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LINEAR STRUCTURE OF WEIGHTED HOLOMORPHIC NON-EXTENDIBILITY

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In this paper, it is proved that, for any domain G of the complex plane, there exists an infinite-dimensional closed linear submanifold M_1 and a dense linear submanifold M_2 with maximal algebraic dimension in the space $H(G)$ of holomorphic functions on G such that G is the domain of holomorphy of every nonzero member f of M_1 or M_2 and, in addition, the growth of f near each boundary point is as fast as prescribed.

1. INTRODUCTION AND NOTATION

Throughout this paper, the following standard terminology and notation will be used. The symbols $\mathbb{N}$, $\mathbb{C}$, $\mathbb{D}, \mathbb{T}$ denote, respectively, the set of positive integers, the complex plane, the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. If $a \in \mathbb{C}$ and $r > 0$ then $B(a,r)$ ($\overline{B}(a,r)$, respectively) denotes the open (closed, respectively) Euclidean ball with centre a and radius r; in particular, $B(0,1) = \mathbb{D}$. For points a, b of $\mathbb{C}$, the line segment joining a with b is $[a,b]$. If $A \subset \mathbb{C}$ then $\overline{A}$ (A^0, ∂A, respectively) denotes its closure (interior, boundary, respectively) in $\mathbb{C}$. Moreover, if $z_0 \in \mathbb{C}$ then $d(z_0, A) := \inf\{|z_0 - z| : z \in A\}$. A domain is a nonempty open subset of $\mathbb{C}$. If G is a domain, then $H(G)$ denotes the Fréchet space (= completely metrisable locally convex space) of holomorphic functions on G, endowed with the topology of uniform convergence on compacta. In particular, $H(G)$ is a Baire space. Finally, if $a \in G$ and $f \in H(G)$ then $\rho(f,a)$ represents the radius of convergence of the Taylor series of f with centre at a. It is well known that $\rho(f,a) \geqslant d(a, \partial G)$.

In 1884 Mittag–Leffler (see [9, Chapter 10]) discovered that for any domain G there exists a function $f \in H(G)$ having G as its domain of holomorphy. Recall that G is said to be a domain of holomorphy for f if f is holomorphic exactly at G, that is, $f \in H(G)$ and f is analytically non-extendible across ∂G or, more precisely, $\rho(f,a) = d(a, \partial G)$ for all $a \in G$. Note that this implies that f has no holomorphic extension on any domain containing G strictly. Both properties are equivalent if, for instance, G is a Jordan domain, but the equivalence is not general (for instance, consider $G := \mathbb{C} \setminus (-\infty, 0]$ and

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$f :=$ the principal branch of the logarithm on G). By $H_e(G)$ we denote the subclass of functions which are holomorphic exactly at G. Hence, the Mittag–Leffler result mentioned above says that $H_e(G) \neq \emptyset$ for any domain G.

In 1933 Kierst and Szpilrajn [**12**] showed that at least for $G = \mathbb{D}$ the property discovered by Mittag–Leffler is generic, in the sense that $H_e(\mathbb{D})$ is not only nonempty but even residual –hence dense– in $H(\mathbb{D})$, that is, its complement in $H(\mathbb{D})$ is of first category. Recently, Kahane ([**11**, Theorem 3.1 and following remarks]; see also [**10**, Proposition 1.7.6] and [**4**, Theorem 3.1]) has observed that Kierst-Szpilrajn's theorem can be extended to every domain G and to rather general topological vector spaces $X \subset H(G)$ (including the full space $X = H(G)$); indeed, under suitable conditions on X, he shows that $H_e(G) \cap X$ is residual in X. In other words, $H_e(G) \cap X$ is *topologically large* in X.

