ricci-flat Kähler metrics on open manifolds, in some sense, extending the Calabi's construction, for instance, Stenzel [34], Dancer-Wang [15], Wang [38].

There are some essential existence theorems for complete Ricci-flat Kähler metrics on open manifolds. Following Yau's solution of the Calabi conjecture [40], the analytic approach to the construction of complete Calabi-Yau metrics on noncompact manifolds was initiated by the seminal works of Tian-Yau [36, 37]. The work by Conlon-Hein [13] establishes the existence results for AC Ricci-flat Kähler metrics. Recently, initiated by Li [28], many nontrivial Calabi-Yau metrics has been constructed in $\mathbb{C}^n$ with Euclidean volume growth by Conlon-Rochon [14], Székelyhidi [35], Apostolov-Cifarelli [3].

For the scalar-flat case, LeBrun [26] adapted the Gibbons-Hawking ansatz in hyperbolic model to construct scalar-flat Kähler metrics with S^1 symmetry. Joyce [23] extended LeBrun's hyperbolic ansatz to toric manifolds and Calderbank-Singer [11] applied Joyce's construction to toric resolutions of $\mathbb{C}^2/\Gamma$ with cyclic quotient singularities and constructed a family of T^2-invariant ALE scalar-flat Kähler metrics. Based on Donaldson's reformulation of Joyce construction in [16], Abreu-Sena-Dias [1] constructed complete scalar-flat toric Kähler metrics on symplectic toric 4-manifolds which are asymptotic to generalized Taub-NUT metrics.

Lock-Viaclovsky [29] constructed scalar-flat Kähler metrics on the minimal resolutions of $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $U(2)$ with no reflections. The existence of ALE scalar-flat Kähler metrics in small deformations of resolutions of $\mathbb{C}^2/\Gamma$ also has been investigated by Honda [19] [20], Lock-Viaclovsky [29] and Han-Viaclovsky [18].

In the cases of complex toric surface, the uniqueness of ALE scalar-flat Kähler metrics have been obtained by Sena-Dias [32], together with work by Wright [39]. However, the general uniqueness of ALE scalar-flat Kähler metric still remains completely open (unlike in the compact case [5, 12], or in the case with cusps [4] or with conical singularities [27]). This paper is concerned with a generalization of LeBrun's existence results to a class of spaces with strong symmetry in all dimensions and proves

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the uniqueness of scalar-flat Kähler metrics in each Kähler class on these spaces under a symmetry assumption.

A compact homogeneous Kähler manifold is a compact Kähler manifold (X, ω) on which the identity component of the bi-holomorphic isometry group acts transitively. The classification of this type of spaces has been known for a long time. By [6], every compact, simply-connected homogeneous Kähler manifold is isomorphic, in the sense of homogeneous complex manifolds, to an orbit of the adjoint representation of some compact semisimple Lie group endowed with a canonical complex structure. Then, the classification of compact homogeneous Kähler manifolds reduces to classifying the orbit space of adjoint representation. In general, each compact homogeneous Kähler manifold is the product of a flat complex torus and a compact simply-connected homogeneous Kähler manifold. In this paper, we are only interested in the compact homogeneous Kähler manifolds without torus part, which we call generalized flag varieties.

Theorem A. *Let X be a generalized flag variety and L a negative homogeneous line bundle over X with $p : L \to X$, the natural projection. Then, all invariant Kähler metrics ω on L can be written as*

$$\omega = p^*\omega_X + dd^c\varphi(r^2),$$

where ω_X is an invariant Kähler form on X and $\varphi(r) \in C^\infty(\mathbb{R}_{\geq 0})$.

Based on Theorem A, we have identified the G-invariant Kähler metrics in a given Kähler class on L with a class of single variable function. To determine the complete scalar-flat Kähler metrics in a given Kähler class, the method of momentum construction developed in Hwang-Singer [22] is applied, which reduces the problem to solving a second order ODE. In conclusion, we have the following theorem,

Theorem B. *Let X be a generalized flag variety and L a negative homogeneous line bundle over X. Then, in each Kähler class on L there exists a unique G-invariant scalar-flat Kähler metric. In particular, this metric is an asymptotically conical Kähler metric.*

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2. Geometry of compact homogeneous Kähler manifolds

In this section, we recall the geometry of compact homogeneous Kähler manifolds. In Section 2.1, 2.2 we discuss the classification of simply-connected compact homogeneous Kähler manifolds and holomorphic line bundles over these manifolds. Section 2.3 dedicates to classify all G-invariant Kähler forms on generalized flag varieties.

Here, we introduce some basic notations, see [24] for more details. Let X be a simply-connected compact homogeneous Kähler manifold and G, the universal covering of the compact semisimple Lie group acts on X. At a distinguished point $p \in X$, let R be the isotropy group of p and S, the identity component of the center of R. $G^{\mathbb{C}}$ denotes the complexification of G. Let T be a fixed maximal torus of G. The corresponding Lie algebra of S, T, R, G are denoted by $\mathfrak{s}$, $\mathfrak{t}$, $\mathfrak{r}$, $\mathfrak{g}$ and $\mathfrak{s}^{\mathbb{C}}$, $\mathfrak{t}^{\mathbb{C}}$, $\mathfrak{r}^{\mathbb{C}}$, $\mathfrak{g}^{\mathbb{C}}$, the complexification of Lie algebras. We write Δ to be the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ and Δ^+, Π, the fixed positive root system and simple root system. For each root $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ denotes the eigen-space of α. Since G is a compact semisimple Lie, the negative Killing form, denoted by $(\cdot, \cdot)$, defines a G-invariant inner product in $\mathfrak{g}$.

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2.1. **Classification of compact homogeneous Kähler manifolds.** According to [30], it was proved that the centralizer of S in G is R. So the maximal tori containing S must be contained in R. In the following, we fix a maximal torus T with $S \subset T \subset R$.

Notice that $\mathfrak{g}^{\mathbb{C}}$ admits a root space decomposition,

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

We introduce a *normalized adapted basis* $\{X_\alpha, Y_\alpha\}$ of each pair of root spaces $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, which satisfies the following:

(a) Let $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ denote the complex eigen-spaces corresponding to the roots α and $-\alpha$. Then

$$X_\alpha - iY_\alpha = E_\alpha \in \mathfrak{g}_\alpha \quad \text{and} \quad X_\alpha + iY_\alpha = E_{-\alpha} \in \mathfrak{g}_{-\alpha}. \tag{2.1}$$

(b) X_α and Y_α are normalized in the sense that,

$$[X_\alpha, Y_\alpha] = -H_\alpha, \tag{2.2}$$

where H_α satisfies $\beta(H_\alpha) = (\beta, \alpha)/2i$ for each root β. Consider the inner product induced by the negative Killing form. One can easily check that $|X_\alpha|^2 = |Y_\alpha|^2 = 1/2$. This normalization will be applied in Section 3.1 to calculate the differential of invariant 1-forms.

Then the compact real sub-algebra $\mathfrak{g}$ can be decomposed as follows:

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta^+} \mathbb{R}\langle X_\alpha, Y_\alpha \rangle. \tag{2.3}$$

Next, we classify simply connected compact homogeneous Kähler manifolds by the structure of root system. Consider a parabolic subgroup P of $G^{\mathbb{C}}$. Then P is determined by a subset of the simple root system. Let $(\Delta^+)' \subset \Delta^+$ be the subset of positive roots generated by Π'. The corresponding Lie algebra $\mathfrak{p}$ can be decomposed as

$$\mathfrak{p} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in (\Delta^+)'} \mathfrak{g}_{-\alpha} = \mathfrak{b} \oplus \sum_{\alpha \in (\Delta^+)'} \mathfrak{g}_{-\alpha}, \tag{2.4}$$

where $\mathfrak{b}$ is the Lie algebra of the Borel subgroup. The *generalized flag variety* with data (G, Π, Π') is defined to be the complex manifold $X = G^{\mathbb{C}}/P$.

Assuming that $X = G^{\mathbb{C}}/P$, we show that X is simply-connected compact homogeneous Kähler manifold. Consider the maximal compact subgroup G of $G^{\mathbb{C}}$. Then G acts transitively on X with stabilizer group $R = G \cap P$. The Lie algebra of the stabilizer group R is

$$\mathfrak{r} = \mathfrak{t} \oplus \sum_{\alpha \in (\Delta^+)'} \mathbb{R}\langle X_\alpha, Y_\alpha \rangle. \tag{2.5}$$

Then, topologically, $X \cong G/R$, and its complex structure can also be described as follows. Let

$$D^+ := \Delta^+ \setminus (\Delta^+)'. \tag{2.6}$$

Then D^+ is a *closed* subset of the root system in the sense that for any $\alpha,\ \beta \in D^+$, if $\alpha + \beta$ is a root, then $\alpha + \beta \in D^+$. The tangent space of X at a distinguished point p can be identified with

$$T_pX = \sum_{\alpha \in D^+} \mathbb{R}\langle X_\alpha, Y_\alpha \rangle. \tag{2.7}$$

We also call $\{X_\alpha, Y_\alpha : \alpha \in D^+\}$ a *normalized adapted basis of* X. There exists a natural R-invariant almost complex structure J on T_pX given by

$$J(X_\alpha) = Y_\alpha, \quad J(Y_\alpha) = -X_\alpha.$$

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Because J is R-invariant, it extends to a G-invariant almost complex structure on the whole tangent bundle TX. The complexified tangent space at p splits into

$$T_p^{(1,0)}X = \sum_{\alpha \in D^+} \mathfrak{g}_\alpha, \quad T_p^{(0,1)}X = \sum_{\alpha \in D^+} \mathfrak{g}_{-\alpha}. \tag{2.8}$$

One can check that J is an integrable almost complex structure because D^+ is closed (see [7, Section 12]), and the complex manifold (X, J) is G-equivariantly biholomorphic to $G^{\mathbb{C}}/P$.

Generalized flag varieties are simply-connected and the proof of this fact will be given in Remark 2.2. There exist Kähler forms on (X, J), as discussed in section 2.3. Hence, each generalized flag variety is a simply-connected compact homogeneous Kähler manifold of G.

