ON EVENTS REPRESENTED BY PROBABILISTIC AUTOMATA OF DIFFERENT TYPES

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1. Introduction. The usual definition of a finite probabilistic automaton (**4**; **5**) does not involve any outputs. In this paper, such automata are referred to as Rabin automata. Various other types of probabilistic automata are obtained by introducing output functions. Similarly, as in connection with finite deterministic automata, the distinction between Moore automata and Mealy automata can then be made. Rabin automata may be regarded as automata where the output function is deterministic. One can also consider the case where the transition function is deterministic.

In § 3, theorems concerning the equivalence and non-equivalence of various types of finite automata are proved. The main result is that for any finite probabilistic automaton, there is an equivalent finite Rabin automation. Hence, an event represented in any finite probabilistic automaton can also be represented in a finite Rabin automaton. In § 4, the representation of events is reduced to the realization of functions, and a necessary and sufficient condition for a function to be realizable in a finite Rabin automaton is established.

2. Preliminaries. We denote by $M(i, j)$ the set of all $i \times j$ matrices consisting of non-negative real numbers whose sum equals 1. Thus $M(1, j)$ denotes the set of all j-dimensional stochastic row vectors. A j-dimensional stochastic vector is termed a *coordinate vector* if one of its components equals 1. The number of elements in a finite set V is denoted by $|V|$.

By a *finite probabilistic automaton* we mean an ordered quintuple

$$A = (S, X, Y, \delta, F)$$

where S, X, and Y are finite non-empty sets (called, respectively, the set of *states*, *input alphabet*, and *output alphabet*), δ is an element of $M(1, |S|)$ (called the *initial distribution*), and F is a function mapping the set $S \times X$ into the set $M(|S|, |Y|)$. Assume that $S = \{s_1, \ldots, s_k\}$, $X = \{x_1, \ldots, x_l\}$, and $Y = \{y_1, \ldots, y_m\}$. The (i, j)th entry in the matrix $F(s_u, x_v)$, where $1 \leqslant i, u \leqslant k$, $1 \leqslant v \leqslant l$, and $1 \leqslant j \leqslant m$, is denoted by

$$(1) \qquad p_A(s_i, y_j/s_u, x_v).$$

The number (1) is referred to as the probability of A entering the state s_i and producing the output y_j, after being in the state s_u and receiving the

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input x_v. Assume that $s^1, \ldots, s^n \in S$, $y^1, \ldots, y^n \in Y$, and $x^1, \ldots, x^n \in X$. Let

$$p_A(s^1 \ldots s^n, y^1 \ldots y^n/x^1 \ldots x^n) = \sum_{s^0 \in S} \delta(s^0) \prod_{i=1}^{n} p_A(s^i, y^i/s^{i-1}, x^i)$$

where $\delta(s^0)$ denotes, for $s^0 = s_j$, the jth component of the vector δ. Furthermore, let

$$(2) \qquad p_A(y^1 \ldots y^n/x^1 \ldots x^n) = \sum p_A(s^1 \ldots s^n, y^1 \ldots y^n/x^1 \ldots x^n),$$

where the sum is taken over all n-tuples $(s^1, \ldots, s^n)$. The number (2) is referred to as the probability of A giving the response $y^1 \ldots y^n$ to the input $x^1 \ldots x^n$. An element of the free semigroup generated by X (Y) is called a *word* over the alphabet X (Y). Two finite probabilistic automata A and B with the same input and output alphabets are termed *equivalent* if

$$p_A(Q/P) = p_B(Q/P),$$

for any input word P and output word Q of the same length.

Consider the sum

$$p_A(y/x^1 \ldots x^n) = \sum p_A(y^1 \ldots y^{n-1}y/x^1 \ldots x^n)$$

which is taken over all $(n-1)$-tuples $(y^1, \ldots, y^{n-1})$. Let λ be a real number, $0 \leqslant \lambda < 1$. We say that the output y *represents* in A the event

$$E(A, y, \lambda) = \{P \mid p_A(y/P) > \lambda\}$$

with *cut-point* λ. It is obvious that two equivalent automata represent the same events.