Recently, we have proved [**4**] for the case $G = \mathbb{D}$ that under adequate hypotheses a topological vector space $X \subset H(\mathbb{D})$ satisfies that $H_e(\mathbb{D}) \cap X$ is also *algebraically large*, in the sense that the last subset contains –except for zero– some "large" (= dense, or closed infinite-dimensional) *linear manifold*. Again, the case $X = H(\mathbb{D})$ is covered. Note that the fact that $H_e(G)$ is not a linear manifold increases the interest in this matter. As for a general domain G, Aron, García and Maestre [**1**, Theorem 8] had already proved in 2001 that $H(G)$ contains a *dense* linear manifold M_1 as well as a *closed infinite-dimensional* linear manifold M_2 such that $M_i \setminus \{0\} \subset H_e(G)$ $(i = 1, 2)$. In fact, their result extends to any domain of holomorphy in $\mathbb{C}^N$ (see also [**4**, Theorem 5.1] for an independent, different proof in the 'dense' case with $N = 1$), and the manifolds M_i $(i = 1, 2)$ are even ideals.

In the terminology of [**8**], a subset S of a linear topological space E is *spaceable* whenever $S \cup \{0\}$ contains some closed infinite-dimensional subspace in E (see [**8**] and [**2**] for nice, recent examples of spaceable sets). Therefore, under this convention, it has been demonstrated in [**1**, Theorem 8] that $H_e(G)$ is spaceable in $H(G)$.

Nevertheless, the approach in [**1**, Theorem 8] does not give any information about *how fast* the functions in M_1 or M_2 can grow near the boundary. In [**4**, note after Theorem 5.1] it is suggested how this can be proved for the manifold M_1 ('dense' case) in $H(G)$, with $G \subset \mathbb{C}$. Hence, it is natural to ask the following:

Given any prescribed ('weight') function $\varphi : G \to (0, +\infty)$, is the set

$$\mathcal{S}_\varphi := \left\{ f \in H_e(G) : \limsup_{z \to t} |f(z)|/\varphi(z) = +\infty \text{ for all } t \in \partial G \right\}$$

spaceable in $H(G)$?

The main aim in this paper is to furnish an affirmative answer to this question. This will be obtained in Section 2. Finally, in Section 3 we shall complete this study by showing the existence of a *dense* linear submanifold M with *maximal algebraic dimension* –that is, $\dim(M) = \chi :=$ the cardinality of the continuum– such that $M \setminus \{0\} \subset \mathcal{S}_\varphi$, where φ is a given weight function as above.

2. Spaceability of the weighted non-extendibility

Before establishing our main result, an auxiliary statement about basic sequences is needed. Let us consider the Hilbert space $L^2(\mathbb{T})$ of all (Lebesgue classes of) measurable functions $f : \mathbb{T} \to \mathbb{C}$ with finite quadratic norm

$$\|f\|_2 := \left(\int_0^{2\pi} \left|f(e^{i\theta})\right|^2 d\theta/(2\pi)\right)^{1/2}.$$

Since $(z^n)_{n=-\infty}^{\infty}$ is an orthonormal basis of $L^2(\mathbb{T})$, we have that $(z^n)_{n\geqslant 1}$ is a basic sequence in $L^2(\mathbb{T})$. Recall that two basic sequences $(x_n)_{n\geqslant 1}$, $(y_n)_{n\geqslant 1}$ in a Banach space $(E, \|\cdot\|)$ are said to be *equivalent* if, for every sequence $(a_n)_{n\geqslant 1}$ of scalars, the series $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_n y_n$ converges. This happens (see [**3**, p. 108]) if and only if there exist two constants $m, M \in (0, +\infty)$ such that, for every finite sequence $(a_j)_{j=1,\dots,J}$ of scalars, we have

$$m\Big\|\sum_{j=1}^{J} a_j x_j\Big\| \leqslant \Big\|\sum_{j=1}^{J} a_j y_j\Big\| \leqslant M\Big\|\sum_{j=1}^{J} a_j x_j\Big\|. \tag{1}$$