Conversely, given a simply-connected compact homogeneous Kähler manifold X that admits a transitive holomorphic action by a simply-connected compact semisimple Lie group G with data (R, T, S) and a point $p \in X$ as above, let $\Delta(\mathfrak{t}, \mathfrak{r})$ denote the root system of $\mathfrak{r}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. This can be viewed as a subset of Δ. Define

$$D := \Delta \setminus \Delta(\mathfrak{t}, \mathfrak{r}).$$

The invariant complex structure J on X determines the set of positive roots D^+ as follows. Since the complexified tangent space of X is identified with

$$T_p^{\mathbb{C}}X = \sum_{\gamma \in D} \mathfrak{g}_\gamma,$$

the R-invariance of J implies that J preserves each root space. Then D^+ is defined to be

$$D^+ := \{\gamma \in D : Jv = iv \ \text{ for all } \ v \in \mathfrak{g}_\gamma\}.$$

The closedness of D^+ follows from the integrability of J. In addition, we can choose a simple root system Π' in $\Delta(\mathfrak{t}, \mathfrak{r})$. Then D^+ and the positive roots, $\Delta^+(\mathfrak{t}, \mathfrak{r})$, generated by Π' determine a positive root system $\Delta^+(\mathfrak{t}, \mathfrak{r})$ in $\Delta(\mathfrak{t}, \mathfrak{g})$, i.e.,

$$\Delta^+(\mathfrak{t}, \mathfrak{g}) = D^+ \cup \Delta^+(\mathfrak{t}, \mathfrak{r}).$$

The set Π' can be extended to a simple root system, Π, of $\Delta(\mathfrak{t}, \mathfrak{g})$ such that Π generates the positive roots $\Delta^+(\mathfrak{t}, \mathfrak{g})$. More details on root systems can be found in [7, Sections 13.6–13.7]. Based on this discussion, the Lie algebra $\mathfrak{r}$ can be written in terms of the simple root set Π' as in (2.5). Thus, X can be identified with the generalized flag variety associated with the data (G, Π, Π').

The notion of a generalized flag variety is actually independent of the choice of a simple root system. By [2, Section 5.13], the different simple root systems are identified by the action of the Weyl group, and then the associated generalized flag varieties are isomorphic via conjugation by an element of G. Hence, without loss of generality, we can fix a simple root system Π at the beginning, and then each generalized flag variety of G is classified in terms of a subset of Π.

In conclusion, we have proved the following theorem, which should be well-known to experts.

Theorem 2.1. *Let X be a compact Kähler manifold. Assume that X is homogeneous under a simply-connected compact semisimple Lie group G. Fix a system of simple roots Π of $\mathfrak{g}^{\mathbb{C}}$. Then X is G-equivariantly biholomorphic to the generalized flag variety of type Π' for some subset $\Pi' \subset \Pi$, i.e.,*

$$X \cong G^{\mathbb{C}}/P, \text{ where } P \text{ is the parabolic subgroup of } G^{\mathbb{C}} \text{ determined by } \Pi'.$$

Remark 2.2. In fact, all homogeneous manifolds discussed in this paper are simply-connected. More precisely, let X be a compact Kähler manifold homogeneous under a simply-connected semisimple compact Lie group G. Then X is simply-connected. This fact follows quickly from the fiber bundle $R \hookrightarrow G \to X$. Let $X = G/R$ and let S be the connected center of R. According to the statement $(*)$, R is the union of all the maximal tori containing S, which implies that R is connected. The fiber

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bundle structure $R \hookrightarrow G \to X$ induces a long exact sequence of homotopy groups

$$\cdots \to \pi_1(G) \to \pi_1(X) \to \pi_0(R) \to \cdots,$$

and this tells us that $\pi_1(X) = 0$.

According to [8, Satz I], each compact homogeneous Kähler manifold is the product of a flat complex torus and a simply-connected compact homogeneous Kähler manifold. Furthermore, the connected component of the identity of the automorphism group of X is a semisimple Lie group [8, Satz 4]. In conclusion, Theorem 2.1 classifies all compact homogeneous Kähler manifolds without torus part.

2.2. **Classification of holomorphic line bundles.** Consider a holomorphic line bundle L over X, with $\pi : L \to X$. The holomorphic line bundle is said to be $G^{\mathbb{C}}$-*homogeneous* (or in many articles $G^{\mathbb{C}}$-*linearizable*) if there exists a $G^{\mathbb{C}}$-action on L such that the projection π is $G^{\mathbb{C}}$-equivariant and the action is linear on fibers. In particular, we can construct $G^{\mathbb{C}}$-homogeneous line bundle as follows.

Given a 1-dimensional holomorphic representation $\chi : P \to \mathbb{C}^*$, a $G^{\mathbb{C}}$-equivariant holomorphic line bundle L_χ over $X = G^{\mathbb{C}}/P$ can be constructed as follows:

$$L_\chi = G^{\mathbb{C}} \times_\chi \mathbb{C} = (G^{\mathbb{C}} \times \mathbb{C})/\sim, \tag{2.9}$$

where $(gh, v) \sim (g, \chi(h)v)$ for all $g \in G^{\mathbb{C}}$, $h \in P$, $v \in \mathbb{C}$. This admits a natural $G^{\mathbb{C}}$-action given by

$$g \cdot [(l, v)] = [(gl, v)] \tag{2.10}$$

for all $g \in G^{\mathbb{C}}$, $l \in G^{\mathbb{C}}$, $v \in \mathbb{C}$. Conversely, given a $G^{\mathbb{C}}$-equivariant holomorphic line bundle L over X, the stabilizer group P at the distinguished point $p \in X$ acts on the fiber L_p, inducing a holomorphic character $\chi : P \to \mathbb{C}^*$ such that $L \cong L_\chi$. In conclusion, there is a one-to-one correspondence between the $G^{\mathbb{C}}$-homogeneous line bundles over X and the characters of P.

Let L be an arbitrary holomorphic line bundle over X. We claim that L is a $G^{\mathbb{C}}$-homogeneous line bundle. To prove this claim, we need to borrow some results from algebraic geometry. Referring to [21, Section 21.3], X is a projective variety. According to a well-known result from GAGA [33], holomorphic line bundles over a projective variety are algebraic; in other words, the line bundle L is algebraic over X. Moreover, since $G^{\mathbb{C}}$ is a simply-connected semisimple Lie group, by [31, Proposition 1], the Picard group of $G^{\mathbb{C}}$ is trivial. According to the key fact that $\operatorname{Pic} G^{\mathbb{C}} = 0$, we can construct a character $\chi : P \to \mathbb{C}^*$ such that $L \cong L_\chi$ (see [31, Theorem 4 or Section 5] for details). Thus, we have the following proposition:

Proposition 2.3. *All holomorphic line bundles over X are $G^{\mathbb{C}}$-homogeneous.*

Fixing a simple root system Π of G, let $X \cong G^{\mathbb{C}}/P$, where the parabolic subgroup P is determined by a subset of simple roots $\Pi' \subset \Pi$ as in (2.4). Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be ordered in such a way that $\alpha_i \in \Pi'$ if and only if $i = k+1, \ldots, n$. Let $\{\omega_1, \ldots, \omega_n\}$ be the set of fundamental weights corresponding to Π, which are defined by

$$\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \ (1 \le i, j \le n).$$

Here the inner product is the bilinear form induced by the Killing form of $\mathfrak{g}$. The lattice generated by $\{\omega_1, \ldots, \omega_n\}$, i.e., all vectors of the form $\sum_{n_i \in \mathbb{Z}} n_i \omega_i$, is called the lattice of algebraically integral weights. Since G is a simply-connected semisimple Lie group, the analytically integral weights coincide with the algebraically integral weights (see [24, Chapter IV.7]). Thus, each algebraically integral weight induces a character of the maximal torus T. Recall that $\mathfrak{s}, \mathfrak{s}^*$ denote the Lie algebra of S and its dual space, respectively. More precisely,

$$\mathfrak{s} = \{H \in \mathfrak{t} : \alpha(H) = 0, \ \forall \alpha \in \Delta(\mathfrak{t}, \mathfrak{r})\}, \quad \mathfrak{s}^* = \{\beta \in \mathfrak{t}^* : (\alpha, \beta) = 0, \ \forall \alpha \in \Delta(\mathfrak{t}, \mathfrak{r})\}.$$

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Then $(\mathfrak{s}^{\mathbb{C}})^* = \mathbb{C}\langle\omega_1, \dots, \omega_k\rangle$ and the intersection of the weight lattice with $(\mathfrak{s}^{\mathbb{C}})^*$ is $\mathbb{Z}\langle\omega_1, \dots, \omega_k\rangle$. We call the elements of this sublattice *integral weights on* X. Given such an element λ, there exists an associated character $\chi^\lambda : S^{\mathbb{C}} \to \mathbb{C}^*$ defined by

$$\chi^\lambda(\exp v) = \exp \lambda(v) \tag{2.11}$$

for all $v \in \mathfrak{s}^{\mathbb{C}} \subset \mathfrak{t}^{\mathbb{C}}$. The character χ^λ can be extended to P in the following way. The Lie subalgebra $\mathfrak{p}$ can be decomposed as follows:

$$\begin{aligned}\mathfrak{p} &= \mathfrak{s}^{\mathbb{C}} \oplus \underbrace{\sum_{\beta\in D^+} \mathfrak{g}_\beta}_{\mathfrak{p}_1} \oplus \underbrace{\sum_{\alpha\in\Pi'} \mathbb{C}\langle H_\alpha\rangle \oplus \sum_{\alpha\in\Pi'} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})}_{\mathfrak{p}_2} \\ &= \mathfrak{s}^{\mathbb{C}} \oplus \mathfrak{n} \ \oplus \mathfrak{p}_2,\end{aligned}$$

where $\mathfrak{p}_1$ is the solvable part of $\mathfrak{p}$, $\mathfrak{n}$ is the nilpotent part of $\mathfrak{p}_1$, and $\mathfrak{p}_2$ is the semisimple part of $\mathfrak{p}$. Each integral weight λ on X can be extended to a complex Lie algebra homomorphism $\sigma : \mathfrak{p} \to \mathbb{C}$ by defining the extension to be 0 on both $\mathfrak{n}$ and $\mathfrak{p}_2$. Then the corresponding holomorphic character χ^σ extends χ^λ from $S^{\mathbb{C}}$ to P. By abuse of notation, we will denote this extension by χ^λ. The following proposition shows that all characters of P arise by extension in this way.

Proposition 2.4. *For all holomorphic characters* $\chi : P \to \mathbb{C}^*$ *there exists an integral weight* λ *on* X *such that* $\chi = \chi^\lambda$.