If each of the matrices $F(s_u, x_v)$ satisfies the condition

$$(3) \qquad F(s_u, x_v) = f(s_u, x_v)\phi(s_u, x_v),$$

where f is a k-dimensional stochastic column vector and ϕ is an m-dimensional stochastic row vector, then A is termed a *finite probabilistic Mealy automaton* (or, for short, a *Mealy automaton*). The components of the vectors f and ϕ are referred to, respectively, as *transition probabilities* and *output probabilities*. By considering the condition (3), we easily obtain the following theorem.

THEOREM 1. *A finite probabilistic automaton* $A = (S, X, Y, \delta, F)$ *is a Mealy automaton if and only if every matrix* $F(s, x)$, *where* $s \in S$ *and* $x \in X$, *is of rank* 1.

Assume that, for each $s_u \in S$ and $x_v \in X$, there is a k-dimensional stochastic vector $f(s_u, x_v)$ and, for each $s_i \in S$, there is an m-dimensional stochastic vector $\phi(s_i)$ such that (1) satisfies the condition

$$(4) \qquad p_A(s_i, y_j/s_u, x_v) = f(s_i/s_u, x_v)\phi(y_j/s_i),$$

where $f(s_i/s_u, x_v)$ denotes the ith component of the vector $f(s_u, x_v)$ and $\phi(y_j/s_i)$ denotes the jth component of the vector $\phi(s_i)$. Then the finite probabilistic automaton A is termed a *finite probabilistic Moore automaton* (or, for short, a *Moore automaton*). If, furthermore, all of the vectors $\phi(s_i)$, $s_i \in S$, are coordinate vectors, then the automaton is termed a *Rabin automaton.*

Remark. The automata considered by Rabin **(5)** differ from the notion introduced above essentially only by the fact that they possess a fixed initial state (instead of an initial distribution of states). The same events are represented in Rabin automata and in the automata considered in **(5)** (where the events are represented by states), provided an initial distribution is allowed in the latter.

A Mealy automaton is said to possess a *deterministic transition function* if all of the vectors $f(s_u, x_v)$ appearing in (3) are coordinate vectors. Then the automaton is called, for short, a *d.t. Mealy automaton.* Similarly, if all of the vectors f appearing in (4) are coordinate vectors, then the automaton is called a *d.t. Moore automaton.* A finite probabilistic automaton is called a *d.t. automaton* if it is either a d.t. Mealy automaton or a d.t. Moore automaton. If in (3) (in (4)) the vectors ϕ, as well as the initial distribution δ, are also coordinate vectors, then the d.t. Mealy (Moore) automaton is an ordinary finite deterministic Mealy (Moore) automaton.

3. Equivalence theorems. We shall consider the problem of whether it suffices to have probabilistic transition functions or probabilistic output functions, i.e., whether, given a finite probabilistic automaton, one can construct an equivalent Rabin automaton or an equivalent d.t. automaton. For Rabin automata, this is always possible. For d.t. automata, such a construction is not, in general, possible. We shall first consider d.t. automata.

THEOREM 2. *Only regular events are represented in a d.t. Moore automaton.*

Proof. Let $E(A, y, \lambda)$ be an event represented in a d.t. Moore automaton $A = (S, X, Y, \delta, F)$ by the output y with cut-point λ. By the definition of a d.t. Moore automaton,

$$p_A(s^1 \ldots s^n, y^1 \ldots y^n/x^1 \ldots x^n) = \sum_{s^0 \in S} \delta(s^0) \prod_{i=1}^{n} f(s^i/s^{i-1}, x_i)\, \phi(y^i/s^i),$$

where each of the numbers $f(s^i/s^{i-1}, x^i)$ equals either 0 or 1. In particular, for each input word $x^1 \ldots x^n$ and each state $s^0 \in S$, there is exactly one sequence of states $s^1, \ldots, s^n$ such that

$$f(s^i/s^{i-1}, x^i) = 1 \qquad (i = 1, \ldots, n). \tag{5}$$

Denote $s^n = s^0x^1 \ldots x^n$. Then, for any input word P,

$$p_A(y/P) = \sum_{s^0 \in S} \delta(s^0)\, \phi(y/s^0P). \tag{6}$$

Let $S = \{s_1, \ldots, s_k\}$, and consider the event $E(i, j)$ represented by the state s_j in the deterministic automaton $A_1 = (S, X, f, s_i)$ without outputs and with the initial state s_i. Each of the events $E(i, j)$, $1 \leqslant i, j \leqslant k$, is regular. Therefore, also each of the intersections

$$E(1, i_1) \cap \ldots \cap E(k, i_k) \qquad (1 \leqslant i_j \leqslant k) \tag{7}$$

is regular. Every input word P belongs to exactly one of the intersections (7) and, by (6), $p_A(y/P)$ assumes the same value for all words P belonging to the same intersection. Thus, the event $E(A, y, \lambda)$ is the union of some of the intersections (7). This implies that the event $E(A, y, \lambda)$ is regular.