Lemma 2.1. *Assume that G is a domain with $\overline{\mathbb{D}} \subset G$ and that $(f_j)_{j\geqslant 1} \subset H(G)$ is a sequence such that it is a basic sequence in $L^2(\mathbb{T})$ that is equivalent to $(z^j)_{j\geqslant 1}$. If*

$$\left\{h_l := \sum_{j=1}^{J(l)} c_{j,l} f_j\right\}_{l\geqslant 1}$$

is a sequence in $\operatorname{span}(f_j)_{j\geqslant 1}$ *converging in $H(G)$, then*

$$\sup_{l\in\mathbb{N}} \sum_{j=1}^{J(l)} |c_{j,l}|^2 < +\infty. \tag{2}$$

Proof: Observe first that, since $\overline{\mathbb{D}}$ is a compact subset of G, convergence in $H(G)$ is stronger than convergence in $L^2(\mathbb{T})$-norm. Therefore $(h_l)_{l\geqslant 1}$ converges in $L^2(\mathbb{T})$, so the sequence $\left(\|h_l\|_2\right)_{l\geqslant 1}$ is bounded, say $\|h_l\|_2 \leqslant \alpha$ $(l \in \mathbb{N})$. Let x_j, y_j, $\|\cdot\|$ be respectively the function $z \mapsto z^j$, the function f_j and the norm $\|\cdot\|_2$. Then, by (1), we get for every $l \in \mathbb{N}$ that

$$m^2 \sum_{j=1}^{J(l)} |c_{j,l}|^2 = m^2 \Big\|\sum_{j=1}^{J(l)} c_{j,l} z^j\Big\|_2^2 \leqslant \Big\|\sum_{j=1}^{J(l)} c_{j,l} f_j\Big\|_2^2 = \|h_l\|_2^2 \leqslant \alpha^2.$$

Hence (2) is satisfied because the supremum is not greater that α^2/m^2. ∎

Now, our main assertion about non-extendibility can be established.

THEOREM 2.2. *Let $G \subset \mathbb{C}$ be a domain and $\varphi : G \to (0, +\infty)$ be a function. Then $\mathcal{S}_\varphi$ is spaceable in $H(G)$.*

PROOF: We must prove the existence of an infinite-dimensional closed linear manifold M in $H(G)$ such that $M \setminus \{0\} \subset \mathcal{S}_\varphi$. The case $G = \mathbb{C}$ being trivial, we may assume $G \neq \mathbb{C}$. We denote by G_* the one-point compactification of G. Recall that in G_* the whole boundary ∂G collapses to a unique point, say ω. Without loss of generality, it can be supposed that $\overline{\mathbb{D}} \subset G$.

We are going to choose countably many pairwise disjoint sequences $\{a(k,n) : n \in \mathbb{N}\}$ ($k \in \mathbb{N}$) of distinct points of $G \setminus \overline{\mathbb{D}}$ such that each of them has no accumulation point in G and every prime end (see [**5**, Chapter 9]) of ∂G is an accumulation point of each such sequence. The last property means, more precisely, the following: For every $k \in \mathbb{N}$, every $a \in G$ and every $r > d(a, \partial G)$, the intersection of $\{a(k,n) : n \in \mathbb{N}\}$ with the connected component of $B(a,r) \cap G$ containing a is infinite. In particular, every point $t \in \partial G$ would be an accumulation point of each sequence $\{a(k,n) : n \in \mathbb{N}\}$.