Proof. The character $\chi : P \to \mathbb{C}^*$ induces a Lie algebra homomorphism $\sigma : \mathfrak{p} \to \mathbb{C}$. It suffices to prove that σ is trivial if restricted to $\mathfrak{n}$ and $\mathfrak{p}_2$. Notice that $\mathfrak{p}_1$ is a solvable Lie algebra and $\mathfrak{n} = [\mathfrak{p}_1, \mathfrak{p}_1]$. The restriction of σ to $\mathfrak{n}$ must be trivial as $\mathbb{C}$ is an abelian Lie algebra. Since $\mathfrak{p}_2$ is a semisimple Lie algebra, for each root $\alpha \in D^+$ there exist $X_\alpha \in \mathfrak{g}_\alpha$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ and $H_\alpha = [X_\alpha, X_{-\alpha}]$ such that

$$\mathfrak{l}_\alpha := \mathbb{C}\langle H_\alpha, X_\alpha, X_{-\alpha}\rangle \cong \mathfrak{sl}(2, \mathbb{C}).$$

By the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, the only possible 1-dimensional representation of $\mathfrak{l}_\alpha$ is trivial. Since the restriction of σ to each $\mathfrak{l}_\alpha$ is trivial, we conclude that $\sigma|_{\mathfrak{p}_2} = 0$. □

Summarizing, we have now proved that every integral weight $\lambda \in \mathbb{Z}\langle\omega_1, \dots, \omega_k\rangle$ induces a character, $\chi^\lambda : P \to \mathbb{C}^*$, hence a homogeneous line bundle L_λ. Conversely, every holomorphic line bundle L is of the form $L \cong L_\lambda$ for some $\lambda \in \mathbb{Z}\langle\omega_1, \dots, \omega_k\rangle$. Recall that an integral weight λ on X is called *dominant* if $\lambda = \sum_{i=1}^k n_i\omega_i$ with $n_i > 0$. By the Highest Weight Theorem, for each dominant integral weight λ there exists a unique finite-dimensional irreducible complex representation $V(\lambda)$ of G with highest weight λ. The Bott-Borel-Weil Theorem [9, Section 7] states that the space of global sections of L_λ is isomorphic to $V(\lambda)$ as a G-module. Also by the Bott-Borel-Weil Theorem, the global sections of L_λ induce an embedding $X \hookrightarrow \mathbb{P}(V(\lambda)^*)$, so L_λ is very ample. In conclusion, we have the following.

Theorem 2.5. *The Picard group of the compact homogeneous Kähler manifold* X *can be identified with the sublattice* $\mathbb{Z}\langle\omega_1, \dots, \omega_k\rangle$ *of the lattice of integral weights. Under this identification, ample line bundles correspond to dominant integral weights, and are automatically very ample.*

Proof. Given the previous discussion, we only need to prove that if L_λ is ample, then the weight λ is dominant. We will calculate the curvature form of L_λ in (3.9). The curvature form is positive if and only if $(\lambda, \alpha) > 0$ for all $\alpha \in D^+$, which implies that λ is dominant. □

2.3. **Invariant Closed (1,1)-forms on** X**.** In this section, we classify the G-invariant closed $(1,1)$-forms on X, which is needed for the proof of Theorem B.

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Throughout this section, let X be a compact homogeneous Kähler manifold with respect to (G,R). Fixing a distinguished point $p\in X$, let $\{X_\alpha, Y_\alpha : \alpha\in D^+\}$ be a normal adapted basis of T_pX as defined in (2.1), (2.2). Then, we write $\{\eta_\alpha,\xi_\alpha : \alpha\in D^+\}$ for the dual basis of T_p^*X.

The results that we will discuss here mainly come from [6, Chapter 8]. A slight difference is that we rewrite the results in more explicit way. Recall that tangent vectors at p can be identified with infinitesimal transformation at p by elements in $\mathfrak{g}$. If $A\in\mathfrak{g}$, we define V_A as a vector field on X by

$$V_A(x) = \frac{d}{dt}\Big|_{t=0}\exp(tA)(x), \qquad \text{for all } x\in X..$$

We call V_A the *fundamental vector field* related to A. Then, by quick calculation, the Lie bracket of two fundamental vector fields is also fundamental as

$$[V_A, V_B] = -V_{[A,B]}. \tag{2.12}$$

The reason we introduce fundamental vector fields is to test invariant 2-forms on X. Recall that the negative Killing form defines a G-invariant metric $(\cdot,\cdot)$ on $\mathfrak{g}$. Fixing an element $S\in\mathfrak{s}$, we can define a 2-form, $\omega_S(V_A,V_B) = (S,[A,B])$. According to the standard calculation (see [6, Prop 8.66]), ω_S is a G-invariant, closed, real $(1,1)$-form. It is observed that the only non-vanishing terms are

$$\omega_S(X_\alpha, Y_\alpha) = (S,[X_\alpha,Y_\alpha]) = (S,-H_\alpha) = \frac{i}{2}\alpha(S). \tag{2.13}$$

The observation (2.13) indicates that we can write down ω_S explicitly in terms of the covector basis $\{\eta_\alpha,\xi_\alpha : \alpha\in D^+\}$ at $p\in X$. Precisely,

$$\omega_S = \sum_{\alpha\in D^+} C_S(\alpha)\eta_\alpha\wedge\xi_\alpha, \qquad C_S(\alpha) = \frac{i}{2}\alpha(S) \tag{2.14}$$

where C_S can be viewed as a linear function of $\mathfrak{s}^*$ (generated by elements in D^+).

Given a G-invariant closed real $(1,1)$ form, ω, in the following, we prove that ω can be written as in (2.14) related to some S. Since ω is G-invariant, for each $A,B,N\in\mathfrak{g}$,

$$0 = (L_{V_N}\omega)(V_A,V_B) = V_N(\omega(V_A,V_B)) - \omega([V_N,V_A],V_B) - \omega(V_A,[V_N,V_B])$$

In particular, by taking N to be an element in $\mathfrak{t}$ and A,B to be elements of $\{X_\alpha,Y_\alpha,\alpha\in D^+\}$, noting that $V_N(\omega(V_A,V_B)) = 0$, we have

- For two different roots in D^+, α,β, $\omega(X_\alpha,X_\beta) = \omega(X_\alpha,Y_\beta) = \omega(Y_\alpha,Y_\beta) = 0$.
- The only nonvanishing case are $\omega(X_\alpha,Y_\alpha) = C_\alpha$, $\alpha\in D^+$.

It suffices to show that C_α, a function defined in D^+, can extend linearly to $\mathfrak{s}^*$. Since 2-form ω and inner product in $\mathfrak{g}$ can be extended linearly to complex field, consider the following complex vectors

$$U = X_\alpha - iY_\alpha, \quad V = X_\beta - iY_\beta, \quad W = X_{\alpha+\beta} + iY_{\alpha+\beta}, \tag{2.15}$$

where $\alpha,\beta,\alpha+\beta\in D^+$ and $X_{\alpha,\beta,\alpha+\beta}$, $Y_{\alpha,\beta,\alpha+\beta}$ are taking from the normal adapted basis as in (2.1), (2.2). Viewing U, V, W as elements in $\mathfrak{g}\otimes\mathbb{C}$, the G-invariant inner product satisfies,

$$(W,[U,V]) = ([W,U],V).$$

Therefore, it is easy to check U,V,W satisfies the following ,

$$[U,V] = \lambda\overline{W}, \quad [V,W] = \lambda\overline{U}, \quad [W,U] = \lambda\overline{V}$$

By closedness and G-invariance of ω, we have

$$\begin{aligned} d\omega(U,V,W) &= \omega([U,V],W) + \omega([V,W],U) + \omega([W,U],V) \\ &= -\lambda\big(\omega(\overline{W},W) + \omega(\overline{U},U) + \omega(\overline{V},V)\big) = 0. \end{aligned} \tag{2.16}$$

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Inserting (2.15) into (2.16), we have

$$\omega(X_{\alpha+\beta}, Y_{\alpha+\beta}) = \omega(X_\alpha, Y_\alpha) + \omega(X_\beta, Y_\beta),$$

hence, $C_{\alpha+\beta} = C_\alpha + C_\beta$ $(*)$. Noticing that $C_\bullet$ defines a function on D^+, the condition $(*)$ implies that $C_\bullet$ can be extended to a linear function on $\mathfrak{s}^*$ as D^+ generates $\mathfrak{s}^*$. In other words, there exists an element $S \in \mathfrak{s}$ such that

$$C_\alpha = \frac{i}{2}\alpha(S)$$

In conclusion, we have the following proposition,

Proposition 2.6. *For any $S \in \mathfrak{s}$, the 2-form ω_S given by (2.14) is a G-invariant closed real $(1,1)$-form. Conversely, any G-invariant closed real $(1,1)$-form, ω, can be related to a unique $S \in \mathfrak{s}$, i.e. $\omega = \omega_S$. The form ω_S is positive if and only if $C_S(\alpha) = \frac{i}{2}\alpha(S)$ is positive for all $\alpha \in D^+$.*

Proposition 2.6 implies a bijection from $\mathfrak{s}$ to the space of G-invariant closed real $(1,1)$ forms on X, $S \to \omega_S$. There is a special element of the G-invariant closed real $(1,1)$-forms, Kähler-Einstein form, and the related element in $\mathfrak{s}$ is given by $S_{KE} = 2\sum_{\alpha\in D^+} H_\alpha$. So we can write ω_{KE} explicitly at the distinguished point $p \in X$ with respect to the normal adapted basis. Indeed, set δ be the sum of all positive roots in D^+, then

$$\omega_{KE} = \frac{1}{2}\sum_{\alpha\in D^+} (\alpha, \delta)\eta_\alpha \wedge \xi_\alpha. \tag{2.17}$$

3. Invariant dd^C-lemmas on homogeneous line bundles

This section is dedicated to the proof of Theorem A. Section 3.1 discusses invariant 1-forms on the unit circle bundle, M, of a homogeneous line bundle L over a compact homogeneous Kähler manifold X. Section 3.2 calculates the differentials of the invariant 1-forms given in Section 3.1. Assuming that L is negative, Section 3.3 proves an invariant dd^C-lemma, and we also show that this dd^C-lemma can be false if L is not negative. Finally, Section 3.4 combines these results to prove Theorem A.

3.1. **Invariant 1-forms on M.** Let X be a homogeneous compact Kähler manifold as before, and $L = L_\lambda$, a homogeneous line bundle over X corresponding an integral weight λ. Given the data (X, L), in this subsection, we will determine all left-invariant 1-forms on the unit circle bundle M of X.

Recall the G-action on L defined by restricting the $G^{\mathbb{C}}$-action of (2.10) to $G \subset G^{\mathbb{C}}$. For each homogeneous line bundle L, there is a natural G-invariant hermitian metric h induced by the standard hermitian metric in $\mathbb{C}$. In particular, according to construction of homogeneous line bundle in (2.9), let $q_0 = \overline{(g,z)} \in L_\lambda$ for $(g,z) \in G^{\mathbb{C}} \times \mathbb{C}$, then $h(q_0, q_0) = |z|^2$. The hermitian metric h induces a radian function r on L. Then, G acts transitively on each level set of r, an S^1 bundle of X, denoted as $M(r)$. Away from zero level set, there is a canonical invariant vector field, $\partial/\partial r$ on L pointing in radius direction. We shall find the set of all invariant vector fields on each level set $M(r)$. Notice that, in general, left-invariant vector fields on G are not well-defined over $M(r)$, as the left action by the stabilizer group on the tangent space at one point can be nontrivial. Let $M = M(1)$ and let $\mathcal{T}_M$ be the space of all global G-invariant vector fields over M. $\mathcal{T}_M$ contains at least one element, $X_0 = J(\partial/\partial r)$, generating a circle action on each fiber. The other elements of $\mathcal{T}_M$ strongly depend on the base manifold X and the integral weight λ. According to (2.7), the tangent space at a distinguished point $p \in X$ can be identified with a subspace of $\mathfrak{g}$. Then, we can choose a normal adapted basis $\{X_\alpha, Y_\alpha\}_{\alpha\in D^+}$ as in (2.1), (2.2).

