THE FC-CHAIN OF A GROUP

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1. Introduction. Baer [**2**] and Neumann [**5**] have discussed groups in which there is a limitation on the number of conjugates which an element may have. For a given group G, let H_1 be the set of all elements of G which have only a finite number of conjugates in G, let H_2 be the set of those elements of G, the conjugates of each of which lie in only a finite number of cosets of H_1 in G; and in this fashion define H_3, H_4, ... We shall show that the H_i are strictly characteristic subgroups of G. The result of Neumann which states that the derivative of G is periodic if $G = H_1$ (that is, if G is a so-called FC-group), and that, in this case, the periodic elements of G form a subgroup reappears in the form that the division hull of H_i in H_{i+1} is a subgroup L_{i+1} such that H_{i+1}/L_{i+1} is abelian. The commutator quotient $H_i \div H_{i+1}$ turns out to be the cross-cut of some collection of subgroups of finite index in G, generalizing a result of Baer [**2**] on the centralizer of H_1 in G. Hall [**6**, p. 114] has proved a strict inclusion theorem on the intersections of some subgroups with the ascending central series. A related result is established for the FC-chain $\{H_i\}$. The concept of FC-nilpotency is introduced ($G = H_n$ for some n), and the relation of FC-nilpotency of a factor group of G to the nilpotency of G itself is discussed. We shall prove that the group of automorphisms of a non-trivial, complete centreless group has no non-trivial FC-chain.

2. The FC-chain. Let G be a non-trivial group, and let $H_1 = H_1\ (G)$ be the set of all $g \in G$, each of which has only a finite number of conjugates in G. By Baer [**2**], H_1 is a characteristic subgroup of G. Indeed it is more; for, let f be an endomorphism of G where $f(G) = G$, and let $x \in H_1$ have the property that $f(x)$ has more than a finite number of distinct conjugates in G. If $\{r_i^{-1} f(x) r_i\}$ $(i = 1, 2, 3, \ldots)$ is a countable subset of the set of distinct conjugates of $f(x)$, the fact that $f(G) = G$ implies the existence of a set $\{s_i\}$, $s_i \in G$, $f(s_i) = r_i$, so that the $f(s_i^{-1} x s_i)$ are all different, whence the $s_i^{-1} x s_i$ are distinct. But this is a contradiction, so that $f(H_1) \subset H_1$, and H_1 is strictly characteristic.

Let $H_0 = H_0\ (G)$ be the subgroup of G consisting of e, the identity element of G, alone. Suppose that $H_n = H_n(G)$ has been defined as a suitable normal subgroup of G. Then form $H_1\ (G/H_n(G))$ and construct its complete inverse image $H_{n+1}(G)$ in G under the natural mapping with kernel $H_n(G)$ which carries G onto $G/H_n(G)$. It is clear that $H_{n+1}(G)$ is a normal subgroup of G and that $H_{n+1}(G)/H_n(G)$ is isomorphic to $H_1\ (G/H_n(G))$. Thus, inductively, we have fashioned the FC-*chain* of normal subgroups $\{H_j(G)\}$ $(j = 0, 1, 2, \ldots)$ of a group

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G. For all such j, $H_j(G) \subset H_{j+1}(G)$. Moreover, each $H_i = H_i(G)$ is strictly characteristic in G. This statement is true for $i = 0$, and, by the above, it is true for $i = 1$. Now let f be any endomorphism of G for which $f(G) = G$, and suppose that $H_i(G)$ is strictly characteristic in G. If h is an element of $H_{i+1}(G)$, then $hH_i(G)$, as an element of $G/H_i(G)$, has only a finite number of conjugates in the latter group. If, now, $f(h)H_i(G)$ has an infinite number of distinct conjugates, choose a countable set of these, each of the form $z_j^{-1} f(h) z_j H_i$, where each $z_j \in G$ $(j = 1, 2, 3, \ldots)$. Construct elements w_j with $f(w_j) = z_j$. Then the $f(w_j^{-1} h w_j) H_i$ are all distinct. If there exist distinct indices j and k for which

$$(w_j^{-1} h w_j) H_i = (w_k^{-1} h w_k) H_i,$$

then there exists $h' \in H_i$ with

$$f(w_j^{-1} h w_j) = f(w_k^{-1} h w_k) f(h'),$$

where $f(h') \in H_i$ since $f(H_i) \subset H_i$, by the induction assumption. However this implies that

$$f(w_j^{-1} h w_j) H_i = f(w_k^{-1} h w_k) H_i,$$

a contradiction. Thus, $f(h)H_i$ has only a finite number of conjugates in G/H_i, so that $f(h) \in H_{i+1}$, and the latter subgroup is strictly characteristic.

Let $Z_i = Z_i(G)$ be the ith member of the ascending central series [**6**] of G. Then $Z_i(G) \subset H_i(G)$; for, proceeding inductively, it is clear that $Z_1(G)$, the centre of G, is included in $H_1(G)$. Suppose that $Z_i \subset H_i$ and[1] that $x \in Z_{i+1}$. Then the coset xZ_i is in the centre $Z_1(G/Z_i)$ of the group of cosets G/Z_i. Since $Z_i \subset H_i$,

$$xH_i \in Z_1(G/H_i) \subset H_1(G/H_i) = H_{i+1}/H_i,$$

so that $x \in H_{i+1}$.

Note that $H_i \subset Z_j$ for $i > j$ implies[1] that $Z_i \subset H_i \subset Z_j \subset H_j$, and $Z_i = Z_j = H_i = H_j$, whence both the FC-chain and the ascending central series break off with the same subgroup $Z_j = H_j$. The possibility of course remains that they have so broken off at an index $k < j$. When both the FC-chain and the ascending central series of a group G terminate with the same subgroup $H_j = Z_j$, we say that G has *mutual truncation at index* $\leqslant j$. We replace $\leqslant$ by $=$ if j is the best possible index.