Let us show how such a family of sequences can be constructed. We begin with $k = 1$. Let $\{c_j : j \in \mathbb{N}\}$ be a dense countable subset of G. For each $j \in \mathbb{N}$ choose $b_j \in \partial G$ such that $|b_j - c_j| = d(c_j, \partial G)$. For every $j \in \mathbb{N}$ let $\{d_{1,j,l} : l \in \mathbb{N}\}$ be a sequence of points in $[c_j, b_j] \setminus \overline{\mathbb{D}}$ such that

$$|d_{1,j,l} - b_j| < 1/(1 + j + l) \quad (j, l \in \mathbb{N}).$$

Then we choose as $\{a(1,n) : n \in \mathbb{N}\}$ a one-fold sequence (without repetitions) consisting of all distinct points of the set $\{d_{1,j,l} : j, l \in \mathbb{N}\}$. It is easy to check that $\{a(1,n) : n \in \mathbb{N}\}$ satisfies the required property. In a second step –that is, for $k = 2$– we can select for every $j \in \mathbb{N}$ a sequence $\{d_{2,j,l} : l \in \mathbb{N}\}$ of points of $[c_j, b_j] \setminus \left(\overline{\mathbb{D}} \cup \{a(1,n) : n \in \mathbb{N}\}\right)$ such that, in addition,

$$|d_{2,j,l} - b_j| < 1/(2 + j + l) \quad (j, l \in \mathbb{N});$$

this is possible due to the denumerability of $\{a(1,n) : n \in \mathbb{N}\}$. Again, we define $\{a(2,n) : n \in \mathbb{N}\}$ as a sequence consisting of all distinct points of the set $\{d_{2,j,l} : j, l \in \mathbb{N}\}$; it then satisfies the required prime end property. It is now clear that this process can be repeated inductively, so yielding the desired disjoint family

$$\left\{\{a(k,n) : n \in \mathbb{N}\} : k \in \mathbb{N}\right\}.$$

Secondly, let us consider the subset $A := \overline{\mathbb{D}} \cup B \subset G$, where

$$B := \{a(k,n) : k, n \in \mathbb{N}\}.$$

Recall that for each $k \in \mathbb{N}$ the sequence $\{a(k,n) : n \in \mathbb{N}\}$ is an enumeration of the distinct points of a certain subset $\{d_{k,j,l} : j, l \in \mathbb{N}\} \subset G$ satisfying

$$|d_{k,j,l} - b_j| < \frac{1}{k + j + l} \quad (j, l \in \mathbb{N}). \tag{3}$$

We have that A is relatively closed in G. Indeed, the set of accumulation points of A in G is just $\overline{\mathbb{D}}$ (which is included in A) because the set of accumulation points of B in G is empty. Let us explain why this is so. Assume, by way of contradiction, that $z_0 \in G$ is an accumulation point of B. Then there is a sequence of distinct points $(d_{k(n),j(n),l(n)})_{n\geqslant 1}$ in B tending to z_0. Then the set $\{(k(n), j(n), l(n)) : n \in \mathbb{N}\}$ is infinite, so at least one of the sets of positive integers $\{k(n) : n \in \mathbb{N}\}$, $\{j(n) : n \in \mathbb{N}\}$, $\{l(n) : n \in \mathbb{N}\}$ is infinite, hence unbounded. Therefore the sequence $(k(n) + j(n) + l(n))_{n\geqslant 1}$ is also unbounded, thus $k(n) + j(n) + l(n) > 2/d(z_0, \partial G)$ for infinitely many $n \in \mathbb{N}$. Consequently,

$$\begin{aligned} \left|d_{k(n),j(n),l(n)} - z_0\right| &\geqslant |z_0 - b_{j(n)}| - \left|d_{k(n),j(n),l(n)} - b_{j(n)}\right| \\ &\geqslant d(z_0, \partial G) - \frac{1}{k(n) + j(n) + l(n)} > \frac{d(z_0, \partial G)}{2} \end{aligned}$$

for infinitely many $n \in \mathbb{N}$, which is absurd.