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Based on the choice of $\{H_\alpha, X_\alpha, Y_\alpha\}$ as in (2.1), (2.2), one can easily check the following Lie algebra structure.

$$[H_\alpha, X_\alpha] = -\frac{|\alpha|^2}{2} Y_\alpha, \quad [H_\alpha, Y_\alpha] = \frac{|\alpha|^2}{2} X_\alpha, \quad [X_\alpha, Y_\alpha] = -H_\alpha.$$

Let q be a distinguished point in M with $\pi(q) = p$, then the tangent space at q can be identified with $\mathbb{R}\langle X_\alpha, Y_\alpha, \alpha \in D^+\rangle \oplus \mathbb{R}X_0$ in the following sense. Consider the G-equivariant bundle projections,

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\pi}} & M \\ & \underset{\pi}{\searrow} & \downarrow{\scriptstyle \pi|_M} \\ & & X \end{array}$$

with $\tilde{\pi}(e) = q \in M$. At the distinguished point $q \in M$ with $\pi(q) = p \in X$, assume that the stabilizer group at $p \in X$ is R and the stabilizer group at $q \in M$ is R_0. By G-action on L, $g \in R_0$ if and only if,

$$g(q) = g(e, \theta) = (e, \chi^\lambda(g)\theta),$$

which implies that $R_0 = \ker \chi^\lambda : R \to S^1 \subset \mathbb{C}^*$ These projections induce the mapping on tangent spaces $\tilde{\pi}_* : \mathfrak{g} \to T_q M$ by

$$N \in \mathfrak{g} \mapsto \frac{d}{dt}\Big|_{t=0} \tilde{\pi} \circ \exp(tN). \tag{3.1}$$

Similarly, we can define the mapping $\pi_* : \mathfrak{g} \to T_p X$. By abusing notation, we write X_α, Y_α instead of $\tilde{\pi}_*(X_\alpha)$, $\tilde{\pi}_*(Y_\alpha)$ and $\pi_*(X_\alpha)$, $\pi_*(Y_\alpha)$ as X_α, Y_α. Notice that the left-invariant vector fields on G can be identified with $\mathfrak{g}$. To determine the space of invariant vector fields, $\mathcal{T}_M$, on M_1, we observe that for any R_0-invariant vector $v \in T_q M$ and $g_1(q) = g_2(q)$, then $g_1 = g_2 r$ with $r \in R_0$ and

$$(g_1)_*(v) = (g_2 r)_*(v) = (g_2)_* r_*(v) = (g_2)_*(v),$$

So each R_0-invariant vector of $T_q M$ determines a left-invariant vector fields on M. There is an one-to-one correspondence between $\mathcal{T}_M$ and the R_0-invariant space of $T_q M$. In particular, in the case that $X = G/T$ with T a maximal torus of G, we have the following proposition

Proposition 3.1. *Let X be a compact Kähler manifold homogeneous under G with stabilizer group R, let Δ be the root system of $(\mathfrak{t}, \mathfrak{g})$. D^+ is defined as in (2.6). Let M be the unit level of homogeneous line bundle determined by an integral weight $\lambda \neq 0$. At a distinguished point $q \in M$, the tangent space $T_q M$ is generated by $\{X_\alpha, Y_\alpha\}_{\alpha \in D^+}$ and X_0. Then, there are the following two possiblities for the space of the left-invariant vector fields on M,*

(a) $\mathcal{T}_M \cong \mathbb{R}\langle X_0\rangle$
(b) $\mathcal{T}_M \cong \mathbb{R}\langle X_0, X_\alpha, Y_\alpha\rangle$*, for some* $\alpha \in D^+$

The case (b) *happens if and only if λ is proportional to α and $\alpha + \beta$, $\alpha - \beta$ are not in Δ, for any nontrivial $\beta \in \Delta(\mathfrak{t}, \mathfrak{r})$. In particular, if the stabilizer group is a maximal torus, then the case (b) happens if and only if λ is proportional to α.*

Proof of Proposition 3.1. Let R_0 be the stabilizer group at $q \in M$. Notice that R_0 is the kernel of character $\chi^\lambda : R \to S^1 \subset \mathbb{C}$ related to the weight λ. Let $V \in T_q M$ be an invariant vector. According to decomposition (2.7) of $T_q M$, the vector V can be written as

$$V = \sum_{\alpha \in D^+} V_\alpha, \qquad V_\alpha \in \mathbb{R}\langle X_\alpha, Y_\alpha\rangle \text{ and } V_\alpha \neq 0.$$

Notice that R_0 preserves each $\mathbb{R}\langle X_\alpha, Y_\alpha\rangle$. If a vector $V_\alpha = aX_\alpha + bY_\alpha \in E_{\pm\alpha}$ is invariant under R_0 action, then, since J is R_0 invariant, which means that $J(aX_\alpha + bY_\alpha) = aY_\alpha - bX_\alpha$ is also R_0 invariant.

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Therefore, all vectors in the space generated by $\langle aX_\alpha + bY_\alpha, -bX_\alpha + aY_\alpha\rangle = \langle X_\alpha, Y_\alpha\rangle = \mathbb{R}\langle X_\alpha, Y_\alpha\rangle$ are R_0 invariant. So we only need to determine the set of all $\alpha \in \Delta^+$ such that X_α, Y_α are invariant under $R_0 = \ker \chi^\lambda$.

Let $\mathfrak{r}_0 = \ker \lambda$, it's easy to see that $\mathfrak{r}_0$ is the Lie algebra of the stabilizer group R_0. Let $T_0 = R_0 \cap T$ associated with Lie algebra $\mathfrak{t}_0 = \mathfrak{r}_0 \cap \mathfrak{t}$. Since X_α, Y_α is T_0 invariant,

$$[H_0, X_\alpha] = 0 = [H_0, Y_\alpha], \qquad \text{for all } H_0 \in \mathfrak{t}_0 \tag{3.2}$$

Noting that

$$[H_0, X_\alpha] = -i\alpha(H_0)Y_\alpha, \qquad [H_0, Y_\alpha] = i\alpha(H_0)X_\alpha.$$

Hence, we have $\ker \alpha = \mathfrak{t}^0 = \ker \lambda|_{\mathfrak{t}}$, hence λ is proportional to α.

Let $Z \in \mathfrak{r}_0$, then X_α, Y_α are R_0 invariant if and only if

$$\tilde{\pi}_*[Z, X_\alpha] = \tilde{\pi}_*[Z, Y_\alpha] = 0,$$

Notice that $\mathfrak{r}_0 = \mathfrak{t}_0 \oplus \sum_{\beta \in \Delta^+(\mathfrak{t},\mathfrak{r})} \mathbb{R}\langle X_\beta, Y_\beta\rangle$, where $\mathfrak{t}_0$ is kernel of λ restricted in $\mathfrak{t}$. Then, X_α and Y_α is R_0-invariant is equivalent to (3.2) and

$$\tilde{\pi}_*[Z_\beta, X_\alpha] = \tilde{\pi}_*[Z_\beta, Y_\alpha] = 0, \qquad \text{for all } Z_\beta \in \mathbb{R}\langle X_\beta, Y_\beta\rangle,\ \beta \in \Delta^+(\mathfrak{t},\mathfrak{r}) \qquad (**)$$

According to [24, Theorem 6.6] and the definition of $\tilde{\pi}_*$, $(**)$ is equivalent to $\alpha+\beta,\ \alpha-\beta \notin D^+ \cup (-D^+)$. Indead, $\alpha + \beta$ and $\alpha - \beta$ are not roots. Assume that

$$\alpha + \beta = \gamma \in \Delta(\mathfrak{t}, \mathfrak{r}),$$

then, $\alpha = \gamma - \beta \in \Delta(\mathfrak{t}, \mathfrak{r})$. But $\alpha \in D^+$, which leads to a contradiction. □

Example 3.2. Let $X = SU(3)/T^2$. Recall the basic notions of semisimple Lie group $SU(3)$. The Lie algebra $\mathfrak{su}(3)$ is the set of trace zero skew-hermitian matrices of order 3. A Cartan sub-algebra $\mathfrak{t}$ is the Lie algebra of diagonal matrices in $\mathfrak{su}(3)$. The set of positive roots with respect to $\mathfrak{t}$ consists of three elements $\{\alpha, \beta, \gamma\}$, and the normal adapted basis is given by $X_{\alpha,\beta,\gamma},\ Y_{\alpha,\beta,\gamma}$.

Consider a distinguished point $q \in M$ with tangent space generated by $\{X_0,\ X_{\alpha,\beta,\gamma},\ Y_{\alpha,\beta,\gamma}\}$. Now, we only focus on the subspace $V \subset T_qM$ generated by $\{X_{\alpha,\beta,\gamma}, Y_{\alpha,\beta,\gamma}\}$. To simplify the notation in calculation, we introduce a complex coordinate system in V; precisely, $z_\alpha = X_\alpha + iY_\alpha$. Let σ_i denote the i-th element of diagonal matrices. Then, α, β, γ can be expressed as, $\alpha = \sigma_1 - \sigma_2,\quad \beta = \sigma_1 - \sigma_3,\quad \gamma = \sigma_2 - \sigma_3$. Hence, let $\{\alpha, \gamma\}$ be the simple root system of $SU(3)$ and $\beta = \alpha + \gamma$. The fundamental weights are given as follows,

$$\omega_1 = \frac{2}{3}\sigma_1 - \frac{1}{3}\sigma_2 - \frac{1}{3}\sigma_3, \qquad \omega_2 = \frac{1}{3}\sigma_1 + \frac{1}{3}\sigma_2 - \frac{2}{3}\sigma_3.$$

Let $\mathcal{O}(p, q)$ denote the line bundle corresponding to the integral weight $\lambda = p\omega_1 + q\omega_2$. In the case of $(p, q) \neq 0$. Noting that that kernel of $\lambda = p\omega_1 + q\omega_2$ is $\mathbb{R}\langle qt_1 - (p + q)t_2\rangle$. Then, the S^1 action is given by $T^{p,q}_\theta = \operatorname{diag}(e^{iq\theta}, e^{-i(p+q)\theta}, e^{ip\theta})$ and if we represent the action on complex coordinate system $(z_\alpha, z_\beta, z_\gamma)$, we have

$$T^{p,q}_\theta(z_\alpha, z_\beta, z_\gamma) = (e^{-i(p+2q)\theta}z_\alpha, e^{i(p-q)\theta}z_\beta, e^{i(2p+q)\theta}z_\gamma).$$

In conclusion, we have the following cases

Conditions on (p, q)	Invariant vector fields over M
$p = -2q$	$X_0,\quad X_\alpha,\quad Y_\alpha$
$p = q$	$X_0,\quad X_\beta,\quad Y_\beta$
$2p = -q$	$X_0,\quad X_\gamma,\quad Y_\gamma$
Others	X_0

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Notice that

$$\alpha = 2\omega_1 - \omega_2, \quad \beta = \omega_1 + \omega_2, \quad \gamma = -\omega_1 + 2\omega_2$$

The integral weights in the above table, $-2n\omega_1 + n\omega_2$, $n\omega_1 + n\omega_2$, $-n\omega_1 + 2n\omega_2$ are proportional to α, β, γ respectively in agreement with Proposition 3.1.