If $x, y \in H_{i+1}$ and if $x \equiv y \bmod Z_{i+1}$, then in the group G/H_i, the elements xH_i and yH_i have the same (finite) number of conjugates.

If $G = H_n(G)$, $G \neq H_{n-1}(G)$, for some positive integer n, we say that G is FC-nilpotent of FC-class n. Hence if G is nilpotent of class m ($G = Z_m(G)$, $G \neq Z_{m-1}(G)$), then G is FC-nilpotent of FC-class $n \leqslant m$.

3. The division hulls. Let K be a subgroup of a group G. By the *division hull* of K in G, $D(K; G)$, we mean the set of all $x \in G$ for which there exist

[1]The author is indebted to the referee for strengthening the argument at this point.

[2]See note 1.

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positive integers $n = n(x)$ with $x^n \in K$. If G is abelian or if the set of all $xyx^{-1}y^{-1}$, where $x, y \in D(K; G)$, is included in K, then $D(K; G)$ is a subgroup of G; but, in general, $D(K; G)$ need not be a subgroup of G. If $D(K; G)$ is a subgroup, and if K is admissible under an endomorphism f ($f(K) \subset K$), then $D(K; G)$ is also admissible under f.

The following is easy to prove: If K is a normal subgroup of G and if A^* and B^* are subsets of G/K with respective complete inverse images A and B in G and if $A^* = D(B^*; G/K)$ then $A = D(B; G)$. Likewise in the immediate category is that K normal in G and G/K abelian imply that $D(K; G)$ is a subgroup of G. If every finitely generated subgroup K of G has the property that $D(K; G)$ is a supgroup of G, then $D(L; G)$ is a subgroup of G for every subgroup L of G.

We use the commutator notation of [**6**]. For instance, G' shall mean (G, G), the subgroup of G generated by the commutators of G, the derivative of G; $G'' = (G', G')$. From one of the above results, $D(G'; G)$ is always a subgroup of G. It follows that if K is any normal subgroup of G' for which G'/K is a periodic group, then $D(K; G) = D(G'; G)$. In particular, if G' is a periodic group, $D(G'; G) = P(G)$, the set of periodic elements of G (as we can see by taking K to be the trivial subgroup of G). $P(G)$ is thus [**5**] a subgroup of G whenever $G' \subset P(G)$.

4. Subgroups of H_{i+1}.

LEMMA 1. *$D(H_i; H_{i+1})$ is a normal subgroup L_{i+1} of H_{i+1} ($i = 0, 1, 2, \ldots$).*

Proof. $H_{i+1}/H_i = H_1$ (G/H_i), an FC-group. By a result due to Neumann [**5**], the set of periodic elements $P = P(H_{i+1}/H_i)$ of H_{i+1}/H_i is a normal subgroup of the latter group. However, $D(H_i; H_{i+1})$ is the complete inverse image in H_{i+1} (under the natural mapping of H_{i+1} onto H_{i+1}/H_i) of P. Consequently, L_{i+1} is a normal subgroup of H_{i+1}.

COROLLARY. *L_{i+1} is a strictly characteristic subgroup of G.*

Proof. If $x \in L_{i+1}$ and if f is an endomorphism of G onto G, then $f(x) \in H_{i+1}$ since H_{i+1} is strictly characteristic. There exists a positive integer n such that $x^n \in H_i$. Hence $f(x^n) = (f(x))^n \in H_i$, since the latter is strictly characteristic. Thus $f(x) \in L_{i+1}$.

Neumann [**5**] has also proved that if G is an FC-group, then $G' \subset P(G)$, the subgroup of periodic elements of G. It follows that, for every FC-group G, $P(G) = D(G'; G)$. Since H'_{i+1} is included in the complete inverse image of $(H_{i+1}/H_i)'$ in H_{i+1}, (under the natural mapping of H_{i+1} onto H_{i+1}/H_i), $H'_{i+1} \subset L_{i+1}$ so that H_{i+1}/L_{i+1} is abelian. If $x \in D(L_{i+1}; H_{i+1})$ then $x^m \in L_{i+1}$ and $x^{mn} \in H_i$ for suitable positive integers m and n. Hence $D(L_{i+1}; H_{i+1}) \subset L_{i+1}$ so that the group H_{i+1}/L_{i+1} is not only abelian but also *torsion-free* in the sense that it has no periodic elements other than its unity.

If G is FC-nilpotent of FC-class n, $n \geqslant 1$, then the fact that H_n/L_n is abelian shows that $G' \subset L_n$.

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Let ϕ_i be the natural homomorphism of G onto G/H_i. For a subgroup S^* of G/H_i, $\phi_i^{-1}(S^*)$ shall mean the complete inverse image in G of S^* under ϕ_i.

LEMMA 2. *H_{i+1}/L_{i+1} is the trivial group or a direct sum of copies of the group of rationals if and only if to each ordered pair (x, m), where $x \in H_{i+1}$ and m is a positive integer, there corresponds an ordered pair (y, n), where $y \in H_{i+1}$ and n is a positive integer, such that $(xy^m)^n \in H_i$.*

Proof. H_{i+1}/L_{i+1} has the required form if and only if it is *complete* in the (abelian group) sense that

$$xL_{i+1} \in H_{i+1}/L_{i+1} \qquad (x \in H_{i+1})$$

implies, for each positive integer m, the existence of $z = z(m) \in H_{i+1}$ with $(zL_{i+1})^m = xL_{i+1}$. If we let $z^{-1} = y$, the result is immediate.