THEOREM 3. *For every d.t. Mealy automaton, there is an equivalent d.t. Moore automaton.*

Proof. Assume that $A = (S, X, Y, \delta, F)$ is a d.t. Mealy automaton such that

$$F(s_u, x_v) = f(s_u, x_v)\phi(s_u, x_v),$$

where the vectors f are k-dimensional coordinate vectors and the vectors ϕ are m-dimensional stochastic vectors. Assume that $|X| = l$. Consider the set $S_1 = S \cup S \times X$. For each $\sigma \in S_1$, we define an m-dimensional stochastic vector $\phi_1(\sigma)$ as follows:

$$\phi_1((s_u, x_v)) = \phi(s_u, x_v).$$

$\phi_1(s_u)$ may be chosen arbitrarily. For each pair (σ, x), where $\sigma \in S_1$ and $x \in X$, we define a $(kl + k)$-dimensional coordinate vector $f_1(\sigma, x)$ as follows. We assume that the elements of S_1 are ordered and thus each component of a $(kl + k)$-dimensional vector corresponds to some state in S_1. In the vector $f_1(s, x)$ the component corresponding to the state (s, x) equals 1. In the vector $f_1((s, x^1), x^2)$ the component corresponding to the state $(\alpha(f(s, x^1)), x^2)$ equals 1, where $\alpha(f(s, x^1))$ denotes the state s_u such that in the vector $f(s, x^1)$ the uth component equals 1. Finally, let δ_1 be the $(kl + k)$-dimensional stochastic vector such that the components corresponding to the states s equal the same components in the vector δ, whereas the components corresponding to the states (s, x) equal 0.

Consider the d.t. Moore automaton $B = (S_1, X, Y, \delta_1, F_1)$ where F_1 is determined by the vectors f_1 and ϕ_1. We claim that A and B are equivalent. In fact, for each state $s^0 \in S$ and each input word $x^1 \ldots x^n$, there is exactly one sequence of states $s^1, \ldots, s^n$ such that (5) is satisfied, where $f(s^i/s^{i-1}, x^i)$ denotes the component of the vector $f(s^{i-1}, x^i)$ corresponding to the state s^i. Hence, we obtain

$$p_A(y^1 \ldots y^n/x^1 \ldots x^n) = \sum_{s^0 \in S} \delta(s^0) \prod_{i=1}^{n} \phi(y^i/s^{i-1}, x^i), \tag{8}$$

where $\phi(y^i/s^{i-1}, x^i)$ denotes the component of the vector $\phi(s^{i-1}, x^i)$ corresponding to y^i. Similarly, for each $\sigma^0 \in S_1$ and each input word $x^1 \ldots x^n$, there is exactly one sequence of states $\sigma^1, \ldots, \sigma^n \in S_1$ such that

$$f_1(\sigma^i/\sigma^{i-1}, x^i) = 1 \qquad (i = 1, \ldots, n),$$

where the left side denotes the component of $f_1(\sigma^{i-1}, x^i)$ corresponding to the state σ^i. Furthermore, if $\sigma^0 = s^0 \in S$, then $\sigma^i = (s^{i-1}, x^i)$ for $i = 1, \ldots, n$, where $s^1, \ldots, s^n$ are defined as above. Therefore, by the definition of the initial distribution, δ_1, we obtain the result

$$p_B(y^1 \ldots y^n/x^1 \ldots x^n) = \sum_{\sigma^0 \in S} \delta_1(\sigma^0) \prod_{i=1}^{n} \phi_1(y^i/\sigma^i),$$

where $\delta_1(\sigma^0)$ denotes the component of δ_1 corresponding to σ^0 and $\phi_1(y^i/\sigma^i)$ denotes the component of $\phi_1(\sigma^i)$ corresponding to y^i. Hence, by (8) and the definition of the vectors ϕ_1, we obtain the result

$$p_B(y^1 \ldots y^n/x^1 \ldots x^n) = p_A(y^1 \ldots y^n/x^1 \ldots x^n).$$

Thus, Theorem 3 follows.