Thus, A is closed in G. But note that $G_* \setminus A$ is connected as well as locally connected at ω, because $\overline{\mathbb{D}}$ is compact (so it is "far" from ω, and we can suppose that the basic connected neighbourhoods of ω do not intersect $\overline{\mathbb{D}}$), $G \setminus \overline{\mathbb{D}}$ is connected and B is countable (so deleting B from $G \setminus \overline{\mathbb{D}}$ makes no influence in connectedness or local connectedness). Let us consider, for every $N \in N$, the function $g_N : A \to \mathbb{C}$ defined as

$$g_N(z) = \begin{cases} z^N & \text{if } z \in \overline{\mathbb{D}}, \\ n\Big(1 + \varphi\big(a(N,n)\big)\Big) & \text{if } z = a(N,n) \text{ and } n \in \mathbb{N}, \\ 0 & \text{if } z = a(k,n) \text{ and } k, n \in \mathbb{N} \text{ with } k \neq N. \end{cases}$$

Observe that g_N is continuous on A and holomorphic on A^0 $(= \mathbb{D})$. Then the Arakelian approximation theorem (see [**7**, pp. 136–144]) guarantees the existence of a function $f_N \in H(G)$ such that

$$\left|f_N(z) - g_N(z)\right| < \frac{1}{3^N} \text{ for all } z \in A.$$

Consequently, one obtains

$$\left|f_N(z) - z^N\right| < \frac{1}{3^N} \text{ for all } z \in \overline{\mathbb{D}}, \tag{4}$$

$$\left|f_N\big(a(N,n)\big) - n\Big(1 + \varphi\big(a(N,n)\big)\Big)\right| < 1 \text{ for all } n \in \mathbb{N}, \text{ and} \tag{5}$$

$$\left|f_N\big(a(k,n)\big)\right| < \frac{1}{3^N} \text{ for all } n \in \mathbb{N} \text{ and all } k \in \mathbb{N} \setminus \{N\}. \tag{6}$$

Finally, we define the sought-after linear manifold M by

$$M := \text{closure}_{H(G)}(\text{span}\{f_N : N \in \mathbb{N}\}).$$

It is clear that M is a closed linear manifold in $H(G)$. On the other hand, we have from (4) that $\|f_N - \varphi_N\|_2 < 3^{-N}$ for all $N \in \mathbb{N}$ (where $\varphi_N(z) := z^N$). By using this last inequality as well as the fact $\sum_{N=1}^{\infty} 3^{-N} < 1$ together with the basis perturbation theorem [**6**, p. 46, Theorem 9], we can derive that $(f_N)_{N\geqslant 1}$ is a basic sequence in $L^2(\mathbb{T})$. Indeed, let $(e_n^*)_{n\geqslant 1}$ be the sequence of coefficient functionals corresponding to the basic sequence $(z^n)_{n\geqslant 1}$. Since $\|e_n^*\|_2 = 1$ $(n \in \mathbb{N})$, one obtains

$$\sum_{N=1}^{\infty} \|e_n^*\|_2 \|f_N - \varphi_N\| < 1.$$

Therefore the perturbation theorem applies because $(\varphi_N)_{N\geqslant 1}$ is a basic sequence.

Since $(f_N)_{N\geqslant 1}$ is a basic sequence, we get that, in particular, the functions f_N $(N \in \mathbb{N})$ are linearly independent. Hence M has infinite dimension.

It remains to show that $M \setminus \{0\} \subset \mathcal{S}_\varphi$. Fix $f \in M \setminus \{0\}$. Since the convergence in $H(G)$ is stronger that the convergence in $L^2(\mathbb{T})$, we have that (the restriction to $\mathbb{T}$ of) f is in $\widetilde{M} := \text{closure}_{L^2(\mathbb{T})}(\text{span}\{f_N : N \in \mathbb{N}\})$. Therefore f has a (unique) representation $f = \sum_{j=1}^{\infty} c_j f_j$ in $L^2(\mathbb{T})$, because $(f_N)_{N\geqslant 1}$ is a basic sequence in this space. But $f \neq 0$, so there is $N \in \mathbb{N}$ with $c_N \neq 0$. On the other hand, there is a sequence