Let $J_{i+1} = D(H'_{i+1} \cap H_i; H_{i+1})$. $x \in J_{i+1}$ implies $x \in H_{i+1}$ and the existence of a positive integer n for which $x^n \in H'_{i+1} \cap H_i$, so that $x^n \in H_i$ and $x \in L_{i+1}$. But $x^n \in H'_{i+1}$ inplies that $x \in D(H'_{i+1}; H_{i+1})$, and $L_{i+1} \subset H_{i+1}$ implies that

$$D(H'_{i+1}; L_{i+1}) \subset D(H'_{i+1}; H_{i+1}),$$

so that both J_{i+1} and $D(H'_{i+1}; L_{i+1})$ are subsets of $D(H'_{i+1}; H_{i+1})$. Conversely, if $x \in D(H'_{i+1}; H_{i+1})$, there exists a positive integer m such that $x^m \in H'_{i+1} \subset L_{i+1}$, and consequently there exists a positive integer n for which $(x^m)^n \in H_i$. This places x in L_{i+1}, hence in $D(H'_{i+1}; L_{i+1})$. Since $x^{mn} \in H'_{i+1}$, $x \in J_{i+1}$, and we have proved that

$$J_{i+1} \equiv D(H'_{i+1} \cap H_i; H_{i+1}) = D(H'_{i+1}; L_{i+1}) = D(H'_{i+1}; H_{i+1}).$$

It is clear that J_{i+1} is a strictly characteristic subgroup of G and that

$$L'_{i+1} \subset H'_{i+1} \subset J_{i+1} \subset L_{i+1} \subset H_{i+1}$$

so that, for instance, $D(J_{i+1}; H_{i+1})$ is a subgroup of G. It is also immediate that the sequences $\{L_{i+1}\}$ and $\{J_{i+1}\}$ are both ascending with i.

5. The commutator quotients. Let S and T be normal subgroups of a group G. Let $S \div T$ be the set of all $x \in G$ which have the property that $(t, x) \in S$ for every $t \in T$. This set is called [**1**] the *commutator quotient* of S by T and is a normal subgroup of G. Let f be an endomorphism of G for which $f(T) = T$ and $f(S) \subset S$. For $x \in S \div T$ and $t \in T$,

$$tf(x)\, t^{-1} f(x^{-1}) = f(uxu^{-1}\, x^{-1}),$$

where $u \in T$; and since $uxu^{-1}\, x^{-1} \in S$, $tf(x)t^{-1} f(x^{-1})$ is likewise in S so that $f(x) \in S \div T$. We have proved that *if S and T are normal subgroups of a group G and if f is any endomorphism of G for which S is admissible and $f(T) = T$, then $S \div T$ is admissible under f.* Moreover it can be shown that $S \div T$ is a characteristic subrgoup of G if both S and T are. Well known is the fact [**1**] that

$S \div T \supset S$ for normal subgroups S and T of G. If S, T and N are normal subgroups of G, it is easy to prove [**1**] that the following are equivalent:

(a) $(T, N) \subset S$; (b) $N \subset S \div T$; (c) $T \subset S \div N$.

The fact that $(N, S \div N) \subset S$ for normal subgroups S and N of G shows that $N \subset S \div (S \div N)$, by the equivalence of (a) and (c). Useful is the result [**1**] that

5.1 $$(N \div G)/N = Z_1\,(G/N)$$

for every normal subgroup N of G.

Since $Z_i \div G = Z_{i+1}$, Z_{i+1} is maximum with respect to being a normal subgroup X for which $(G, X) \subset Z_i$. One would like to have a similar result for the FC-chain, but the facts are otherwise. If we define W_{i+1} by $W_{i+1}\,(G) = H_i(G) \div G$ $(i = 0, 1, 2, \ldots)$ it is easy to see that W_{i+1}, maximum with respect to the property of being a normal subgroup X of G for which $(G, X) \subset H_i$, can be represented by $W_{i+1}\,(G) = \phi_i^{-1}\,(Z_1\,(G/H_i\,(G)))$, upon application of 5.1. W_{i+1}/H_i is abelian, whence $W'_{i+1} \subset H_i \subset W_{i+1}$. Since

$$Z_1\,(G/H_i) \subset H_1\,(G/H_i) \qquad W_{i+1} \subset H_{i+1}.$$

Since

$$(Z_{i+1}, G) \subset Z_i \subset H_i \qquad Z_{i+1} \subset W_{i+1}.$$

It is clear that the W_{i+1} form an ascending chain of subgroups of G which "intertwines" with the FC-chain, where $W_1\,(G) = Z_1\,(G)$ and each W_{i+1} is a strictly characteristic subgroup of G. The last remark follows from the fact that $W_{i+1} = H_i \div G$, and that if f is an endomorphism on G onto G, H_i is admissible, so that, by an earlier remark on the admissibility of the commutator quotient, so is W_{i+1} admissible. Note that $H_i \div W_{i+1} = G$, since $W_{i+1} = H_i \div G$ implies $G \subset H_i \div W_{i+1}$.

Let us define $V_{i+1}(G) = H_i\,(G) \div H_{i+1}\,(G)$ $(i = 0, 1, 2, \ldots)$. It follows that $H_{i+1} \subset H_i \div V_{i+1}$. By the symbol $C(H < G)$ for a subgroup H of G we mean [**2**] the centralizer of H in G.

THEOREM 1. *V_{i+1} is maximum with respect to the property of being a normal subgroup X of G for which*

$$(H_{i+1}, X) \subset H_i; \quad W_{i+1} \subset V_{i+1}; \quad V_{i+1} = \phi_i^{-1}(C(H_{i+1}/H_i < G/H_i));$$

and V_{i+1} is the cross-cut of some collection of finite-indexed subgroups of G.

Proof. The first statement derives from the definition of commutator quotient. As a function on the cartesian square of the lattice of normal subgroups of the group G into that same lattice, $X \div Y$ is monotonically increasing in X and monotonically decreasing in Y. Since $V_{i+1} = H_i \div H_{i+1}$ and $W_{i+1} = H_i \div G$, $W_{i+1} \subset V_{i+1}$. As for the third statement, $y \in V_{i+1}$ if and only if $(y, h) \in H_i$ for every $h \in H_{i+1}$. But this is equivalent to the commuting of $\phi_i(y)$ with every $\phi_i(h)$. Since, however, the $\phi_i(h)$ range over all of H_{i+1}/H_i, the third statement is established. For the last statement, we recall that Baer [2] has showed that, for any group K, $C(H_1\ (K) < K)$ can be represented as the cross-cut of some collection of finite-indexed subgroups of K. Thus

$$C(H_{i+1}/H_i < G/H_i) = \bigcap_\alpha N_\alpha^*,$$

where each N_α^* is a normal subgroup of finite index in G/H_i. Then

$$\phi_i^{-1}C(H_{i+1}/H_i < G/H_i) = \bigcap_\alpha \phi_i^{-1} N_\alpha^*.$$

Each $\phi_i^{-1}\ N_\alpha^* = N_\alpha$ is a normal subgroup of G. Since

$$G/N_\alpha \cong (G/H_i)/(N_\alpha/H_i) = (G/H_i)/N_\alpha^*,$$

each N_α has a finite index in G, and the proof is complete.