The next theorem is an immediate consequence of Theorems 2 and 3.

THEOREM 4. *Only regular events are represented in a d.t. automaton.*

On the other hand, it is well known **(5)** that there are Rabin automata where non-regular events can be represented. Thus, we obtain the following

THEOREM 5. *There is a Rabin automaton A such that no d.t. automaton is equivalent to A.*

It is interesting to note that Theorem 5 does not hold for infinite probabilistic automata **(2**, Theorem 9**)**.

We shall now prove that it suffices to have a probabilistic transition function in a finite probabilistic automaton.

THEOREM 6. *For any finite probabilistic automaton, there is an equivalent Rabin automaton.*

(Hence, all events represented in a finite probabilistic automaton can also be represented in a Rabin automaton.)

Proof. Let $A = (S, X, Y, \delta, F)$ be a finite probabilistic automaton, where we use our earlier notations: $|S| = k$, $|Y| = m$. Consider the set $S_1 = S \times Y$. For each $\sigma \in S_1$, we define an m-dimensional coordinate vector $\phi(\sigma)$ such that, in each $\phi((s, y))$, the component corresponding to the output y equals 1. For each pair (σ, x), where $\sigma = (s, y) \in S_1$ and $x \in X$, we define a km-dimensional stochastic vector $f(\sigma, x)$ such that

$$f((s', y')/\sigma, x) = F(s', y'/s, x),$$

where we use the same notation as before. (Thus, the vector $f((s, y), x)$ is independent of y.) Finally, we define a km-dimensional stochastic vector δ_1 arbitrarily in such a way that

$$\sum_{y \in Y} \delta_1((s, y)) = \delta(s),$$

where $\delta_1((s, y))$ denotes the component corresponding to the element $(s, y) \in S_1$ and $\delta(s)$ denotes the component corresponding to the element $s \in S$.

Consider the Rabin automaton $B = (S_1, X, Y, \delta_1, F_1)$ where F_1 is determined by the vectors f and ϕ. By the definition of F_1 and δ_1, we obtain the result

(9) $$p_B((s^1, y^1) \ldots (s^n, y^n), y^1 \ldots y^n/x^1 \ldots x^n)$$

$$= \sum_{(s^0, y^0) \in S_1} \delta_1(s^0, y^0) \prod_{i=1}^{n} F_1((s^i, y^i), y^i/(s^{i-1}, y^{i-1}), x^i)$$

$$= \sum_{s^0 \in S} \delta(s^0) \prod_{i=1}^{n} F(s^i, y^i/s^{i-1}, x^i)$$

$$= p_A(s^1 \ldots s^n, y^1 \ldots y^n/x^1 \ldots x^n).$$

On the other hand,

$$p_B((s^1, y^{i_1}) \ldots (s^n, y^{i_n}), y^1 \ldots y^n/x^1 \ldots x^n) = 0$$

if, for some j, $y^{i_j} \neq y^j$. Thus, by (9), the equation

$$p_B(y^1 \ldots y^n/x^1 \ldots x^n) = p_A(y^1 \ldots y^n/x^1 \ldots x^n)$$

follows. This completes the proof of Theorem 6.

As an immediate corollary, we obtain a result established also by Buharajev (**2**, Theorem 8):

THEOREM 7. *For any Mealy automaton, there is an equivalent Moore automaton.*

It is shown in (**4**) that non-regular events can be represented in a three-state Rabin automaton where the input alphabet consists of a single element. We shall now prove that non-regular events cannot be represented in any finite two-state probabilistic automaton whose input alphabet consists of a single element. (It is well known (cf. **5**) that non-regular events can be represented in a two-state Rabin automaton whose input alphabet consists of two elements.)