$$\left\{h_l := \sum_{j=1}^{J(l)} c_{j,l} f_j\right\}_{l\geqslant 1}$$

in $\text{span}\{f_j : j \in \mathbb{N}\}$ (without loss of generality, we can assume that $J(l) \geqslant N$ for all l) that converges to f compactly in G. By Lemma 2.1,

$$C := \sup_{l\in\mathbb{N}} \sum_{j=1}^{J(l)} |c_{j,l}|^2 < +\infty.$$

But $(h_l)_{l\geqslant 1}$ also converges to f in $L^2(\mathbb{T})$, so the continuity of each projection

$$\sum_{j=1}^{\infty} d_j f_j \in \widetilde{M} \mapsto d_m \in \mathbb{C} \quad (m \in \mathbb{N})$$

yields that $\lim_{l\to\infty} c_{N,l} = c_N$. In particular, there exists $l_0 \in \mathbb{N}$ such

$$|c_{N,l}| \geqslant \frac{|c_N|}{2} \text{ for all } l \geqslant l_0. \tag{7}$$

Let us fix $n \in \mathbb{N}$. Since the singleton $\{a(N,n)\}$ is a compact subset of G, we get the existence of a positive integer $l = l(n) \geqslant l_0$ such that

$$\left|h_l\big(a(N,n)\big) - f\big(a(N,n)\big)\right| < 1. \tag{8}$$

By using (5), (6), (7), (8), the triangle inequality and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\left|f\big(a(N,n)\big)\right| &\geqslant \left|h_l\big(a(N,n)\big)\right| - 1 \\
&\geqslant \left|c_{N,l} f_N\big(a(N,n)\big)\right| - \sum_{\substack{j=1 \\ j\neq N}}^{J(l)} \left|c_{j,l} f_j\big(a(N,n)\big)\right| - 1 \\
&\geqslant \frac{|c_N|}{2}\Big(n\Big(1+\varphi\big(a(N,n)\big)\Big) - 1\Big) - \sum_{\substack{j=1 \\ j\neq N}}^{J(l)} |c_{j,l}| \frac{1}{3^j} - 1 \\
&\geqslant \frac{|c_N|}{2}\Big(n\Big(1+\varphi\big(a(N,n)\big)\Big) - 1\Big) - \Big(\sum_{j=1}^{\infty}\Big(\frac{1}{3^j}\Big)^2\Big)^{1/2}\Big(\sum_{\substack{j=1 \\ j\neq N}}^{J(l)} |c_{j,l}|^2\Big)^{1/2} - 1 \\
&\geqslant \frac{|c_N|}{2}\Big(n\Big(1+\varphi\big(a(N,n)\big)\Big) - 1\Big) - C^{1/2} - 1.
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} f\big(a(N,n)\big) = \infty = \lim_{n\to\infty} f\big(a(N,n)\big)/\varphi\big(a(N,n)\big).$$

The second equality shows that $\limsup_{z\to t} \left|f(z)\right|/\varphi(z) = +\infty$ for all $t \in \partial G$, because each boundary point is a limit point of $\big(z_n := a(N,n)\big)_{n\geqslant 1}$.

Now, it is time to use the prime end approximation property of the sequence (z_n). Suppose, by way of contradiction, that $f \notin \mathcal{S}_\varphi$. Then $f \notin H_e(G)$, so there must be a point $c \in G$ such that $\rho(f,c) > d(c, \partial G)$. Choose r with $d(c,\partial G) < r < \rho(f,c)$. By the construction of the sequences $\big(a(k,n)\big)_{n\geqslant 1}$ $(k \in \mathbb{N})$, there exists a sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ for which $z_{n_j} \in G \cap B(c,r)$ $(j \in \mathbb{N})$. Finally, the sum $S(z)$ of the Taylor series of f with centre c is bounded on $B(c,r)$. But $S = f$ on $G \cap B(c,r)$, so $S(z_{n_j}) = f(z_{n_j}) \to \infty$ $(j \to \infty)$, which is absurd. This contradiction finishes the proof. ∎