Since $(H_i . H_i) \subset (V_{i+1}, H_{i+1}) \subset H_i$, $H_i/(V_{i+1}, H_{i+1})$ is abelian, and, by making the normal subgroup X in $(X, H_{i+1}) \subset H_i$ as large as possible, (X, H_{i+1}) itself is moved "above" the derivative H'_i. Likewise $H_i/(G, W_{i+1})$ is abelian. There is, however, a point of dissimilarity between W_{i+1} and V_{i+1}. For normal subgroups X satisfying $W_{i+1} = H_i \div X$, $X = G$ is the obvious maximum which can be obtained. On the other hand, with $V_{i+1} = H_i \div X$, the maximum which X takes on is

$$M_{i+1} = H_i \div V_{i+1} \supset H_i.$$

For $y \in G$, $yH_i = \phi_i(y)$ commutes with every $\phi_i(v) \in V_{i+1}/H_i$ if and only if $y \in M_{i+1}$. Hence

$$M_{i+1} = \phi_i^{-1}\ (C(V_{i+1}/H_i < G/H_i)).$$

Likewise, it is easy to show that

$$V_{i+1} = \phi_i^{-1}\ (C(M_{i+1}H/_i < G/H_i)).$$

Thus, for normal subgroups X of G satisfying $V_{i+1} = H_i \div X$, the maximum is obtained by, essentially, forming centralizers twice from H_{i+1}/H_i.

THEOREM 2. *$W_{i+1} \cap Z_{i+1} = (H_i \cap Z_j) \div G$, so that $W_{i+1} \cap Z_{j+1}$ is maximum with respect to being a normal subgroup X of G for which $(G, X) \subset H_i \cap Z_j$, and*

$$(W_{i+1} \cap Z_{j+1})/(H_i \cap Z_j) = Z_1\ (G/H_i \cap Z_j).$$

Proof. $x \in W_{i+1} \cap Z_{j+1}$ implies that $\phi_i(x)$ and $\phi_i(g)$ commute for every $g \in G$ and that $xgx^{-1}\ g^{-1} \in Z_j$, since $(G, Z_{j+1}) \subset Z_j$. Thus

$$W_{i+1} \cap Z_{j+1} \subset (H_i \cap Z_j) \div G.$$

Conversely, $(H_i \cap Z_j) \div G \subset H_i \div G,\ Z_j \div G$. But $H_i \div G = W_{i+1}$ and $Z_j \div G = Z_{j+1}$, so that the first statement of the theorem follows. Apply 5.1 as before.

COROLLARY. (a) *If $Z_j \subset H_i$, then $Z_{j+k} \subset W_{i+k}$* $(k = 0, 1, 2, \ldots)$. (b) *If $H_i \subset Z_j$, then $W_{i+1} \subset Z_{j+1}$.* (c) *If $H_i = Z_j$, then $W_{i+1} = Z_{j+1}$.* (d) *If each*

$W_i = H_i$ $(i = 1, 2, 3, \ldots)$ then each $H_i = Z_i$ (whence each H_{i+1}/H_i is abelian, and G is FC*-nilpotent under these conditions if and only if G is nilpotent).*

6. A strict inclusion theorem. In the case of the ascending central series, Hall [**6,** p. 114] has proved a strict inclusion theorem. In Theorem 3 below, we shall obtain a similar result for the FC-series.

LEMMA 3. *Let N be a normal subgroup of G for which $N \subset W_{i+1}$ and $N \not\subset H_i$, where $i \geqslant 1$. Then the following inclusions are strict:*

$$N \supset N \cap H_i \supset N \cap H_{i-1}.$$

Proof. $(G, N) \subset N \cap (G, W_{i+1}) \subset N \cap H_i$. If $N \cap H_i \subset N \cap H_{i-1}$, then $(G, N) \subset N \cap H_i$ would imply $(G, N) \subset H_{i-1}$. By the maximum character of W_i, $N \subset W_i \subset H_i$, a contradiction, so that the inclusion $N \cap H_i \supset N \cap H_{i-1}$ is strict. Also, if $N = N \cap H_i$, then $N \subset H_i$, a contradiction, so that the inclusion $N \supset N \cap H_i$ is strict.

THEOREM 3. *If $Z_{i+1} \not\subset H_i$ then the following inclusions are strict:*

$$Z_{i+1} \supset Z_{i+1} \cap H_i \supset Z_{i+1} \cap H_{i-1} \supset Z_{i+1} \cap H_{i-2} \supset \ldots \supset Z_{i+1} \cap H_1 \supset (e),$$

where e is the identity of G.

