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107.06 Proving inequalities via definite integration: a visual approach

Fascination with inequalities has encouraged numerous visual proofs. It is quite interesting to see and feel the beauty. There are several techniques to do these proofs logically. Definite integration of one variable is seemed to be a greater tool in this case. Geometrically, definite integration means area under a given curve. So, basically it will assign a number. If we use different curves in the same region then it will give us different numerical expressions and we can compare between them. We can use this tool in such a way that it will give us the required expression for an inequality. Also, it will give us a clear visual representation in order to prove our claims. In this Note, we provide another area argument on the general inequality (see [1])

$$e \leqslant A < B \Rightarrow A^B > B^A$$

and also two visual proofs of two different inequalities using area under the curves.

Inequality 1

The constants e and π have encouraged numerous visual proofs of the inequality $\pi^e < e^\pi$ (see [2]). In [3], Gallant provided the most general proof for which this inequality is a consequence, showing that when $e \leqslant A < B$, we have $A^B > B^A$; he used slopes of secant lines connecting the origin to points on the curve $y = \ln(x)$. We provide an alternate visual proof for this general inequality.

Claim: $e \leqslant A < B \Rightarrow A^B > B^A$.

Proof

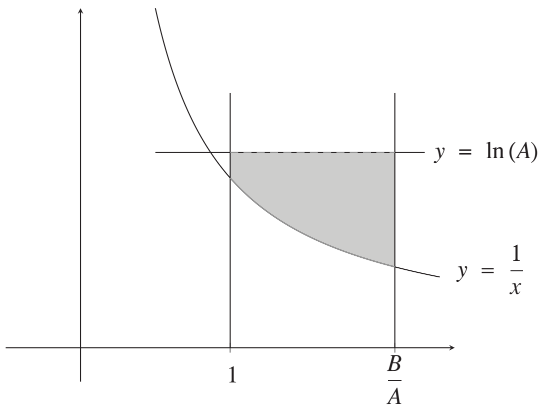

FIGURE 1

If $e \leqslant A < B$, then

$$\int_1^{\frac{B}{A}} \ln(A)\,dx > \int_1^{\frac{B}{A}} \frac{1}{x}\,dx \Rightarrow \ln(A)\left(\frac{B}{A} - 1\right) > \ln\frac{B}{A}$$

$$\Rightarrow \frac{B}{A} > \frac{\ln(B)}{\ln(A)} \Rightarrow A^B > B^A.$$

Corollary: $e < \pi \Rightarrow e^\pi > \pi^e$.

Inequality 2

This visual proof is of the famous *Jordan's inequality*, named after Camille Jordan, which states that $\frac{2}{\pi}x < \sin x$ for $x \in \left(0, \frac{1}{2}\pi\right)$. The first *Proof without words* of this inequality was given by Yuefeng [4], using the geometry of circles. The second visual proof was given by Nelsen [5] by putting a concave curve between two straight lines. We give another visual proof of this inequality using an area argument.

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Claim: $0 < \alpha < \dfrac{\pi}{2} \Rightarrow \sin\alpha > \dfrac{2\alpha}{\pi}$.

Proof

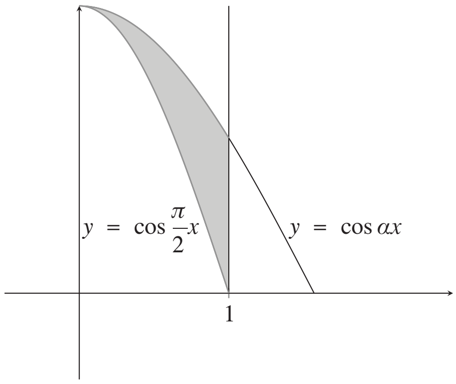

FIGURE 2

If $0 < \alpha < \frac{1}{2}\pi$, then

$$\int_0^1 \left(\cos\alpha x - \cos\frac{\pi x}{2}\right)dx > 0 \Rightarrow \frac{\sin\alpha}{\alpha} - \frac{2}{\pi} > 0 \Rightarrow \sin\alpha > \frac{2\alpha}{\pi}.$$

Corollary: $1 - \dfrac{2}{\pi}x < \cos x$ for $x \in \left(0, \frac{1}{2}\pi\right)$.