The left-invariant 1-forms on M can be viewed as the dual space of left-invariant vector fields. More precisely, if we apply the previous notion of adapted basis at T_qM, given by $\{X_0, X_\alpha, Y_\alpha; \alpha \in D^+\}$, we write the dual basis of T_q^*M as $\{\eta_0, \eta_\alpha, \xi_\alpha; \alpha \in D^+\}$. According to Proposition 3.1, the space of left-invariant vector fields is in one-to-one correspondence with the space generated by $\{X_0\}$ or $\{X_0, X_\alpha, Y_\alpha\}$ for some $\alpha \in D^+$. Consider the subset, $\{\eta_0\}$ or $\{\eta_0, \eta_\alpha, \xi_\alpha\}$ of the dual basis, whose elements are R_0 invariant. Therefore, $\{\eta_0\}$ or $\{\eta_0, \eta_\alpha, \xi_\alpha\}$ generates the space of G-invariant 1 forms over M.

3.2. **The differentials of left invariant 1-forms on M.** According to proposition 3.1, the left invariant 1-forms are generated by $\{\eta_0\}$ or $\{\eta_0, \xi_\alpha, \eta_\alpha\}$. We only consider the second case in this subsection.

Let M be an S^1 bundle associated with line bundle L. Notice that there is a natural projection $\tilde{\pi}: G \to M$. If we write Ω_1, Ω_2 as the space of smooth 1-forms and 2-forms respectively, then we have the following commutative graph,

$$\begin{array}{ccc} \Omega_1(M) & \xrightarrow{\tilde{\pi}^*} & \Omega_1(G) \\ \downarrow d & & \downarrow d \\ \Omega_2(M) & \xrightarrow{\tilde{\pi}^*} & \Omega_2(G). \end{array} \tag{3.3}$$

Since the left-invariant vector fields are globally generated in G, with a natural basis corresponding to $\{H_\alpha, X_\beta, Y_\beta : \alpha \in \Pi, \beta \in \Delta\}$ and its dual basis $\{h_\alpha, \eta_\beta, \xi_\beta : \alpha \in \Pi, \beta \in \Delta\}$, then the pull back of η_α, ξ_α in $\Omega_1(M)$ under $\tilde{\pi}$ are exactly η_α, ξ_α in $\Omega_1(G)$. And the pull-back of η_0 is a certain combination of h_α determined by weight λ. To calculate the differentials of left invariant 1-forms, The technique is to apply the Maurer-Cartan equations on G with respect to the natural basis, Since Maurer-Cartan equations gives us the derivative of left-invariant 1-form on G, combining the commutative graph (3.3), we have the derivative of left-invariant 1 form on M.

Proposition 3.3. *Let X be the homogeneous space and M, the S^1 bundle of X associated with λ. Assuming that, at the distinguished point $q \in M$, the space of left-invariant 1-forms can be identified with the spaces generated by $\{\eta_0, \eta_\alpha, \xi_\alpha\}$, $\alpha \in D^+$. In this case, λ is proportional to α, assuming that $\alpha = -l\lambda$. then the derivative are given by*

$$d\eta_0 = -\frac{1}{2}\sum_{\alpha \in D^+} (\lambda, \alpha)\eta_\alpha \wedge \xi_\alpha, \tag{3.4}$$

$$\begin{aligned} d\eta_\alpha = -l\eta_0 \wedge \xi_\alpha - \frac{C^\alpha_{\beta,-\gamma}}{2} &\sum_{\substack{\beta,\gamma \in D^+ \\ \beta-\gamma=\alpha}} (\eta_\beta \wedge \eta_\gamma + \xi_\beta \wedge \xi_\gamma) \\ &- \frac{C^\alpha_{\beta,\gamma}}{2} \sum_{\substack{\beta,\gamma \in D^+ \\ \beta+\gamma=\alpha}} (\eta_\beta \wedge \eta_\gamma - \xi_\beta \wedge \xi_\gamma) \end{aligned} \tag{3.5}$$

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$$d\xi_\alpha = l\eta_0 \wedge \eta_\alpha + \frac{C^\alpha_{\beta,-\gamma}}{2} \sum_{\substack{\beta,\gamma\in D^+ \\ \beta-\gamma=\alpha}} (\eta_\beta \wedge \xi_\gamma - \xi_\beta \wedge \eta_\gamma)$$
$$- \frac{C^\alpha_{\beta,\gamma}}{2} \sum_{\substack{\beta,\gamma\in D^+ \\ \beta+\gamma=\alpha}} (\eta_\beta \wedge \xi_\gamma + \xi_\beta \wedge \eta_\gamma). \tag{3.6}$$

where the coefficients $C^\alpha_{\beta,-\gamma}$ are the coefficients from Maureur-Cartan equation. Let E_α, E_β, $E_{-\gamma}$ be the root vector of α, β, $-\gamma$ satisfying (2.1), then

$$[E_\beta, E_{-\gamma}] = C^\alpha_{\beta,-\gamma} E_\alpha$$

Proof. Let $\{\omega_1, \cdots, \omega_k\}$ be fundamental integral weights on X and $\lambda = \sum_i n_i\omega_i$. By definition of fundamental weights and our setting of H_{α_i}, we can evaluate H_{α_i} under λ.

$$\lambda(H_{\alpha_i}) = \sum_j n_j\omega_j(H_{\alpha_i}) = \frac{1}{2i}\sum_j n_j(\omega_j,\alpha_i) = -i\frac{|\alpha_i|^2}{4}\sum_j \frac{2n_j(\omega_j,\alpha_i)}{(\alpha_i,\alpha_i)} = -i\frac{|\alpha_i|^2}{4}n_i$$

By the definition of L_λ, at distinguished point $q \in M$ with $\tilde{\pi}(e) = q$, we have

$$\tilde{\pi}_*(H_{\alpha_i}) = \frac{d}{dt}\Big|_{t=0}\chi^\lambda(\exp(tH_{\alpha_i})) = \frac{d}{dt}\Big|_{t=0}\exp(t\lambda(H_{\alpha_i})) = -\frac{|\alpha_i|^2}{4}n_iX_0. \tag{3.7}$$

For each i,

$$\tilde{\pi}^*(\eta_0)(H_{\alpha_i}) = \eta_0(\tilde{\pi}_*(H_{\alpha_i})) = -\frac{|\alpha_i|^2}{4}n_i\eta_0(X_0) = -\frac{|\alpha_i|^2}{4}n_i.$$

Since the previous calculation shows that $\omega_i(H_{\alpha_j}) = -i|\alpha_j|^2\delta_{ij}/4$, then the pullback of η_0 can be represented by $\tilde{\pi}^*\eta_0 = -i\lambda$. To get the formula (3.4), notice that

$$\tilde{\pi}^*d\eta_0(X_\alpha, Y_\alpha) = d\tilde{\pi}^*(\eta_0)(X_\alpha, Y_\alpha) = \tilde{\pi}^*\eta_0([X_\alpha, Y_\alpha]) = -i\lambda(H_\alpha) = -\frac{1}{2}(\lambda, \alpha).$$

To get the formulas (3.5), (3.6), if we pull back both sides of formulas by $\tilde{\pi}$,

$$\tilde{\pi}^*(d\eta_\alpha)(H_{\alpha_i}, Y_\alpha) = -\tilde{\pi}^*\eta_\alpha([H_{\alpha_i}, Y_\alpha]) = -\frac{(\alpha,\alpha_i)}{2},$$
$$\frac{2(\alpha,\alpha_i)}{n_i|\alpha_i|^2}\tilde{\pi}^*(\eta_0\wedge\xi_\alpha)(H_{\alpha_i}, Y_\alpha) = -i\frac{2(\alpha,\alpha_i)}{n_i|\alpha_i|^2}\sum_i \lambda\wedge\xi_\alpha(H_{\alpha_i}, Y_\alpha) = -\frac{(\alpha_i,\alpha)}{2}.$$

Since $\tilde{\pi}^*$ is injective, $d\eta_\alpha$ admits the term $\frac{2(\alpha,\alpha_i)}{n_i|\alpha_i|^2}\eta_0\wedge\xi_\alpha$. We can show that the coefficient is independent of the index i and related to the factor l. Notice that λ is proportional to α, $\alpha = -l\lambda$, then

$$\frac{2(\alpha,\alpha_i)}{n_i|\alpha_i|^2} = \frac{2(-l\lambda,\alpha_i)}{n_i|\alpha_i|^2} = \frac{2(-l\sum_j n_j\omega_j,\alpha_i)}{n_i|\alpha_i|^2} = -l.$$

To compute the remaining cross terms of $d\eta_\alpha$ and $d\xi_\alpha$, we shall understand the structure of Lie algebra. Notice that $d\eta_\alpha$ has a nonvanishing 2-form related with root β, γ only if (1) $\alpha = \beta+\gamma$ or (2) $\alpha = \beta-\gamma$. Both in the case (1) and (2), we argue that $\beta, \gamma \in D^+$. For instance, in the case (1), assume that $\gamma \in \Delta(\mathfrak{t},\mathfrak{r})^+$, then we have $\beta = \alpha-\gamma$ is a root, which contradicts the condition in proposition 3.1.

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In the case (1), notice that $[E_\beta, E\gamma] = C^\alpha_{\beta,\gamma} E_\alpha$ and $[E_\beta, \overline{E_\gamma}]$ has no E_α terms. Combining with the relation (2.1), we have,

$$\eta_\alpha([X_\beta, X_\gamma]) = -\eta_\alpha([Y_\beta, Y_\gamma]) = \frac{C^\alpha_{\beta,\gamma}}{2},$$
$$\xi_\alpha([X_\beta, Y_\gamma]) = \xi_\alpha([Y_\beta, X_\gamma]) = \frac{C^\alpha_{\beta,\gamma}}{2}.$$

The above equation implies that

$$\tilde{\pi}^*(d\eta_\alpha)(X_\beta, X_\gamma) = -\pi^*\eta_\alpha([X_\beta, X_\gamma]) = -\frac{C^\alpha_{\beta,\gamma}}{2}$$
$$\tilde{\pi}^*(d\eta_\alpha)(Y_\beta, Y_\gamma) = -\pi^*\eta_\alpha([Y_\beta, Y_\gamma]) = \frac{C^\alpha_{\beta,\gamma}}{2}$$

Hence, $d\eta_\alpha$ has the term $(C^\alpha_{\beta,\gamma}/2)(-\eta_\beta \wedge \eta_\gamma + \xi_\beta \wedge \xi_\gamma)$. Likewise, we can find the formulas for $d\eta_\alpha$ and $d\xi_\alpha$ as (3.5) and (3.6) and we completes the proof. □

Fixing a line bundle related to a weight λ, the Chern class of L_λ can be represented by the form $-i\partial\overline{\partial}\log r^2$, where r is the radius function induced by some Hermitian metric on L_λ. Then we have

$$\frac{1}{2} dd^c \log r^2 = -d\big(J\frac{dr}{r}\big). \tag{3.8}$$

Indeed, the 1-form $-Jdr/r$ is induced by circle action along each fiber and more details will be discussed in the next section. Referring to Proposition 3.3, the curvature form can be represented as

$$-\frac{1}{2} dd^c \log h = \frac{1}{2} \sum_{\alpha\in\Delta^+} (\lambda, \alpha)\eta_\alpha \wedge \xi_\alpha. \tag{3.9}$$

Comparing with (2.17), the Kähler-Einstein metric is the curvature form associated with the anti-canonical bundle, L_δ.