THEOREM 8. *Assume that $A = (S, X, Y, \delta, F)$ is a finite probabilistic automaton such that $|S| = 2$ and $|X| = 1$. Then only regular events can be represented in A.*

Proof. Let $|Y| = m$. Following the proof of Theorem 6, we construct a Rabin automaton B equivalent to A. Because the vectors $f((s, y), x)$ are independent of y, we conclude that there are at most two distinct vectors $f((s, y), x)$. This implies that the rank of the $2m$-dimensional stochastic matrix formed by these vectors is at most 2. Hence, the characteristic values of the matrix are real. By a result of Paz (**4**, Corollary 8), we conclude that only regular events are represented in B. Theorem 8 now follows because A and B are equivalent.

4. Realization of functions. In the theory of finite deterministic automata, one considers mappings from the set of input words into the set of output words realized by a given automaton. Several results about such mappings are known (cf. **3** and the references given there). In this section, we shall consider analogous problems for finite probabilistic automata. Considerations are restricted to Rabin automata.

For a Rabin automaton $A = (S, X, Y, \delta, F)$, we define

$$p_A(s/x^1 \dots x^n) = \sum p_A(s^1 \dots s^{n-1}s, y^1 \dots y^n/x^1 \dots x^n),$$

where the sum is taken over all $(2n - 1)$-tuples $(s^1, \dots, s^{n-1}, y^1, \dots, y^n)$. We say that a *subset* S_1 of S *represents* in A the event

$$E(A, S_1, \lambda) = \{P | \textstyle\sum_{s \in S_1} p_A(s/P) > \lambda\}$$

with cut-point λ. It is obvious that an event can be represented in a Rabin automaton by a set of states if and only if it can be represented in a Rabin automaton by an output.

Denote by X^* the free semigroup generated by X and containing the identity e, called the *empty word*. (Thus, X^* consists of all words over the alphabet X and of the empty word e. Note that in the earlier sections of this paper we have not included the empty word in our discussions.) Again, let $S = \{s_1, \dots, s_k\}$ and $X = \{x_1, \dots, x_l\}$. We define the function

$$Z_A \colon X^* \to M(1, k)$$

as follows:

$$Z_A(e) = \delta,$$

$$Z_A(P) = (p_A(s_1/P), \dots, p_A(s_k/P)), \qquad \text{for } P \neq e.$$

The function Z_A is said to be *realized* by the automaton A. Obviously, a function $Z \colon X^* \to M(1, k)$ is realized by a Rabin automaton if and only if there are k-dimensional stochastic matrices $N(x_1), \dots, N(x_l)$ such that

$$(10) \qquad Z(Px_i) = Z(P)N(x_i),$$

for all $P \in X^*$ and all i, $1 \leqslant i \leqslant l$.

An event $E \subset X^* - \{e\}$ is representable in a Rabin automaton if and only if there is a function $Z \colon X^* \to M(1, k)$ realized by a Rabin automaton such that, for some λ $(0 \leqslant \lambda < 1)$ and some k-dimensional column vector g whose components are 0's and 1's, the following condition is satisfied: $P \in E$

if and only if $Z(P)g > \lambda$, where $P \in X^* - \{e\}$. By Theorem 6, this statement is valid also as regards representability in any finite probabilistic automata. We shall now establish a criterion for a function to be realized in a Rabin automaton.

THEOREM 9. *A function $Z: X^* \to M(1, k)$ is realized by a Rabin automaton if and only if the following two conditions are satisfied:*

(i) *For all $P \in X^*$ and all $x \in X$, if*

$$Z(P) = \sum_{i=1}^{n} \gamma_i Z(P_i),$$

then also

$$Z(Px) = \sum_{i=1}^{n} \gamma_i Z(P_i x).$$

(ii) *If $Z(P_1), \ldots, Z(P_v)$, where $1 \leqslant v \leqslant k$, are linearly independent and $x \in X$, then there are k-dimensional row vectors $Z_{v+1}, \ldots, Z_k$ and $U_{v+1}, \ldots, U_k$ such that*

$$(11) \qquad \left\|\begin{matrix} Z(P_1) \\ \cdot \\ \cdot \\ \cdot \\ Z(P_v) \\ Z_{v+1} \\ \cdot \\ \cdot \\ \cdot \\ Z_k \end{matrix}\right\|^{-1} \left\|\begin{matrix} Z(P_1 x) \\ \cdot \\ \cdot \\ \cdot \\ Z(P_v x) \\ U_{v+1} \\ \cdot \\ \cdot \\ \cdot \\ U_k \end{matrix}\right\|$$

is a stochastic $k \times k$ matrix.

