THE REPRESENTATION TYPE OF ALGEBRAS AND SUBALGEBRAS

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1. Introduction. For Λ an associative algebra with identity over a field K, $[\Lambda : K] < \infty$, and d an integer, we define $g_\Lambda(d)$ to be the number of inequivalent indecomposable Λ-modules of degree d over K. Following **(6)**, we define Λ to be of *finite* representation type if

$$\textstyle\sum_d g_\Lambda(d) < \infty.$$

Λ is said to be of *bounded* representation type if there exists d_0 such that $g_\Lambda(d) = 0$ for $d \geqslant d_0$; Λ is of *unbounded* representation type if not of bounded type. We shall say that Λ is of *strongly unbounded* type if $g_\Lambda(d) = \infty$ for an infinite number of integers d. See **(6)** and **(7)** for a number of conditions showing algebras to be of strongly unbounded type.

Now let Λ be a subalgebra of an algebra Γ with $[\Gamma : K] < \infty$ also. In this paper we give conditions under which the representation type of Λ can be related to that of Γ.

To do this, we must have a process for inducing Γ-modules from Λ-modules and conversely. Such processes date back to Frobenius (in the case of group representations), have been studied extensively by D. G. Higman **(4)**, and are used by Cartan and Eilenberg **(2**, II, § 6**)** under the heading "change of rings." We shall consider conditions on the algebra Γ and the subalgebra Λ under which every indecomposable Γ-module is obtained from an indecomposable Λ-module or conversely. In this way we may relate their representation types.

It should be noted that, unlike **(2)** and **(4)**, we do not require that the identity of Λ also be the identity of Γ. This allows consideration of a wider class of subalgebras. Also, for M to be a Λ-module we do not require that the identity $1 \in \Lambda$ act like the identity on M. By an indecomposable Λ-module M, however, we mean indecomposable and non-trivial (that is, $1M \neq (0)$). With these two assumptions, an indecomposable Λ-module M does have the property that $1m = m$ for all $m \in M$, for otherwise, a trivial direct summand could be split off.

2. Algebras and subalgebras. Let M be a two-sided (associative) Λ-module. It is convenient to regard M as a left Λ^e-module, where $\Lambda^e = \Lambda \otimes_K \Lambda'$ and Λ' is anti-isomorphic to Λ **(2**, IX, §3**)**. Thus if Λ is a subalgebra of Γ then Γ is a left Λ^e-module and Λ is a Λ^e-submodule of Γ.

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THEOREM 1. *If* Λ *is a subalgebra of* Γ *such that* $\Gamma = \Lambda + C$ (Λ^e*-direct sum*), *and if* Λ *is of* (*strongly*) *unbounded type then so is* Γ.

Proof. Let M be an indecomposable Λ-module then $\Gamma \otimes_\Lambda M$ is a Γ-module (called the induced module $I(M)$ in **(4)** or the covariant ϕ-extension of M in **(2**, II, §6**)**) which in turn can be considered as a Λ-module. By the assumptions on Λ and Γ, $\Gamma \otimes_\Lambda M = (\Lambda + C) \otimes_\Lambda M = \Lambda \otimes_\Lambda M + C \otimes_\Lambda M = M + C \otimes_\Lambda M$ (Λ-direct sum). For the last equality we need the fact that Λ has an identity which acts as an identity on M. This follows from the indecomposability of M.

We then have that $\Gamma \otimes_\Lambda M$ contains an indecomposable Γ-direct summand $P(M)$ such that $[M : K]\,[\Gamma : K] \geqslant [P(M) : K] \geqslant [M : K]$. To exhibit $P(M)$, first decompose $\Gamma \otimes_\Lambda M$ into indecomposable Γ-direct summands and then decompose each of those into indecomposable Λ-direct summands. By the Krull-Schmidt theorem **(5**, V, §13**)** one of these Λ-direct summands is M. P(M) is the Γ-direct summand which contains M.

Thus if Λ is of unbounded type so is Γ. If Λ has an infinite number of indecomposable modules, (M_i) $i \in I$, all of degree d over the field K, then each of the Γ-indecomposable modules $P(M_i)$ can be isomorphic to no more than a finite number of $P(M_j)$. Thus if Λ is of strongly unbounded representation type so is Γ.