3. Manifolds with maximal algebraic dimension

We conclude this note with a theorem that completes our Theorem 2.2 as well as Theorem 5.1 in [**4**] and (in the one-dimensional case) Theorem 8 in [**1**]. Specifically, we are able to construct –for a prescribed function $\varphi : G \to (0,+\infty)$– a linear submanifold $M \subset H(G)$ with $M \setminus \{0\} \subset \mathcal{S}_\varphi$ that is not only dense, but even it satisfies $\dim(M) = \chi$ (notice that the dense linear manifold M whose construction is suggested in [**4**, note following Theorem 5.1] was only of countably infinite dimension; in the opposite direction, the dense manifold X provided in [**1**, Theorem 8] does satisfy $\dim(X) = \chi$, but the fact $X \setminus \{0\} \subset \mathcal{S}_\varphi$ does not hold). Observe that, as an easy consequence of Baire's

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category theorem and of the fact that $H(G)$ is infinite-dimensional, metrisable, separable and complete, we have $\dim(H(G)) = \chi$. Hence χ is the maximal algebraic dimension which is permitted for the submanifolds of $H(G)$. For instance, the linear manifold M constructed in the proof of Theorem 2.2 satisfies $\dim(M) = \chi$ (because it is a closed subspace of $H(G)$, so M is also infinite-dimensional, metrisable, separable and complete) but it is not dense.

THEOREM 3.1. *Let $G \subset \mathbb{C}$ be a domain and $\varphi : G \to (0, +\infty)$ be a function. Then there is a dense linear manifold M in $H(G)$ such that $\dim(M) = \chi$ and $M \setminus \{0\} \subset \mathcal{S}_\varphi$.*

PROOF: Again, the case $G = \mathbb{C}$ is trivial, so we suppose $G \neq \mathbb{C}$. First, we consider pairwise disjoint sequences $\{a(k,n) : n \in \mathbb{N}\}$ $(k \in \mathbb{N})$, and then we select a sequence $\{f_N : N \in \mathbb{N}\} \subset H(G)$. This is made exactly as in the proof of Theorem 2.2, with the sole exception that instead of (5) we have

$$\Big|f_N\big(a(N,n)\big) - n^{1/2}\Big(1 + \varphi\big(a(N,n)\big)\Big)\Big| < 1 \quad \text{for all } n \in \mathbb{N}. \tag{9}$$

In other words, with the notation of the proof of Theorem 2.2 we would define

$$g_N\big(a(N,n)\big) := n^{1/2}\Big(1 + \varphi\big(a(N,n)\big)\Big) \quad (N, n \in \mathbb{N})$$

before the application of Arakelian's theorem. The key point will be that $n^{1/2}$ tends to infinity as $n \to \infty$, but less rapidly than any power n^N $(N \in \mathbb{N})$. Let us define

$$M_1 := \operatorname{closure}_{H(G)}\big(\operatorname{span}\{f_N : N \in \mathbb{N}\}\big).$$

Therefore we obtain as in the proof of Theorem 2.2 that $M_1 \setminus \{0\} \subset \mathcal{S}_\varphi$. As observed at the beginning of this section, we have $\dim(M_1) = \chi$.

Second, fix an increasing sequence $\{K_n : n \in \mathbb{N}\}$ of compact subsets of G such that each compact subset of G is contained in some K_n and each component of the complement of every K_n contains some connected component of the complement of G (see [**13**, Chapter 13]). Choose a dense countable subset $\{\psi_n : n \in \mathbb{N}\}$ of $H(G)$. Now consider for each $N \in \mathbb{N}$ the set $A_N := K_N \cup \{a(k,n) : k, n \in \mathbb{N}\}$. In a similar way to the proof of [**4**, Theorem 5.2], we have that A_N is closed in G and that $G_* \setminus A_N$ is connected and locally connected at ω. The function $h_N : A_N \to \mathbb{C}$ defined as