The proof is by substituting $\frac{1}{2}\pi - x$ for x into Jordan's inequality. The inequality is called *Kober's inequality*.

Inequality 3

Though there are slight differences, the third inequality is quite similar to Napier's inequality, which states that,

$$0 < a < b \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}.$$

The following is an alternative visual proof of a conclusion drawn by Plaza [6, 7].

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Claim: $0 < a < b \Rightarrow \dfrac{2}{b + a} < \dfrac{\ln b - \ln a}{b - a} < \dfrac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$.

Proof

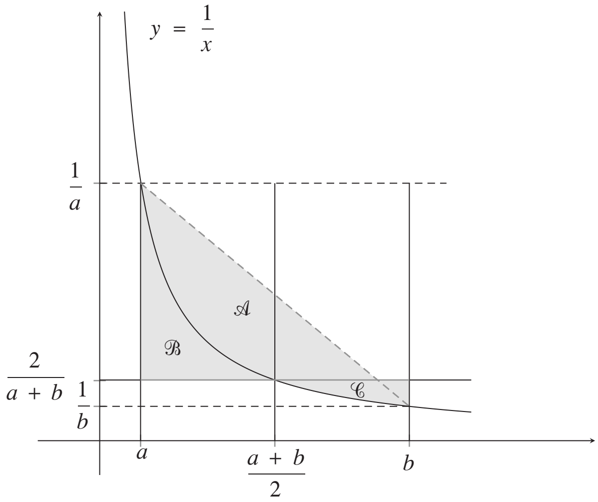

FIGURE 3

Using the formula for the area of a trapezium, region $\mathscr{A}$ of the diagram shows

$$\int_a^b \frac{1}{x}\,dx < \frac{1}{2}(b - a)\left(\frac{1}{a} + \frac{1}{b}\right).$$

Also, comparing the areas of regions $\mathscr{B}$ and $\mathscr{C}$

$$\int_a^{\frac{1}{2}(a+b)} \frac{1}{x}\,dx - \frac{2}{a+b}\left[\frac{a+b}{2} - a\right] > \frac{2}{a+b}\left[b - \frac{a+b}{2}\right] - \int_{\frac{1}{2}(a+b)}^b \frac{1}{x}\,dx$$

$$\Rightarrow \int_a^b \frac{1}{x}\,dx > 2\frac{b-a}{b+a}.$$

Clearly, from the above inequalities we get

$$2\frac{b - a}{b + a} < \int_a^b \frac{1}{x}\,dx < \frac{b - a}{2}\left(\frac{1}{a} + \frac{1}{b}\right),$$

$$\frac{2}{b + a} < \frac{\ln b - \ln a}{b - a} < \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right),$$

$$[A(a, b)]^{-1} < [L(a, b)]^{-1} < [H(a, b)]^{-1}$$

where, $A(a, b) = \frac{1}{2}(a + b)$; $L(a, b) = \dfrac{b - a}{\ln b - \ln a}$; $H(a, b) = \dfrac{2}{\frac{1}{a} + \frac{1}{b}}$ are, respectively, the arithmetic mean, logarithmic mean and harmonic mean of two positive different numbers.

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107.07 A geometric illustration for infinite sequences and series

We can use a positive vanishing sequence to construct a sequence of squares. This so-called *square set* can then be used to visualise sums involving the sequence. As a first example, we use the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ to construct the arrangement of squares seen in Figure 1, which we call the *square set for* $\frac{1}{2^n}$. We define l to be the length of this set, h to be the height to which the squares converge, and A to be the total area of the set.

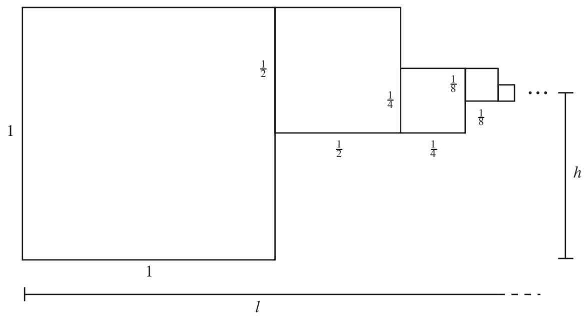

FIGURE 1: The square set for $\dfrac{1}{2^n}$