Proof. Taking N in Lemma 3 to be Z_{i+1}, we have

$$Z_{i+1} \supset Z_{i+1} \cap H_i \supset Z_{i+1} \cap H_{i-1}$$

with strict inclusions. Since $Z_{i+1} \not\subset H_i$, $Z_{i+1-k} \not\subset H_{i-k}$ (by Corollary (a) of Theorem 2), where $k = 1, 2, 3, \ldots, i$. Suppose that the inclusion

$$Z_{i+1} \cap H_{i+1-k} \supset Z_{i+1} \cap H_{i-k}$$

is strict. Take N in Lemma 3 to be Z_{i+1-k}. Then

$$Z_{i+1-k} \supset Z_{i+1-k} \cap H_{i-k} \supset Z_{i+1-k} \cap H_{i-k-1}$$

with strict inclusions. But $Z_{i+1} \supset Z_{i+1-k}$, so that if

$$Z_{i+1} \cap H_{i-k} = Z_{i+1} \cap H_{i-k-1}$$

then $Z_{i+1} \cap H_{i-k} \subset H_{i-k-1}$, $Z_{i+1-k} \cap H_{i-k} \subset H_{i-k-1}$, and

$$Z_{i+1-k} \cap H_{i-k} \subset Z_{i+1-k} \cap H_{i-k-1},$$

a contradiction with the above strict inclusion. Hence

$$Z_{i+1} \cap H_{i-k} \supset Z_{i+1} \cap H_{i-k-1}$$

with strict inclusion, and the result is established by induction.

We can define for each ordinal α a subgroup H_α of G as follows: H_1 is defined as above. If α is not a limit ordinal, let $\alpha(-)$ be the predecessor of α. If $H_{\alpha(-)}$

is defined, then define H_α by $H_\alpha/H_{\alpha(-)} = H_1\ (G/H_{\alpha(-)})$. If α is a limit ordinal, let

$$H_\alpha = \bigcup_{\beta<\alpha} H_\beta,$$

the set-theoretic union of the H_β. With appropriate but entirely trivial[3] modifications, the prior statements of this paper can be adapted for this extended FC-chain. Similar modifications can be made throughout the remainder of the paper, but these latter are not of such uniform simplicity. Since a detailed discussion at this time of the properties of the extended FC-chain would obscure the central issues, we shall not return to this point in the present work.

7. FC-nilpotency.

LEMMA 4. $$\phi_i^{-1}\ (H_k\ (G/H_i(G)) = H_{i+k}(G) \qquad (k = 0, 1, 2, \ldots).$$

Proof. We use induction on k. For $k = 0$, $\phi_i^{-1}(e^*) = H_i(G)$ (where e^* is the identity of G/H_i), so that the result holds for $k = 0$. $\phi_i^{-1}\ (H_1\ (G/H_i)) = H_{i+1}$, so that the result holds also for $k = 1$. Let us now assume its validity for k. Then $H_{i+k}\ (G)/H_i(G)$ is $H_k\ (G/H_i(G))$. Let Φ_k be the natural mapping on $G/H_i(G)$ onto $G/H_{i+k}(G)$ with kernel $H_k(G/H_i(G)) = H_{i+k}(G)/H_i(G)$.

$$\Phi_k^{-1}(H_1(G/H_{i+k}(G))) = \Phi_k^{-1}\,\phi_{i+k}(H_{i+k+1}),$$

since the case $k = 1$ has been established. But

$$\begin{aligned}\Phi_k^{-1}\,\phi_{i+k}(H_{i+1+k}) &= \Phi_k^{-1}(H_{i+k+1}/H_{i+k})\\ &= \Phi_k^{-1}((H_{i+k+1}/H_i)/(H_{i+k}/H_i)) = H_{i+k+1}/H_i.\end{aligned}$$

However,

$$\begin{aligned}\Phi_k^{-1}(H_1(G/H_{i+k}(G))) &= \Phi_k^{-1}(H_1((G/H_i(G))/(H_{i+k}(G)/H_i(G))))\\ &= \Phi_k^{-1}(H_1((G/H_i(G))/H_k(G/H_i(G))))\\ &= \Phi_k^{-1}(H_{k+1}(G/H_i(G))/H_k(G/H_i(G)))\\ &= H_{k+1}(G/H_i(G)),\end{aligned}$$

and the result is established.

LEMMA 5. *Let Θ be the natural map of G onto G/N where N is a normal subgroup of G. Then $\Theta^{-1}\,H_k(G/N) \supset H_k\ (G)$ $(k = 0, 1, 2, \ldots)$.*

Proof. For $k = 0$, the result is obvious. $H_1(G/N)$ is an FC-group so that $\Theta^{-1}H_1(G/N) \supset H_1(G)$. Now suppose that $R_k = \Theta^{-1}H_k(G/N) \supset H_k(G)$.

$$\begin{aligned}H_1(G/R_k) &\cong H_1((G/N)/(R_k/N)) = H_1((G/N)/H_k(G/N))\\ &\cong H_{k+1}(G/N)/H_k(G/N) \cong (R_{k+1}/N)/(R_k/N)\\ &\cong R_{k+1}/R_k \cong (R_{k+1}/H_k(G))/(R_k/H_k(G)).\end{aligned}$$

But

$$H_1(G/R_k) \cong H_1((G/H_k(G))/(R_k/H_k(G)))$$

[3]See note 1.

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so that the latter group is isomorphic to

$$(R_{k+1}/H_k(G))/(R_k/H_k(G)).$$

Hence

$$R_{k+1}/H_k(G) \supset H_1(G/H_k(G)) \cong H_{k+1}(G)/H_k(G),$$

and $R_{k+1} \supset H_{k+1}(G)$, so that the proof is complete.

THEOREM 4. *Let N be a normal subgroup of a group G such that* (1) $N \subset H_n(G)$ *and* (2) *there exists a positive integer k for which G/N is* FC-*nilpotent of* FC-*class k. Then G is* FC-*nilpotent of* FC-*class* $\leqslant n + k$.

Proof. $H_k(G/N) = G/N$.

$$G/H_n(G) \cong (G/N)/(H_n(G)/N);$$

and

$$H_k(G/H_n(G)) = H_{n+k}(G)/H_n(G),$$

by Lemma 4. Hence

$$H_k((G/N)/H_n(G)/N)) \cong (H_{n+k}(G)/N)/(H_n(G)/N).$$

By Lemma 5 (taking G/N for G and $H_n(G)/N$ for N),

$$H_{n+k}(G)/N \supset H_k(G/N) = G/N.$$

Hence $H_{n+k}(G) = G$.