$$h_N(z) = \begin{cases} \psi_N(z) & \text{if } z \in K_N, \\ n^N\Big(1 + \varphi\big(a(k,n)\big)\Big) & \text{if } z = a(k,n)\ (k, n \in \mathbb{N}) \text{ and } z \notin K_N \end{cases}$$

is continuous on A_N and holomorphic on A_N^0 $(= K_N^0)$. We now use again the Arakelian approximation theorem to obtain this time a function $F_N \in H(G)$ such that

$$\big|F_N(z) - h_N(z)\big| < \frac{1}{N} \quad \text{for all } z \in A_N. \tag{10}$$

From (10) we derive that $|F_N(z) - \psi_N(z)| < 1/N$ for all $z \in A_N$ and all $N \in \mathbb{N}$. These inequalities together with the denseness of $\{\psi_N : N \in \mathbb{N}\}$ and the exhaustion property of the family $\{K_N : N \in \mathbb{N}\}$ yield the denseness of the sequence $\{F_N : N \in \mathbb{N}\}$ in $H(G)$.

Finally, we define M as

$$M := \operatorname{span}\big(M_1 \cup \{F_N : N \in \mathbb{N}\}\big).$$

Since $M \supset \{F_N : N \in \mathbb{N}\}$ and $M \supset M_1$, it is evident that M is a dense linear submanifold of $H(G)$ and $\dim(M) = \chi$. It remains to show that $M \setminus \{0\} \subset \mathcal{S}_\varphi$. For this, fix a function $f \in M \setminus \{0\}$. If $f \in M_1$ then we already know that $f \in \mathcal{S}_\varphi$. Thus, we can assume that $f \in M \setminus M_1$. Then there are finitely many scalars $c_1, \dots, c_N, d_1, \dots, d_\mu$ with $c_N \neq 0$ such that

$$(11) \qquad f = \sum_{j=1}^{N} c_j F_j + \sum_{j=1}^{\mu} d_j f_j.$$

Recall that according to the proof of Theorem 2.2 the set $B := \{a(k,n) : k, n \in \mathbb{N}\}$ has no accumulation point in G. In particular, each compact set K_j may contain only finitely many points $a(k,n)$. Therefore we can derive from (10) the existence of a number $n_0 \in \mathbb{N}$ such that

$$(12) \qquad \Big|F_j\big(a(N,n)\big) - n^j\Big(1 + \varphi\big(a(N,n)\big)\Big)\Big| < 1 \quad \text{for all } n \geqslant n_0 \ (j = 1, \dots, N).$$

On the other hand, we obtain by (6) and (9) that

$$(13) \qquad \Big|f_j\big(a(N,n)\big)\Big| < n^{1/2}\Big(1 + \varphi\big(a(N,n)\big)\Big) + 1 \quad (j = 1, \dots, \mu;\ n \in \mathbb{N}).$$

To finish, from (11), (12), (13) and the triangle inequality it is deduced for $n \geqslant n_0$ that

$$\begin{aligned} \Big|f\big(a(N,n)\big)\Big| \geqslant |c_N|\Big[n^N\Big(1 + \varphi\big(a(N,n)\big)\Big) - 1\Big] - \sum_{j=1}^{N-1} |c_j|\Big[n^j\Big(1 + \varphi\big(a(N,n)\big)\Big) + 1\Big] \\ - \Big(\sum_{j=1}^{\mu} |d_j|\Big)\Big[n^{1/2}\Big(1 + \varphi\big(a(N,n)\big)\Big) + 1\Big]. \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} f\big(a(N,n)\big) = \infty = \lim_{n\to\infty} f\big(a(N,n)\big)/\varphi\big(a(N,n)\big).$$

Then the desired conclusion may be achieved as in the last paragraph of the proof of Theorem 2.2. ∎

FINAL QUESTION. Do the analogues of Theorems 2.2 and 3.1 hold for a domain of holomorphy in $\mathbb{C}^N$?

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