In the proof of Theorem 1, we could have used the module $\mathrm{Hom}_\Lambda(\Gamma, M)$ (called the produced module in **(4)**, the contravariant ϕ-extension of M in **(2**, II, §6**)**) instead of $\Gamma \otimes_\Lambda M$. Under the assumptions of Theorem 1, $\mathrm{Hom}_\Lambda(\Gamma, M) = \mathrm{Hom}_\Lambda(\Lambda, M) + \mathrm{Hom}_\Lambda(C, M)$ (Λ-direct). If $1M = M$ (for example, M indecomposable) then $\mathrm{Hom}_\Lambda(\Lambda, M) = M$. Again using the Krull-Schmidt Theorem, there exists a Γ-indecomposable module $P'(M)$ with the same properties as $P(M)$.

There are several occasions when a subalgebra Λ of Γ will satisfy the hypotheses of Theorem 1. For instance, if H is a subgroup of a finite group G, then the subgroup algebra Λ is a Λ^e-direct summand of the group algebra Γ. The Λ^e-complement of Λ in Γ has a K-basis consisting of group elements not in the subgroup. Theorem 1 for group algebras and subgroup algebras was proved by Higman **(3)**.

Another important case where the conditions of Theorem 1 are satisfied is when e is an idempotent of Γ and Λ is taken to be $e\Gamma e$. Here $\Gamma = \Lambda + C$ (Peirce decomposition) is Λ^e-direct. This case was pointed out to us by Higman and is contained in the following:

COROLLARY 2. *If* e *is an idempotent of* Γ *and* $e\Gamma e$ *is of* (*strongly*) *unbounded type then so is* Γ.

A restricted version of Corollary 2 is used in **(6)** and **(7)** in the case that $e\Gamma e$ is the basic algebra of Γ.

Higman also noted that the condition $\Gamma = \Lambda + C(\Lambda^e$-direct sum) of Theorem 1 is equivalent to the condition $e\Gamma e = \Lambda + eCe$ (Λ^e-direct sum) where e is the identity of Λ. When these conditions hold and Λ is of (strongly) unbounded type then both $e\Gamma e$ and Γ are also of (strongly) unbounded type by Theorem 1.

Theorem 1 is also applicable to the tensor product of two algebras.

COROLLARY 3. *If* $\Gamma = \Lambda \otimes_K \Sigma$, *and either* Λ *or* Σ *is of (strongly) unbounded type then so is* Γ.

Proof. We need only note that $\Lambda \otimes_K 1 = \Lambda$ is a subalgebra of Γ and

$$\Gamma = \sum_1^n \Lambda \otimes_K \alpha_i$$

(Λ^e-direct sum) where (α_i) is a K-basis for Σ and $\alpha_1 = 1$ in Σ.

Now suppose that the subalgebra is of finite (bounded) type. We consider conditions under which the containing algebra is also of finite (bounded) type. The following theorem gives one such condition.

THEOREM 4. *If* Λ *has an identity* e *and* $\Gamma e \otimes_\Lambda e\Gamma = \Gamma + C$ (Γ^e-*direct sum*) *then if* Λ *is of finite (bounded) type so is* Γ.

Proof. Let M be an indecomposable Γ-module, $\Gamma M = M$. Consider M as a Λ-module and form the Γ-module $\Gamma e \otimes_\Lambda M$. Then $\Gamma e \otimes_\Lambda M = \Gamma e \otimes_\Lambda eM = \Gamma e \otimes_\Lambda (e\Gamma \otimes_\Gamma M) = (\Gamma e \otimes_\Lambda e\Gamma) \otimes_\Gamma M = (\Gamma + C) \otimes_\Gamma M = M + C \otimes_\Gamma M$ all as Γ-modules. The first equality results from the following: $\Gamma e \otimes_\Lambda M_o = (0)$ where M_o is the Λ-sub-module of M annihilated by e and $\Gamma e \otimes_\Lambda M = \Gamma e \otimes eM + \Gamma e \otimes_\Lambda M_o$. The other equalities follow from distributivity and associativity of the tensor product and from the assumptions of the theorem.