COROLLARY 1. *If $G/Z_n(G)$ is* FC-*nilpotent of* FC-*class k, then $G = H_{n+k}(G)$.*

COROLLARY 2. *If $W_n(G)$ has finite index in G, then $G = H_n(G)$.*

Proof. For $n = 1$, $G/W_1 = G/Z_1$. Since G/Z_1, a finite group, is isomorphic to the group of inner automorphisms [4] of G, there are only a finite number of inner automorphisms of G, and G is an FC-group. For $n > 1$,

$$G/W_n \cong (G/H_{n-1})/(W_n/H_{n-1}).$$

Since $W_n/H_{n-1} = Z_1(G/H_{n-1})$, G/H_{n-1} is an FC-group, by the argument employed for $n = 1$. By the theorem, G is FC-nilpotent of FC-class $\leqslant n$.

COROLLARY 3. *If $G' \subset H_n(G)$ for some non-negative integer n, then G is* FC-*nilpotent of* FC-*class* $\leqslant n + 1$.

Note that if G is FC-nilpotent of FC-class k, then G/N is FC-nilpotent of FC-class $\leqslant k$, where N is a normal subgroup of G. For, by Lemma 5,

$$\Theta^{-1}(H_k(G/N)) \supset H_k(G) = G,$$

so that $H_k(G/N) = G/N$. Immediate is

COROLLARY 4. *Let $N \subset H_n(G)$ where N is a normal subgroup of G. Let G be* FC-*nilpotent of* FC-*class t so that G/N is* FC-*nilpotent of* FC-*class k. Then* $k \leqslant t \leqslant k + n$.

8. The FC-chain of a "large" normal subgroup.

THEOREM 5. *Let K be a normal subgroup of finite index in G for which $H_i(G) \subset K (i = 0, 1, 2, \ldots)$. Then $H_i(K) = H_i(G)$ for all such i.*

Proof. Clearly $H_1(K) \supset H_1(G)$. For $x \in H_1(K)$, there exist a finite number of conjugates of x in K. Let the t_i $(i = 1, 2, 3, \ldots, n)$ be the set of representatives of the cosets of K in G. Let g be any element of G. Then there exist $h \in K$ and a positive integer $i \leqslant n$ such that $g = ht_i$, whence $g^{-1}xg = t_i^{-1}(h^{-1}xh)t_i$. There are only a finite number of possibilities for the $h^{-1}xh$ since $x \in H_1(K)$ and $h \in K$. Hence there are only a finite number of $g^{-1}xg$ for fixed $x \in H_1(K)$. Thus $x \in H_1(G)$, and $H_1(K) = H_1(G)$.

Now suppose that $H_i(K) = H_i(G)$. Since G/K is a finite group, the index of $K/H_i(G)$ in $G/H_i(G)$ is finite. Since $K \supset H_{i+1}(G)$,

$$K/H_i(G) \supset H_{i+1}(G)/H_i(G), \qquad H_1(K/H_i(G)) = H_1(G/H_i(G))$$

by the above argument on H_1. Then

$$H_{i+1}(G) = \phi_i^{-1}(H_1(K/H_i(G))) = \phi_i^{-1}(H_1(K/H_i(K))),$$

since $H_i(K) = H_i(G)$. Since $H_{i+1}(G) \subset K$, it follows that $H_{i+1}(K) = H_{i+1}(G)$.

COROLLARY. *Let G be an extension of an FC-nilpotent group K by a finite, non-trivial group F. Then there exists a positive integer i for which $H_i(G) \not\subset K$.*

Proof. If each $H_j(G) \subset K$ $(j = 1, 2, 3, \ldots)$ then, by the theorem, each $H_j(G) = H_j(K)$. In particular, $H_n(G) = H_n(K) = K$, where n is the FC-class of K. But then

$$H_1(G/K) = H_1(G/H_n(G)) = H_{n+1}(G)/H_n(G).$$

Since $H_1(G/K) = F$ is a non-trivial group, $H_{n+1}(G) \neq H_n(G) = K$. But $H_{n+1}(G) = H_{n+1}(K)$, by the theorem, and $H_{n+1}(K) = H_n(K) = K$, a contradiction.

9. Groups for which H_1 is trivial.

THEOREM 6. *Let $H_n(G)$ be a direct summand of the group G. Then*

$$H_{n+k}(G) = H_n(G) \qquad (k = 1, 2, 3, \ldots).$$

Proof. $G = H_n(G) \oplus K$, where $K \cong G/H_n(G)$. Hence $H_1(K) \cong H_{n+1}(G)/H_n(G)$. Consider ordered pairs (e, x), where e is the identity of $H_n(G)$ and $x \in H_1(K)$. It follows that $(e, x) \in H_1(G)$. Hence $(e, x) \in H_n(G)$, so that $x = e'$, the identity of K. Thus $H_1(K) = (e')$ and $H_{n+1}(G) = H_n(G)$. The result follows at once.

If the FC-chain breaks off before or at $H_n(G)$, then $H_1(G/H_n(G))$ is the trivial group, and conversely. Thus $G/H_n(G)$ has no non-trivial H_1-group and has, as a consequence, no non-trivial centre and is isomorphic to the group of its inner automorphisms.

For an automorphism α of G, let $F(\alpha)$ denote the set of all points which are fixed under α. This set of fixed points is, as is well known, a subgroup of G. Let $J(G)$ be the group of inner automorphisms of G, and let $A(G)$ be the group of automorphisms of G. Recall the definition of mutual truncation in §2. We then have

THEOREM 7. *Let G be a group with mutual truncation at index $\leqslant n$. Then $J(G)$ has mutual truncation at index $n - 1$ (if $n \geqslant 1$).*

Proof. For any index k, $H_k(G/H_n(G))$ is trivial, by Lemma 4. Since $H_n(G) = Z_n(G)$,

$$H_k(G/H_n(G)) \cong H_k(J(G)/Z_{n-1}(J(G))).$$

By Lemma 5, $Z_{n-1}(J(G)) \supset H_k(J(G))$. Take $k = n$ for the result.