Proof. We shall first prove the "only if" part. Assume that, for each $x_i \in X$, there is a $k \times k$ stochastic matrix $N(x_i)$ such that (10) is satisfied. Condition (i) follows, by the distributive law for matrix multiplication. Given linearly independent vectors $Z(P_1), \ldots, Z(P_v)$, we choose k-dimensional row vectors Z_i, $v + 1 \leqslant i \leqslant k$, such that all of the vectors Z are linearly independent. For $x \in X$, let $U_i = Z_i N(x)$, $v + 1 \leqslant i \leqslant k$. Then, by (10),

$$\left\|\begin{matrix} Z(P_1) \\ \cdot \\ \cdot \\ \cdot \\ Z(P_v) \\ Z_{v+1} \\ \cdot \\ \cdot \\ \cdot \\ Z_k \end{matrix}\right\| \cdot N(x) = \left\|\begin{matrix} Z(P_1 x) \\ \cdot \\ \cdot \\ \cdot \\ Z(P_v x) \\ U_{v+1} \\ \cdot \\ \cdot \\ \cdot \\ U_k \end{matrix}\right\|$$

and hence, by the assumption concerning $N(x)$, the matrix (11) is a $k \times k$ stochastic matrix. Thus, also condition (ii) is satisfied.

For the "if" part, assume that (i) and (ii) are satisfied. We choose elements $P_1, \ldots, P_v \in X^*$, $1 \leqslant v \leqslant k$, such that $Z(P_1), \ldots, Z(P_v)$ are linearly independent and, for no elements $Q_1, \ldots, Q_{v+1} \in X^*$, $Z(Q_1), \ldots, Z(Q_{v+1})$ are linearly independent. This implies that, for any $P \in X^*$, there are real numbers γ_j, $1 \leqslant j \leqslant v$, such that

$$Z(P) = \sum_{j=1}^{v} \gamma_j Z(P_j). \tag{12}$$

By (ii) there is, for $x \in X$, a stochastic matrix $N(x)$ (not necessarily unique) such that

$$Z(P_j)N(x) = Z(P_j x) \qquad (j = 1, \ldots, v). \tag{13}$$

Let $P \in X^*$ and $x \in X$ be arbitrary. Because the condition (i) is satisfied, we obtain, by (12),

$$Z(Px) = \sum_{j=1}^{v} \gamma_j Z(P_j x)$$

and thus, by (13), equation (10) follows. Hence, we have established Theorem 9.

Remark 1. The condition (ii) is satisfied if and only if it is satisfied for the maximal number v of linearly independent vectors $Z(P_1), \ldots, Z(P_v)$. If $v = k$, then the stochastic matrices $N(x)$ defining the automaton are unique.

Remark 2. For an analogous result concerning deterministic automata, condition (i) is both necessary and sufficient. In fact, consider functions $Z\colon X^* \to M'(1, k)$, where $M'(1, k)$ denotes the set of k-dimensional coordinate (row) vectors. A function Z is realized by a finite deterministic automaton if and only if, for all $P_1, P_2 \in X^*$ and $x \in X$, the equation $Z(P_1) = Z(P_2)$ implies the equation $Z(P_1 x) = Z(P_2 x)$. Clearly, condition (i) and this condition are equivalent in the deterministic case.

Remark 3. Buharajev **(1)** has mentioned without proof the result that two conditions (i)′ and (ii)′ are necessary and sufficient for the realizability of functions Z. Condition (i)′ is essentially the same as condition (i). Condition (ii)′ is as follows: If $\sum_{i=1}^{n} \gamma_i Z(P_i)$ is a stochastic vector, then also $\sum_{i=1}^{n} \gamma_i Z(P_i Q)$ is a stochastic vector, for any $Q \in X^*$. If one considers mappings $Z\colon X^* \to M(1, k)$ such that there are k linearly independent vectors $Z(P)$, then Buharajev's result follows from Theorem 9 (because coordinate vectors can be expressed as linear combinations of the vectors $Z(P)$). For mappings Z possessing less than k linearly independent images $Z(P)$, it seems a plausible conjecture that condition (i) (or (i)′) alone is sufficient for the realizability of Z.

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