3.3. **Invariant dd^c-lemma on L_λ.** Consider the data (X, L_λ) as in the previous section. According to proposition 3.1, the space of left invariant vector fields on M have two different cases (a) and (b). Suppose that λ satisfies one of the following conditions,

- The space of left invariant vector fields on M satisfies case (a) in Proposition 3.1;
- The space of left invariant vector fields on M satisfies case (b) in Proposition 3.1. And λ is proportional to some positive root α with $\lambda = -l\alpha$, $l > 0$.

Then, we have the following invariant dd^c lemma.

Proposition 3.4. *Let (X, L_λ) satisfy the conditions above. If ω is an G-invariant closed real $(1,1)$-form on L, $[\omega] = 0$, then there exists a G-invariant Kähler potential $\Phi \in \mathcal{C}^\infty(L_\lambda)$ such that.*

$$\omega = dd^c\Phi. \tag{3.10}$$

Proof. Since ω is exact, there exists an 1-form θ such that $d\theta = \omega$. Moreover θ can choose to be G invariant. Notice that $\omega = d(g^*\theta)$, then, by taking integral over G, we obtain a G-invariant 1-form θ with $\omega = d\theta$.

The main idea of proof is to represent $d\theta$ and $\partial\overline{\partial}\phi$ with respect to G-invariant coframe, then we can reduce the proof of Proposition 3.4 to solving a system of ODE.

Let L_λ be the line bundle such that the space of left invariant vector fields on M satisfies case (b). Suppose that λ is proportional to α. The basis of left-invariant 1-form on level set M is $\{\eta_0, \eta_\alpha, \xi_\alpha\}$. Then, we can extend the these invariant 1-forms in radian direction by rescaling $1/r$ on each level $M(r)$. Precisely, There is a natural projection $p : L^\times \to M$. By identifying $L^\times \cong G \times_\lambda \mathbb{C}^\times$, the projection can

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be written explicitly,

$$p((g,z)) = (g, |z|^{-1}z).$$

Thus, p^* extends $\{\eta_0, \eta_\alpha, \xi_\alpha\}$ to $L^\times$. By abusing the notion, we write $\{\eta_0, \eta_\alpha, \xi_\alpha\}$ as the extended vector fields over $L^\times$. Also let $\mu = dr/r$, hence $\{\mu, \eta_0, \eta_\alpha, \xi_\alpha\}$ forms a basis of invariant 1-forms over $L^\times$. Then the invariant 1-form θ can be represented as

$$\theta = \varphi_r(r)\mu + \varphi_0(r)\eta_0 + \varphi_\alpha(r)\eta_\alpha + \phi_\alpha(r)\xi_\alpha. \tag{3.11}$$

Applying Proposition 3.3, we can take derivative of θ in (3.11),

$$d\theta = r\varphi_0'(r)\mu \wedge \eta_0 - \frac{\varphi_0(r)}{2} \sum_{\alpha \in D^+} (\lambda, \alpha)\eta_\alpha \wedge \xi_\alpha \tag{3.12}$$

$$+ r\varphi_\alpha'(r)\mu \wedge \eta_\alpha + r\phi_\alpha'(r)\mu \wedge \xi_\alpha - l\varphi_\alpha\eta_0 \wedge \xi_\alpha + l\phi_\alpha\eta_0 \wedge \eta_\alpha \tag{3.13}$$

$$- \varphi_\alpha \frac{C^\alpha_{\beta,-\gamma}}{2} \sum (\eta_\beta \wedge \eta_\gamma + \xi_\beta \wedge \xi_\gamma) + \phi_\alpha \frac{C^\alpha_{\beta,-\gamma}}{2} \sum (\eta_\beta \wedge \xi_\gamma - \xi_\beta \wedge \eta_\gamma) \tag{3.14}$$

$$- \varphi_\alpha \frac{C^\alpha_{\beta,\gamma}}{2} \sum (\eta_\beta \wedge \eta_\gamma - \xi_\beta \wedge \xi_\gamma) - \phi_\alpha \frac{C^\alpha_{\beta,\gamma}}{2} \sum (\eta_\beta \wedge \xi_\gamma + \xi_\beta \wedge \eta_\gamma) \tag{3.15}$$

Let J be complex structure. Note that $d\theta$ is real $(1,1)$-form if and only if $Jd\theta = d\theta$. Notice that $J\eta_0 = \mu$, $J\xi_\alpha = \eta_\alpha$, then

$$\begin{aligned}
Jd\theta = {} & r\varphi_0'(r)\mu \wedge \eta_0 - \frac{\varphi_0(r)}{2} \sum_{\alpha \in D^+} (\lambda, \alpha)\eta_\alpha \wedge \xi_\alpha \\
& + r\varphi_\alpha'\eta_0 \wedge \xi_\alpha - r\phi_\alpha'\eta_0 \wedge \eta_\alpha - l\varphi_\alpha\mu \wedge \eta_\alpha - l\phi_\alpha\mu \wedge \xi_\alpha \\
& - \varphi_\alpha \frac{C^\alpha_{\beta,-\gamma}}{2} \sum (\eta_\beta \wedge \eta_\gamma + \xi_\beta \wedge \xi_\gamma) + \phi_\alpha \frac{C^\alpha_{\beta,-\gamma}}{2} \sum (\eta_\beta \wedge \xi_\gamma - \xi_\beta \wedge \eta_\gamma) \\
& + \varphi_\alpha \frac{C^\alpha_{\beta,\gamma}}{2} \sum (\eta_\beta \wedge \eta_\gamma - \xi_\beta \wedge \xi_\gamma) + \phi_\alpha \frac{C^\alpha_{\beta,\gamma}}{2} \sum (\eta_\beta \wedge \xi_\gamma + \xi_\beta \wedge \eta_\gamma)
\end{aligned}$$

Then, $Jd\theta = d\theta$ implies the following ODE

$$r\varphi_\alpha' = -l\varphi_\alpha, \qquad r\phi_\alpha' = -l\phi_\alpha \tag{3.16}$$

and the terms of line (3.15) are vanishing. To ensure the (3.15) vanishes,

$$C^\alpha_{\beta,\gamma} = 0 \qquad \text{or} \qquad \varphi_\alpha = \phi_\alpha = 0. \tag{3.17}$$

Indeed, referring to ([24], Theorem 6.6), if $\alpha = \beta + \gamma$, for some $\alpha, \beta, \gamma \in \Delta$, then the corresponding constant $C^\alpha_{\beta,\gamma} \neq 0$. Assuming that there exist positive roots, β and γ, satisfying $\alpha = \beta + \gamma$, by (3.17), we have $\varphi_\alpha = \phi_\varphi = 0$, which automatically satisfies (3.16); hence, if $d\theta$ is a real $(1,1)$ form with some structure constants $C^\alpha_{\beta,\gamma}$ nonvanishing, then $d\theta$ can be written as

$$d\theta = r\varphi_0'(r)\mu \wedge \eta_0 - \frac{\varphi_0(r)}{2} \sum_{\alpha \in D^+} (\lambda, \alpha)\eta_\alpha \wedge \xi_\alpha$$

When it comes to the cases that the corresponding weight λ is proportional to a simple root α, i.e. α cannot be written as the sum of two positive roots, $d\theta$ should satisfy the equation (3.16) and its solution is given by $\varphi = C/r^l$ with C an arbitrary constant. In the sequel, it suffices to show that the constant C in the expression of solution φ should equal zero. To prove this, we need to apply the condition that the form, $d\theta$, is well-defined across the zero level of line bundle. Firstly, we take a reference metric ω_0 near the zero level of line bundle L. Let h be the canonical invariant hermitian metric defined as before and r be the radial function related to h. Also, given a bundle coordinate u,

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we have

$$\begin{aligned}\omega_\epsilon =&\pi^*\omega_{KE} + \epsilon \cdot dd^c r^2\\ =&\pi^*\omega_{KE} + \epsilon r^2 \cdot dd^c \log h + \epsilon h \cdot i du \wedge d\overline{u}.\end{aligned}$$

It is easy to see that ω_ϵ is positive around zero level. To simplify calculation, let ϵ tends to 0, then we obtain a semi-positive form $\omega_0 = \pi^*\omega_{KE}$. Consider the following integration,

$$\int_K \omega_0^{n-1} \wedge d\theta \wedge d\theta, \tag{3.18}$$

where the K is a compact neighborhood of zero level, which is defined as $K = \{x \in L,\ r(x) \leq \delta\}$. In the sequel, we write (3.12), (3.13) and (3.14) as Θ_1, Θ_2, Θ_3. Noticing that all crossing terms $\Theta_i \wedge \Theta_j \wedge \omega_0^{n-1} = 0$, $(i \neq j)$, and $\Theta_3 \wedge \Theta_3 \wedge \omega_0^{n-1} = 0$. Only two nonvanishing terms of the integration (3.18) are the following,

$$\int_K \omega_0^{n-1} \wedge d\theta \wedge d\theta = \int_K \omega_0^{n-1} \wedge \Theta_1 \wedge \Theta_1 + \int_K \omega_0^{n-1} \wedge \Theta_2 \wedge \Theta_2 \tag{3.19}$$