COROLLARY 1.[4] *If G has mutual truncation at index $\leqslant 1$, and if U is any group extension of $J(G)$, then each $J(G) \cap H_n(U)$ is trivial.*

Proof. By the theorem, $J(G)$ has mutual truncation at index 0. If S and T are groups with $S \subset T$, it is easy to see that $S \cap H_n(T) \subset H_n(S)$ for every n. Take $S = J(G)$ and $T = U$ for the result.

We should note[4] that the condition $H_1(G) \cap G' = (e)$ implies mutual truncation for G at index $\leqslant 1$. For, if N is a normal subgroup of G, then $G' \cap N = (e)$ implies that $gxg^{-1}x^{-1} = e$ for every $g \in G$ and for every $x \in N$. Hence $N \subset Z_1(G)$, and $H_1(G) = Z_1(G)$. Then $Z_1(G) \cap G'$ is trivial. But the latter has one of two consequences: (1) $Z_1(G) = (e)$, whence we have mutual truncation at index 0, or (2) $G' = (e)$, whence G is abelian, so that we have mutual truncation at index $\leqslant 1$.

COROLLARY 2. (a) *Let G have mutual truncation at index $\leqslant 1$, and let $J(G)$ be* FC-*nilpotent. Then G is abelian.* (b) *If $H_1(G)$ is trivial, then $A(G)$ is* FC-*nilpotent if and only if G is trivial.*

Proof. (a) By Corollary 1, each $H_n(J(G))$ is trivial. Since $J(G)$ is FC-nilpotent, $J(G)$ must be trivial so that G is abelian. (b) If $A(G)$ is FC-nilpotent, its subgroup $J(G)$ is also FC-nilpotent. $H_1(G) = (e)$ implies that $Z_1(G) = (e)$. By (a), G is abelian, so that $G = H_1(G) = (e)$.

Part (b) of the above corollary shows that if G is "badly" non-abelian and infinite, then its automorphism group cannot be finite, abelian, nilpotent, FC or, in general, FC-nilpotent.

COROLLARY 3. *If the* FC-*chain of G breaks off after a finite number of steps (say, at index n), then G is* FC-*nilpotent if and only if $A(G/H_n(G))$ is* FC-*nilpotent.*

[4]See note 1.

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Proof. $H_1(G/H_n(G))$ is trivial, since $H_{n+1}(G) = H_n(G)$. By Corollary 2(b), $G/H_n(G)$ is trivial, and $G = H_n(G)$.

THEOREM 8. *Let G be a group with mutual truncation at index $\leqslant r$. Let U be any group extension of $J(G)$ with the property $J(G) \subset U \subset A(G)$, where the inclusions need not be strict. Let α be in $H_j(U)$, and let α_r be the automorphism induced on $G/Z_r(G)$ by α. Then*

$$D(F(\alpha_r);\quad G/Z_r(G)) = G/Z_r(G).$$

Proof. If $j = 0$ or if G is trivial, the proof is immediate. If G is non-trivial and if $j \geqslant 1$, consider a fixed $g \in G, g \neq e. \alpha \in H_j(U)$ then implies the existence of integers $m < n$ and of $\phi \in H_{j-1}(U)$ such that

9.1 $$g^{-m}\,\alpha(g^m\,xg^{-m})\,g^m = g^{-n}\,\alpha(g^n\,\phi(x)g^{-n})\,g^n$$

for every $x \in G$. If we write $\alpha(u) = g$ and $k = g^{-m}\,u^{m-n}\,g^n$, then $\phi(x) = k^{-1}xk$ for every $x \in G$. Thus

$$\phi \in J(G) \cap H_{j-1}(U) \subset H_{j-1}(J(G)).$$

But the latter group is included in $Z_{r-1}(J(G))$, by the proof of Theorem 7. Thus $k \in Z_r(G)$. Now $k^{-1}xk = xv(x)$ where $v(x) \in Z_{r-1}(G)$ if $r \geqslant 1$ and $v(x) = e$ if $r = 0$. If we write $\alpha(g) = h$, 9.1 can be simplified to

9.2 $$g^{n-m}\,h^m\,\alpha(x)\,h^{-m} = h^n\,\alpha(x)\,\alpha(v)\,h^{-n}\,g^{n-m},$$

or

9.3 $$h^{-n}\,g^{n-m}\,h^m\,\alpha(x) \equiv \alpha(x)\,h^{-n}\,g^{n-m}\,h^m \qquad \text{mod } Z_{r-1}(G).$$

Since α is an automorphism, $\alpha(x)$ ranges over all of G, and $h^{-n}\,g^{n-m}\,h^m \in Z_r(G)$. Thus $g^{n-m} \equiv h^{n-m}$ mod $Z_r(G)$ for every $g \in G$ (where we understand that m and n are functions of g and α). Remembering that $h = \alpha(g)$, we see that the conclusion of the theorem follows at once.

COROLLARY 1. *If G has mutual truncation at index $\leqslant r$, if $Z_r(G)$ is a periodic group, and if $\alpha \in H_j(U)$ where $J(G) \subset U \subset A(G)$, then $D(F(\alpha); G) = G$.*

COROLLARY 2. (a) *Let G be a group for which $Z_1(G)$ is trivial. For $\alpha \in H_1(U)$ where $J(G) \subset U \subset A(G)$, $D(F(\alpha); G) = G$.* (b) *Let G be a finite group for which $Z_1(G)$ is trivial. For $\alpha \in A(G)$, $D(F(\alpha); G) = G$.*

Proof. (a) In the proof of the theorem we can take $\phi = \mathrm{I}$, the identity automorphism. Then 9.1 in the proof reduces to

$$h^{-n}\,g^{n-m}\,h^m\,y = yh^{-n}\,g^{n-m}\,h^m$$

where $\alpha(x) = y$. Since α is an automorphism and since $Z_1(G) = (e)$, $h^{-n}\,g^{n-m}\,h^m = \mathrm{e}$, and (a) follows directly. (b) is a trivial consequence of (a).