Thus, under the assumptions of the theorem, we have shown that every indecomposable Γ-module M appears as a direct summand of a module of the form $\Gamma e \otimes_\Lambda eM$, where eM is a Λ-module (on which e acts as identity). Further, if $eM = eM' + eM''$ (Λ-direct sum) then $\Gamma e \otimes_\Lambda eM = \Gamma e \otimes_\Lambda eM' + \Gamma e \otimes_\Lambda eM''$ (Γ-direct sum). Hence every indecomposable Γ-module appears as a direct summand of some $\Gamma e \otimes_\Lambda M$ where M is an indecomposable Λ-module. Note also that $[\Gamma e \otimes_\Lambda M : K] \leqslant [\Gamma e : K]\,[M : K]$ Thus if Λ is of finite (bounded) type so is Γ.

The above proof could have been altered to use the module

$$\mathrm{Hom}_\Lambda(\Gamma e, eM) \text{ instead of } \Gamma e \otimes_\Lambda eM.$$

The conditions of Theorem 4 are difficult to apply because the structure of the Γ^e-module $\Gamma e \otimes_\Lambda e\Gamma$ is complicated. We do, however, make use of this condition in the following:

THEOREM 5. *If* Σ *is a separable algebra over the field* K, *then* Λ *and* $\Lambda \otimes_K \Sigma$ *are of the same representation type.*

Proof. By corollary 3, if Λ is of (strongly) unbounded type so is $\Lambda \otimes_K \Sigma$, regardless of Σ.

If Λ is of finite (bounded) type, we apply the condition of Theorem 4. Note that here the identity of $\Lambda \otimes 1 = \Lambda$ is also the identity of $\Lambda \otimes_K \Sigma$. From **(2**, IX, 7.10**)**, a necessary and sufficient condition that Σ be separable is that the exact sequence

$$(0) \rightarrow C \rightarrow \Sigma \otimes_K \Sigma' \xrightarrow{p} \Sigma \rightarrow (0)$$

split, where $p(\alpha \otimes \beta') = \alpha\beta$. This means that the Σ^e-module $\Sigma \otimes_K \Sigma$ (which is Σ^e-isomorphic to $\Sigma \otimes_K \Sigma'$) can be written as a Σ^e-direct sum $\Sigma + C$. Now tensor-multiply over K the Λ^e-module $\Lambda = \Lambda \otimes_\Lambda \Lambda$ with $\Sigma \otimes_K \Sigma = \Sigma + C$ to obtain the $\Lambda^e \otimes_K \Sigma^e$-modules

$$(\Lambda \otimes_\Lambda \Lambda) \otimes_K \Sigma \otimes_K \Sigma = \Lambda \otimes_K \Sigma + \Lambda \otimes_K C.$$

But $\Lambda^e \otimes_K \Sigma^e = \Gamma^e$, the left hand side of the above equation is Γ^e-isomorphic with $(\Lambda \otimes_K \Sigma) \otimes_\Lambda (\Lambda \otimes_K \Sigma)$ or $\Gamma \otimes_\Lambda \Gamma$, and the right hand side is Γ^e-isomorphic with $\Gamma + \Lambda \otimes_K C$. The condition of Theorem 4 is satisfied and Γ is of finite (bounded) type.

3. Fields. In the following we consider the representation type of the tensor product of two fields. Algebras obtained in this way are commutative. In **(6)**, it is shown that for Λ a commutative K-algebra, $[\Lambda : K] < \infty$, where N is the radical of Λ and Λ/N is a direct sum of fields isomorphic to K, Λ is of finite type if and only if Λ is the direct sum of ideals Λ_i where $\Lambda_i = K[X]/(X^{t_i})$, X an indeterminant over K. If K is infinite and Λ is not of finite type then Λ is of strongly unbounded type.

In the following, the degree of any containing field over the base field is always finite.

LEMMA 1. *If J is a field purely inseparable over the field K and J cannot be obtained from K by the adjunction of a single element then $J \otimes_K J$ is of strongly unbounded type. If $J = K(\alpha)$ then $J \otimes_K J$ is of finite type.*

Proof. In either case we have the mapping $p : J \otimes_K J \rightarrow J$, $p(\alpha \otimes \beta) = \alpha\beta$ which gives rise to the exact sequence

$$0 \rightarrow N \rightarrow J \otimes_K J \rightarrow J \rightarrow 0.$$

The ideal N, generated by elements of the form $n = \alpha \otimes 1 - 1 \otimes \alpha$, is the radical of $J \otimes_K J$ because

$$n^{p^a} = 0, \quad [J{:}K] = p^a,$$

p the characteristic of K. Thus $J \otimes_K J$, considered as a J-algebra is of the form considered in **(6)**.