Inserting the solution of ODE (3.16),

$$\Theta_2 = -C_1 l r^{-l} \mu \wedge \eta_\alpha - C_2 l r^{-l} \mu \wedge \xi_\alpha - C_1 l r^{-l} \eta_0 \wedge \xi_\alpha + C_2 l r^{-l} \eta_0 \wedge \eta_\alpha,$$

then we can compute the two terms in (3.19) separately,

$$\begin{aligned}\int_K \Theta_2 \wedge \Theta_2 \wedge \omega_0^{n-1} &= -\int_K (C_1^2 + C_2^2) l^2 r^{-2l} \mu \wedge \eta_0 \wedge \eta_\alpha \wedge \xi_\alpha \wedge \omega_0^{n-1}\\ &= -(C_1^2 + C_2^2) l^2 \int_0^\delta r^{-2l-1} dr \int_{M_1} \eta_0 \wedge \eta_\alpha \wedge \xi_\alpha \wedge \omega_0^{n-1} = -\infty\end{aligned}$$

and assume λ is proportional to a positive root by a negative constant,

$$\begin{aligned}\int_K \Theta_1 \wedge \Theta_1 \wedge \omega_0^{n-1} &= -\frac{1}{2} \sum_{\alpha \in \Delta^+} \int_K r \varphi_0'(r) \varphi_0(r) (\lambda, \alpha) \eta_\alpha \wedge \xi_\alpha \wedge \mu \wedge \eta_0 \wedge \omega_0^{n-1}\\ &= -\frac{1}{2} \sum_{\alpha \in \Delta^+} (\lambda, \alpha) \int_0^\delta \varphi_0'(r) \varphi_0(r) dr \int_{M_1} \eta_0 \wedge \eta_\alpha \wedge \xi_\alpha \wedge \omega_0^{n-1}\\ &= C \lim_{\epsilon \to 0} (\varphi_0(r))^2 \Big|_\epsilon^\delta < C', \qquad (C,\ C' \text{ are nonnegative constants})\end{aligned}$$

Hence, , $\int_K d\theta \wedge d\theta \wedge \omega_0^{n-1} = -\infty$, which leads to a contradiction. We obtain that $\varphi_\alpha = \phi_\alpha = 0$. According to the discussion of two cases, we have $\varphi_\alpha = \phi_\alpha = 0$ and $d\theta$ can be represented as,

$$d\theta = r\varphi_0'(r) \mu \wedge \eta_0 - \frac{\varphi_0(r)}{2} \sum_{\alpha \in \Delta^+} (\lambda, \alpha) \eta_\alpha \wedge \xi_\alpha \tag{3.20}$$

Take a function $\Phi(r)$ such that $\Phi'(r) = \varphi_0(r)/r$, we compute $i\partial\overline{\partial}\Phi$,

$$dd^c\Phi = -d \cdot J d\Phi = -d \cdot J(\varphi_0(r)\mu) = d(\varphi_0(r)\eta_0) = r\varphi_0'(r)\mu \wedge \eta_0 + \varphi_0(r) d\eta_0.$$

Combining with Proposition 3.3, we obtain

$$dd^c\Phi = r\varphi_0'(r)\mu \wedge \eta_0 - \frac{\varphi_0(r)}{2} \sum_{\alpha \in D^+} (\lambda, \alpha) \eta_\alpha \wedge \xi_\alpha \tag{3.21}$$

Now, we find $\Phi \in C^\infty(L^\times)$ such that $\omega = dd^c\Phi$. It suffices to prove that Φ can extend smoothly across the zero level of L. To prove this, we need to apply a basic fact from complex functions: Let $f : [0, \infty) \to \mathbb{R}$ be a smooth function, then, $g(z) = f(|z|)$ is smooth in $\mathbb{C}$ if and only if there exits a

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smooth function $h : [0,\infty) \to \mathbb{R}$ such that $f(r) = h(r^2)$. Since ω is defined in the whole line bundle L, on each fibre, the terms of ω,

$$r\varphi_0'(r)\mu \wedge \eta_0, \qquad \varphi_0(r)\eta_\alpha \wedge \xi_\alpha \tag{3.22}$$

can extend smoothly across the zero level. Notice that $\mu \wedge \eta_0 = Cr^{-2} du \wedge d\bar{u}$, where u denotes the fiber coordinate of L. Then, it is easy to see that the terms in (3.22) is smooth on each fiber if and only if $r^{-1}\varphi'(r)$ and $\varphi(r)$ are smooth on $\mathbb{C}$ if and only if there exists a smooth function $h : [0,\infty) \to \mathbb{R}$ such that $h(r^2) = \varphi(r)$. According to the definition of Φ, we expand Φ near 0.

$$\Phi'(r) = h(r^2)/r = C_{-1}/r + C_1 r + C_2 r^3 + \dots .$$

we have

$$\Phi(r) = C_{-1} \log r + C_0 + C_1 r^2 + C_2 r^4 + \dots .$$

We claim that C_{-1} is vanishing. Otherwise, $\varphi(0) \neq 0$, which implies that $\varphi_0(r)\eta_0$ is not well defined on the zero level. Recalling the expression of θ, this contradicts against the fact that θ is well defined on L. Therefore, we find a global Kähler potential for ω, which completes the proof. □

Remark 3.5. The invariant $\partial\overline{\partial}$-lemma does not hold for all line bundles. For instance, let α be a simple root with $\alpha = 2\lambda$, for instance, the line bundle $\mathcal{O}(1) \to \mathbb{CP}^1$. Consider the following invariant 1-form

$$\theta = r^2\eta_0 + r^2\eta_\alpha + r^2\xi_\alpha.$$

Then, we can check that $d\theta$ is an invariant exact real (1,1) form. However, comparing with (3.21), there is no potential function for this form.

3.4. **Proof of Theorem A.** Firstly, it is easy see that the invariant dd^c lemma can be applied when L is a negative line bundle over X. Notice that the homogeneous line bundle L can shrink to the base manifold X. Furthermore, we can require the shrinking process to be G-equivalent. In particular, we have the following isomorphism between cohomology groups.

$$p^* : \; H^{1,1}_G(X) \;\cong\; H^{1,1}_G(L),$$

which implies that all invariant Kähler classes of L arise from the invariant Kähler classes of X. In each invariant Kähler class of X, there exists exactly one invariant Kähler form, which follows directly from dd^c-lemma on X. Given an invariant Kähler form ω on L, by previous discussion, there exists an invariant Kähler form ω_X on X such that $[\omega] = p^*[\omega_X]$ then, by Propositions 3.4, there exists a smooth function Φ defined on L such that,

$$\omega = p^*\omega_X + dd^c\Phi.$$

Hence, we complete the proof of theorem A.

4. Momentum profiles and the classification of invariant cscK metrics

This section is dedicated to the proof of Theorem B. Our main tool of this section comes from [22], in which the momentum construction is applied to investigate the Calabi ansatz. In Section 4.1, we will build up momentum construction by tracking the idea in [22]. Then, momentum construction will be applied in Section 4.2 to get the existence and uniqueness of G-invariant scalar-flat Kähler metric. We also prove the scalar-flat Kählers are asymptotically conical metrics. This proves Theorem B.

4.1. **Momentum construction of Calabi ansatz.** Let (X, ω_X) be a Kähler manifold and $\pi : (L, h) \to (X, \omega_X)$ be a holomorphic line bundle of M with Hermitian metric h. Let t be the logarithm of fibre norm function related to h; i.e., given a local line bundle coordinate chart (u, v), where

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u represents the fibre coordinate, $t = \log[r(u)^2] := \log h(u, \overline{u})$. Then, Kähler metrics arise from Calabi Ansatz is given by

$$\omega = \pi^* \omega_X + \frac{1}{2} dd^c f(t), \tag{4.1}$$

where f is a smooth function of one real variable. According to theorem A, all invariant Kähler metrics comes from Calabi ansatz (4.1).

It is well-known that the problem of prescribed scalar curvature is equivalent to a fourth order PDE of potential function, specially, in this case, a fourth order ODE. However, Hwang and Singer comes up the method of momentum profile in [22] by which it can be reduced to be a second order ODE, as the curvature formula related to momentum profile is of second order. In the following, we study the momentum profile associated with the given original data.

The Kähler metric, ω, arising from Calabi ansatz (4.1) may not exist in the whole line bundle L. For instance, in some cases, it might be blowing up in a finite domain with respect to fibre coordinate. We use L' to denote the possible existence region of L for ω. Notice that Kähler metric ω constructed in (4.1) admits a natural Killing field X_0, which generates a circle action on each fibre and can be written in local bundle coordinates, $X_0 = rJ(\partial/\partial r)$. In aspects of symplectic geometry, X_0 generates a Hamiltonian action on (L, ω) by

$$i_{X_0} \omega = -d\tau. \tag{4.2}$$

At each point on $p \in X$, there exists a coordinate chart around p such that $\partial \log h|_p = \overline{\partial} \log h|_p = 0$. By computing ω at each point in the chart,

$$\omega = \pi^* \omega_X + f'(t) \frac{1}{2} dd^c \log h + f''(t) i \frac{du \wedge d\overline{u}}{|u|^2} \tag{4.3}$$

and inserting (4.3) into (4.2), we have $\tau = f'(t)$. Let the interval I be the image of moment map τ. Noting that $||X_0||_\omega$ is a constant along each level of τ, we can define the function $\varphi : I \to \mathbb{R}_{\geq 0}$ by factoring through τ,

$$\varphi(\tau) = \frac{1}{2} ||X_0(\tau)||_\omega^2.$$

The interval I together with the function φ is called *momentum profile* related to (L', ω). The essential relation between $\varphi(\tau)$ and the potential f is given by,

$$\varphi(\tau) = \frac{1}{2} \omega(X_0, JX_0) = -\frac{1}{2} JX_0(f'(t)) = -\frac{1}{2} f''(t) \cdot JX_0(\log r^2) = f''(t) \tag{4.4}$$

Also by observing (4.3), the positivity of Kähler form implies the following two things: f is a convex function; hence the moment map $\tau = f'(t)$ induces a Legendre transformation from t to τ. Moreover, if we denote $\gamma = -i\partial\overline{\partial} \log h$, the positivity of ω also requires $\omega - \tau\gamma$ to be positive. An interval, I, is defined to be a *momentum interval* if for all $\tau \in I$, $\omega(\tau) = \omega - \tau\gamma$ is positive. In the following, we shall reconstruct (L', ω) by momentum profile (I, φ) with a momentum interval I and $\varphi : I \to \mathbb{R}_{\geq 0}$.

Based on the inverse Legendre transformation, we can rebuild the Kähler metric ω explicitly by momentum profile (I, φ) as follows. Let (a, b) be the interior of I with $-\infty \leq a < b \leq \infty$ and fix $\tau_0 \in I$.

(a) Fibre domain: Let T be the defining domain of $f(t)$ with $T^\circ = (t_1, t_2)$, then,

$$t_1 = \lim_{\tau \to a+} \int_{\tau_0}^{\tau} \frac{dx}{\varphi(x)} \quad \text{and} \quad t_2 = \lim_{\tau \to b-} \int_{\tau_0}^{\tau} \frac{dx}{\varphi(x)}.$$

(b) Potential function: $f(t)$ is given by data (I, φ)

$$f(t) = \int_{\tau_0}^{\tau(t)} \frac{x dx}{\varphi(x)}.$$

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(c) Fibre metric: The metric ω induces the metric on each fibre in terms of coordinate u, g_{fibre} and the ω-distance between τ_0 level and $\tau(t)$ level, $s(t)$

$$g_{\text{fibre}} = \varphi(\tau)\left|\frac{du}{u}\right|^2, \qquad s(t) = \int_{\tau_0}^{\tau(t)} \frac{dx}{2\sqrt{\varphi(x)}}.$$

where the formulas in (b), (c) can be obtained by change the variable though Legendre transformation.