THEOREM 9. *Let G be a group with mutual truncation at index $\leqslant r$. Let U be any group extension of $J(G)$ with the property $J(G) \subset U \subset A(G)$, where the* *inclusions need not be strict. If* $\alpha \in H_j(U)$, *then* $F(\alpha_r)$ *has a finite index in* $G/Z_r(G)$.

Proof. There exists a finite (but not necessarily unique) set of elements $\{g_i\}$ $(i = 1, 2, \ldots, N)$ in G such that to $g \in G$, there exists an index i and a mapping $\phi \in H_{j-1}(U)$ with

$$g^{-1}\,\alpha(gxg^{-1})\,g = g_i^{-1}\,\alpha(g_i\,\phi(x)\,g_i^{-1})\,g_i$$

for every $x \in G$. As in the proof of Theorem 8, $\phi(x) = xv(x)$, where $v(x) \in Z_{r-1}(G)$ if $r \geqslant 1$, and $v(x) = e$ if $r = 0$. It follows that

$$g_i^{-1}\,\alpha^{-1}(g_i)\,\alpha^{-1}(g^{-1})\,g \in Z_r(G)$$

or that $\alpha(gg_i^{-1}) \equiv gg_i^{-1} \bmod Z_r(G)$. The theorem follows at once. A trivial rearrangement of the last step shows that

$$g^{-1}\,\alpha(g) \equiv g_i^{-1}\,\alpha(g_i) \qquad \bmod Z_r(G),$$

as we should expect in light of [**3,** p. 165, (c′)].

COROLLARY 1.[5] *If* $H_1(G)$ *is trivial and if* $\alpha \in H_j(U)$, *where* $J(G) \subset U \subset A(G)$, *then* $F(\alpha)$ *has finite index in* G.

COROLLARY 2. *Let* G *be a group for which* $Z_1(G)$ *is trivial. If* $\alpha \in H_1(U)$, *where* $J(G) \subset U \subset A(G)$, *then* $F(\alpha)$ *has finite index in* G.

Proof. In the proof of the theorem we can take ϕ to be the identity map. The rest of the argument follows without difficulty.

Following common custom, a group will be called *complete* if for each positive integer n, the set of all x^n $(x \in G)$, is a set of generators for G. By $T_n(G)$, where n is a fixed positive integer, we shall mean the set of all $\alpha \in A(G)$ for which $\alpha(x) \equiv x \bmod Z_n(G)$ for every $x \in G$. If $n = 1$, we have the so-called *normal* or *central* automorphisms [**6**]. $T_0(G)$ is to consist of the identity automorphism of G, alone.

THEOREM 10. *Let* G *be a complete group which has mutual truncation at index* r. *Let* U *be an extension of* $J(G)$ *with the property* $J(G) \subset U \subset A(G)$, *where the inclusions need not be strict. Then* $H_j(U) \subset T_r(G)$ $(j = 1, 2, \ldots)$.

Proof. Suppose $\alpha \in H_j(U)$. To each $x \in G$, there exists, by the proof of Theorem 8, a positive integer $t(x)$ such that

$$\alpha(x^{t(x)}) \equiv x^{t(x)} \qquad \bmod Z_r(G).$$

Moreover, there exists a uniform bound $M = M(\alpha) \geqslant t(x)$ for all $x \in G$, since $\alpha \in H_j(U)$. Let $N = M!$. Then $\alpha(x^N) \equiv x^N \bmod Z_r(G)$. Since the set of all x^N is a set of generators of G, $\alpha(g) \equiv g \bmod Z_r(G)$ for every $g \in G$, and $\alpha \in T_r(G)$.

[5]See note 1.

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COROLLARY 1. *If $H_1(G)$ is trivial for a complete group G, then $H_1(U)$ is trivial, where $J(G) \subset U \subset A(G)$.*

Proof. Note that the index r of truncation is 0. Alternately, we can prove a stronger result:

COROLLARY 2. *If $Z_1(G)$ is trivial for a complete group G, then $H_1(U)$ is trivial, where $J(G) \subset U \subset A(G)$.*

Proof. Using the proof of Corollary 2 to Theorem 8, we can modify the proof of the present theorem to show that $\alpha(x^N) = x^N$ for every x.

10. Examples of FC-nilpotent groups. Consider two countable classes of copies of I_2, the group of integers modulo 2. Let the generators of these groups be denoted by the e_1 $(i = 1, 2, 3, \ldots)$ and the f_j $(j = 1, 2, 3, \ldots)$. Form the free product F of all the members of these two classes of copies of I_2. Impose the relations (1) $e_i x = x e_i$, for every generator e_i of the first class and for every $x \in F$; and (2) $f_i f_j = f_j f_i e_i e_j$ for all i and j. Call the resulting group G. Then every word in G can be given the unique canonical form $f_{i_1} f_{i_2} \ldots f_{i_n} E$ where E is a word in the e_i's and $i_1 < i_2 < \ldots < i_n$ if the class of the i_j's is non-void. It is easy to prove that $Z_1(G)$ is the set of all elements generated by the e's alone. If the number of f's in the canonical form of a word is even, then the word has no more than two conjugates in G, while if the number is odd, there are an infinite number of conjugates of the word in G. All the words of even "f-length" form the subgroup $H_1(G)$, and this subgroup is distinct from both $Z_1(G)$ and G. It is clear that $x^2 \in Z_1(G)$ for every $x \in G$, so that $G = D(Z_1(G); G)$. It is not difficult to show that $G/H_1(G) \cong I_2$, so that $G = H_2(G)$. We thus have an example of an FC-nilpotent group of FC-class 2.

The referee has pointed out the following: Let G be a free group on two or more generators. We construct the *lower central series* [**6**] of G as follows: $G(0) = G$; $G(1) = (G, G)$; $G(i + 1) = (G, G(i))$. Then $G/G(c)$ is an example of an FC-nilpotent group of FC-class c for every positive integer c.

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