If J is not obtained by a single adjunction then K and J are infinite **(1**, Theorem 26**)** and for each α in J there exists $b < a$ such that α^{p^b} belongs to K. Let m be the largest of these b's, m strictly $< a$. Thus $[N : J] = p^a - 1$ and for every n in N,

$$n^{p^m} = 0.$$

Hence $J \otimes_K J$ cannot be isomorphic to

$$J[X]/(X^{p^a})$$

so it is of strongly unbounded type. $J \otimes_K J$ is also of strongly unbounded type when considered as a K-algebra.

In the case $J = K(\alpha)$, the radical element $\alpha \otimes 1 - 1 \otimes \alpha = n$, non-zero powers of which are linearly independent, generates N. For the powers $\alpha^r (r = 1, \ldots, p^a)$ form a K-basis for J and the elements $\alpha^i \otimes \alpha^j (i, j = 1, \ldots, p^a)$ form a K-basis for $J \otimes_K J$. But

$$n^{p^a-1} = (\alpha \otimes 1 - 1 \otimes \alpha)^{p^a-1} = \alpha^{p^a-1} \otimes 1 + * + 1 \otimes \alpha^{p^a-1},$$

where * indicates a sum of terms of the form $t\alpha^i \otimes \alpha^j$ with both i and j greater than 1. This is not zero so in this case

$$J \otimes_K J = J[X]/(X^{p^a})$$

and considered as a J-algebra is of finite type. Both K and J are subalgebras in the centre of $J \otimes_K J$ and any J-linear transformation is also K-linear so $J \otimes_K J$ considered as a K-algebra is also of finite type.

Using Lemma 1 and the structure theory for fields, we may drop the purely inseparable condition.

LEMMA 2. *If F can be obtained from K by a single adjunction then $F \otimes_K F$ is of finite type. If not, $F \otimes_K F$ is of strongly unbounded type.*

Proof. Let J be the field of elements of F purely inseparable over K, then F is separable over J, J purely inseparable over K. F can be obtained by a single algebraic adjunction if and only if J can be so obtained. This is seen by using (**1**, Theorem 26) and the fact that every field between F and K is the unique composite of a field separable over K and a field purely inseparable over K (that is, between J and K).

Hence by Lemma 1 $J \otimes_K J$ is of strongly unbounded type if F is not obtainable by a single adjunction, of finite type if F can be so obtained. But by Theorem 5, $J \otimes_K J$, $(J \otimes_K J) \otimes_J F = J \otimes_K F$, and $F \otimes_J (J \otimes_K F) = F \otimes_K F$ are all of the same representation type because F is separable over J.

Combining Lemmas 1 and 2, we prove the following:

THEOREM 6. *Let $\Omega \geqslant L$, $F \geqslant K$, $L \cap F = J$ all be fields. If L, F are separable over J and J is obtained from K by a single adjunction then $L \otimes_K F$ is of finite type. If J is not obtained from K by a single adjunction then $L \otimes_K F$ is of strongly unbounded type.*

Proof. In the first case $J \otimes_K J$ is of finite type by Lemma 2. Using Theorem 5, the algebras $J \otimes_K J$, $(J \otimes_K J) \otimes_J F = J \otimes_K F$ and $L \otimes_J (J \otimes_K F) = L \otimes_K F$ are all of the same (that is, finite) type because L and F are assumed separable over J.

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In the second case $J \otimes_K J$ is of strongly unbounded type by Lemma 2, thus so are the algebras $J \otimes_K F$ and $L \otimes_K F$ by Corollary 2.

We believe that Theorem 6 could be sharpened to read "$L \otimes_K F$ and $J \otimes_K J$ are of the same representation type," however, in the case $J \otimes_K J$ is of finite type, the condition of Theorem 5 is too weak to be used without additional assumptions on L and F.

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