The next step is to work out the curvature formula in terms of momentum profiles.To make curvature formula fit in the momentum profile, define $(\tau, \pi) : L' \to I \times X$. Here are some notations that will be used in the following:

- Let ω_φ represent the Kähler metric constructed by momentum profile (φ, I), and we can rewrite (4.3) in terms of τ,
$$\omega_\varphi = p^*\omega_X(\tau) + \varphi(\tau)\frac{idu \wedge d\overline{u}}{|u|^2}, \tag{4.5}$$
where $\omega_X(\tau) = \omega_X - \tau\gamma$.
- Let B denote the endomorphism $\omega_X^{-1}\gamma$, ρ_X be the Ricci curvature form of X, define the following functions on $I \times X$,
$$\begin{aligned} Q(\tau) &= \det(I - \tau B), \\ R(\tau) &= \operatorname{tr}[(I - \tau B)^{-1}(\omega_X^{-1}\rho_X)]. \end{aligned}$$

Then, the Ricci curvature, Laplacian and scalar curvature have the following representation in terms of momentum profile and notations above

- The Ricci form of ω_φ,
$$\rho_\varphi = p^*\rho_X - i\partial\overline{\partial}\log\varphi Q(\tau) \tag{4.6}$$
- Scalar curvature S_φ,
$$S_\varphi = R(\tau) - \Delta_{\omega_{X(\tau)}}\log Q(\tau) - \frac{1}{Q}\frac{\partial^2}{\partial\tau^2}(\varphi Q)(\tau) \tag{4.7}$$

In the case of G-invariance, $Q(\tau)$ is a polynomial in τ and $\Delta_{X(\tau)}\log Q(\tau) = 0$. Then, we can assume that $R(\tau) = P(\tau)/Q(\tau)$ for some polynomial P in τ. Therefore, we reduce the problem of prescribed scalar curvature to a second order ODE,

$$(\varphi Q)'' + QS_\varphi = P. \tag{4.8}$$

4.2. **Proof of Theorem B.** In the following, we compute the explicit formula of the polynomials $Q(\tau)$, $P(\tau)$ in terms of ω_X and corresponding weight λ. Recall the formulas (2.14) and (3.9) and ω_X, γ can be expressed in terms of $dz_\alpha = \eta_\alpha + i\xi_\alpha$ and $d\overline{z}_\alpha = \eta_\alpha - i\xi_\alpha$,

$$\omega_X = \frac{i}{2}C_{\alpha,S}\ dz_\alpha \wedge d\overline{z}_\alpha, \qquad \gamma = -i\partial\overline{\partial}\log h = \frac{i}{4}\sum_{\alpha\in D^+}(\lambda,\alpha)\ dz_\alpha \wedge d\overline{z}_\alpha,$$

where $S \in \mathfrak{s}$ such that $C_{\alpha,S} > 0$. Then, the matrix B is diagonal, and $Q(\tau)$ has the following expression,

$$Q(\tau) = \det(I - \tau B) = \prod_{\alpha\in D^+}\left[1 - \tau\frac{(\lambda,\alpha)}{2C_{\alpha,S}}\right],$$

Since the Ricci curvature ρ_X has the expression,

$$\rho_X = \frac{i}{4}\sum_{\alpha\in D^+}(\alpha,\delta)\ dz_\alpha \wedge d\overline{z}_\alpha$$

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Then,

$$R(\tau) = \operatorname{tr}\left[(I-\tau B)^{-1}\,\omega_X^{-1}\rho_X\right] = \sum_{\alpha\in D^+}\frac{(\alpha,\delta)}{2C_{\alpha,S}-\tau(\lambda,\alpha)}.$$

Hence, the ODE (4.8) can be rewrite as follows,

$$\Big(\varphi\prod_{\alpha\in D^+}\big(2C_{\alpha,S}-\tau(\lambda,\alpha)\big)\Big)'' + S_\varphi\prod_{\alpha\in D^+}\big(2C_{\alpha,S}-\tau(\lambda,\alpha)\big) = \prod_{\alpha\in D^+}\big(2C_{\alpha,S}-\tau(\lambda,\alpha)\big)\sum_{\beta\in D^+}\frac{(\beta,\delta)}{2C_{\beta,S}-\tau(\lambda,\beta)} \tag{4.9}$$

To determine the initial data of the ODE 4.9, we need to apply the following completeness proposition in ([22], Proposition 2.2-2.3).

Proposition 4.1. (Hwang, Singer [22]) *Let (I,φ) be a given momentum profile. Then the associated fibre metric is complete if and only if the following conditions hold at each endpoint of I*

- *Finite Endpoints: φ satisfies one of the following conditions,*
  (i) φ vanishes to first order with $|\varphi'| = 1$; or
  (ii) φ vanishes to order at least two.
- *Infinite Endpoints: φ grows at most quadratically, i.e., $\varphi \le K\tau^2$.*

And the corresponding (L',ω) behaves differently under different decay conditions provided in Proposition 4.1. We conclude the corresponding relations in the table 1, where we consider finite ends at $\tau = 0$ and infinite ends as $\tau \to \infty$. The proof of these bundle behaviors directly follows from the reconstruction of the data (L',ω) by moment profile (φ, I), (a)–(c).

Type of Ends	Decay (Growth) conditions	Fibre Range (t)	Distance to Ends (ω)
finite ends	$\varphi = 0, \quad \varphi' = 1,$	$[-\infty, t_0]$	finite
finite ends	$\varphi = \varphi' = 0$	$(-\infty, t_0]$	infinite
infinite ends	$C\tau < \varphi \le K\tau^2$	$[t_0, t_{\text{end}})$, $(t_{\text{end}} < \infty)$	infinite
infinite ends	$\varphi \le C\tau$	$[t_0, \infty)$	infinite

Table 1. Behaviors of (L',ω)

To fit in our cases, we define the momentum interval $I = [0,b)$ with $b \le +\infty$. The reason we take the left ends to be 0 is to ensure $\omega|_X = \omega_S$. Assume that the corresponding scalar curvature of ω_φ is constant. Combining with Proposition 4.1 and table 1, we shall solve the ODE with initial condition $\varphi(0) = 0$, $\varphi'(0) = 1$ and $S_\varphi = 0$. It is obvious that there is a unique solution φ satisfies (4.9) and the initial condition.

4.2.1. *The asymptotic behavior of scalar-flat Kähler metrics.* In the scalar-flat cases, φ satisfies

$$\Big(\varphi\prod_{\alpha\in D^+}\big(2C_{\alpha,S}-\tau(\lambda,\alpha)\big)\Big)' = \int_0^\tau\prod_{\alpha\in D^+}\big(2C_{\alpha,S}-t(\lambda,\alpha)\big)\sum_{\beta\in D^+}\frac{(\beta,\delta)}{2C_{\beta,S}-t(\lambda,\beta)}dt + \prod_{\alpha\in D^+}2C_{\alpha,S}.$$

Assuming that λ is negative, φ is a strictly increasing function with initial value $\varphi(0) = 0$ with degree one. According to table 1, ω_φ is well-defined over the whole bundle L_λ. By solving the ODE (4.9), the

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leading coefficient of the solution is given as follows,

$$i_{\lambda,X} = \frac{1}{(n-1)n} \sum_{\alpha\in D^+} \frac{(\alpha,\delta)}{(\alpha,-\lambda)}.$$

We call $i_{\lambda,X}$ the *metric index* of line bundle of (L_λ, X). Recall the Kähler metric associated to φ is given as in (4.5). Let $g_X(\tau)$, g_X, g_γ be the metric corresponding to $\omega_X(\tau) = \omega_X - \tau\gamma$, ω_X, $-\gamma$ respectively then

$$g_\varphi = g_X + \tau g_\gamma + 2\varphi(\tau)\eta_0^2 + 2\varphi(\tau)\mu^2.$$

where μ and η_0 are the dual of $r\partial/\partial r$ and X_0 respectively. Then, on each level set of L, $M(\tau)$, there is a metric induced by g_φ, denoted by $g(\tau)$,

$$g_{M(\tau)} = g_X + \tau g_\gamma + 2\varphi(\tau)\eta_0^2.$$

Let C_λ be the cone associated with L_λ by collapsing the base manifold X. Then, we should determine the radial function l of cone C_λ such that the scalar-flat Kähler metric is asymptotically conical to $Ai\partial\bar{\partial}l^2$, where A is the constant coefficient and can be canceled by rescaling. Based on the discussion in Section 4.1, we have the following relationship between τ and t,

$$t = \int_{\tau_0}^{\tau(t)} \frac{dx}{\varphi(x)} = \int_{\tau_0}^{\tau(t)} \frac{dx}{i_{\lambda,X}x} + \frac{a_1 dx}{x^2} + \ldots = a_0 + \frac{1}{i_{\lambda,X}} \log\tau - \frac{a_1}{\tau} + \ldots$$

where the second equality is just the Taylor expansion of $1/\varphi(x)$. Taking exponential and solve for τ, we can see that τ admits the following expansion at infinity,

$$\tau = b_1 r^{2i_{\lambda,X}} + b_0 + b_{-1} r^{-2i_{\lambda,X}} + \ldots \tag{4.10}$$

Now, let $l = r^{i_{\lambda,X}}$, then the model Kähler metric over C_λ is defined by the radial function l,

$$\omega_{\text{mod}} = i\partial\bar{\partial}l^2 = -i_{\lambda,X}l^2\gamma + 2i_{\lambda,X}^2 l^2 \mu\wedge\eta_0$$

Rewrite the model Kähler form in terms of metric,

$$\begin{aligned} g_{\text{mod}} &= i_{\lambda,X}l^2 g_\gamma + 2i_{\lambda,X}^2 l^2\eta_0^2 + 2i_{\lambda,X}^2 l^2\mu^2 \\ &= l^2(i_{\lambda,X}g_\gamma + 2i_{\lambda,X}^2\eta_0^2) + 2dl^2 \end{aligned}$$

And the metric can be represented by dl as follows,

$$\begin{aligned} g_\varphi &= l^2\Big(\frac{1}{l^2}g_X + \frac{\tau}{l^2}g_\gamma + \frac{2\varphi(\tau)}{l^2}\eta_0^2\Big) + \frac{2\varphi(\tau)}{i_{\lambda,X}^2 l^2}dl^2 \\ &= b_1\big[l^2(i_{\lambda,X}g_\gamma + 2i_{\lambda,X}^2\eta_0^2) + dl^2\big] + O(l^{-2}) \\ &= b_1 g_{\text{mod}} + O(l^{-2}) \end{aligned}$$

Therefore, all scalar-flat Kähler metrics on L_λ are asymptotically conical to $(C_\lambda, g_{\text{mod}})$, which complete the proof of Theorem B.

In general, g_φ decays to g_{mod} by order -2, which can be improved in some special cases. For instance, let the base metric ω_X be equal to the curvature form γ, then, the similar calculation shows that the metric g_φ with scalar-flat curvature decays to order $-2n+2$.

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