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CONVERGENCE OF APPROXIMATE SOLUTIONS OF A QUASILINEAR PARTIAL DIFFERENTIAL EQUATION

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This article is a sequel to a paper in which a quasilinear partial differential equation with nonlinear boundary condition was approximated using mollifiers, and the existence of solutions to the approximating problem shown under quite general conditions. In this paper we show that standard *a priori* Hölder estimates ensure the convergence of these solutions to a classical solution of the original problem. Some partial results giving such estimates for special cases are described.

1. Introduction

As in [**2**], we consider the problem

$$\begin{aligned} Qu &= a^{ij}(x,u,Du)D_{ij}u + a(x,u,Du) = 0 \quad \text{in } \Omega \\ Gu &= g(x,u,Du) = u - q(x,Du) = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an bounded domain such that $\partial\Omega \in C^{1,\alpha}$, and we assume that a^{ij}, $a \in C^{0,1}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ and q, $q_p \in C^{1,\alpha}(\partial\Omega \times \mathbb{R}^n)$.

The existence of a solution to (1.1) is equivalent to the existence of a pair of functions (u,ω) such that

$$Qu = 0 \quad \text{in } \Omega \tag{1.2}$$

$$u = \omega \quad \text{on } \partial\Omega$$

$$\omega - q(x,Du) = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Using the standard freezing of coefficients to rewrite the differential operator, (1.2) may be written in the form $u - T(u,\omega)$, allowing the full problem to be written as

$$(I-K)\begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} u - T(u,\omega) \\ \omega - q(x,Du) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.4}$$

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The results described in this paper, and its precusor [**2**], are based upon the author's doctoral thesis [**1**]. I wish to thank Dr. H.B.Thompson for his supervision during the course of my doctoral studies.

where it is assumed that $(u,\omega) \in C^{1,\gamma}(\overline{\Omega}) \times C^{1,\rho}(\partial\Omega)$ for $0 < \gamma < \rho < \alpha$.

We considered in [**2**] an approach in which degree theory is applied to equations which in some sense approximate (1.4) by using a mollification operator to replace the Du term in the boundary condition with a smoother approximation, denoted $(Du)_\eta$. Once the existence of solutions to the approximating problems has been guaranteed (as in [**2**]), the question remains: Under what circumstances do these approximate solutions converge in the limit to a solution of the original problem? It is this question we address here.

We assume in all that follows that the differential and boundary operators satisfy the conditions

$$0 < \lambda |\xi|^2 \leqslant a^{ij}(x,z,p)\xi_i\xi_j \leqslant \Lambda |\xi|^2 < \infty \quad \forall \xi \in \mathbb{R}^n \setminus \{0\} \tag{1.5}$$

$$-g_p(x,z,p)\cdot\vec{n} > \chi$$

$$-g_p(x,z,p)\cdot\vec{n} > \ \chi\, |g_p(x,z,p)|, \tag{1.6}$$

for some positive constants λ, Λ, χ, and the natural structure conditions

$$\begin{aligned} \Lambda &\leqslant \lambda\mu(|z|) \\ |a| &\leqslant \lambda\mu_0(|z|)\left(1+|p|^2\right), \\ |a_x|, |a_z|, |p|\,|a_p| &\leqslant \lambda\mu_1(|z|)\left(1+|p|^2\right), \\ \left|a_x^{ij}\right|, \left|a_z^{ij}\right|, (1+|p|)\left|a_p^{ij}\right| &\leqslant \lambda\mu_1(|z|), \end{aligned} \tag{1.7}$$

where μ, μ_0, μ_1 are positive non-decreasing functions, and that there exists a positive constant M_1 such that

$$(sgn\ z)\,a(x,z,0) < 0 \quad \text{in } \overline{\Omega}, \tag{1.8}$$

for $|z| > M_1$.

The mollification process used is an adaptation of the standard one. Let $\rho \in C^\infty(\mathbb{R}^n,\mathbb{R})$ be a nonnegative function such that $\rho(x) = 0$ if $|x| \geqslant 1$ and $\int \rho(\cdot)\,dx = 1$. For $f \in C^0(\Omega)$, and $\eta > 0$, consider the operator $\varphi_\eta *$ defined by

$$(\varphi_\eta * f)(x) = \eta^{-n} \int_\Omega \rho\left(\frac{x-y}{\eta}\right) f(y)dy, \tag{1.9}$$

provided that $dist(x,\partial\Omega) > \eta$. We call η the *mollification parameter*, and $\varphi_\eta * f$ the *mollification* of f. For the properties of the mollifier see [4] or [**7**].

For technical reasons, in [**2**] we used an adaptation of the standard mollifier, denoted by $(\ )_\eta$ and defined by taking $(f)_\eta$ to be $S(\eta, f)\,\varphi_\eta * f$, where

$$
(1.10)\qquad
\begin{aligned}
S(\eta,f) &\stackrel{\text{def}}{=} (1-\eta)\max\{0,\min\{s_1(\eta,f),s_2(\eta,f)\}\}\\
s_1(\eta,f) &\stackrel{\text{def}}{=} 1-\frac{\|(\varphi_\eta * f - f)\|_{0;\overline{\Omega}}}{S_1/2}\\
s_2(\eta,f) &\stackrel{\text{def}}{=} \sup\Big\{t\in\mathbb{R}\ \Big|\ (1-t)\tau \geqslant t\ \|\varphi_\eta * f - f\|_{0;\overline{\Omega}}\ \|g_p\|_{0;\partial\Omega(A)}\Big\},
\end{aligned}
$$

where S_1, τ, C are positive constants, $A = \|f\|_{0;\partial\Omega} + C + 1$ and the set $\partial\Omega(A)$ is defined to be $\big\{(x,z,p)\in\partial\Omega\times\mathbb{R}\times\mathbb{R}^n\ \big|\ |z|+|p|<A\big\}$.

2. Convergence to a classical solution

The following result is from [**2**].

Theorem 2.1. *If Q and G satisfy (1.5), (1.6), (1.7) and (1.8),* then *for each $\eta\in(0,\eta_0)$ (η_0 some positive constant) there exists a function $u\in C^{2,\alpha}(\Omega)\cap C^{1,\alpha}(\overline{\Omega})$ such that $\|u\|_{0;\Omega} < M_1 + 1 + \operatorname{diam}\Omega$ and*

$$
(2.1)\qquad
\begin{aligned}
Qu &= 0 && \text{in } \Omega\\
g\Big(x,u,(Du)_\eta\Big) &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

This solution we denote by u_η, since there is an obvious dependence upon the mollification parameter.

The question we consider here is: Under what circumstances do these solutions to the approximating problem (2.1) give in the limit a classical solution of (1.1) as η tends to zero? (We follow the convention of referring to the 'convergence' of the u_η when in fact we shall require only the convergence of any subsequence.) The following result gives a sufficient condition.

Theorem 2.2. *Assume for some sequence of $\eta_i \searrow 0$ there exist u_{η_i} solving (2.1). If there exists constants $0<\sigma$, $C<\infty$ such that*

$$
(2.2)\qquad \|u_{\eta_i}\|_{1,\sigma;\overline{\Omega}} \leqslant C,
$$

then *there exists a classical solution $\overline{u}$ of Equation (1.1).*

Proof: We denote the restriction of u_η to $\partial\Omega$ by ω_η, with $\overline{\omega}$ defined similarly. It suffices to show that $u_\eta - T(u_\eta,\omega_\eta) = 0$ in Ω for η converging to zero implies that $\overline{u} - T(\overline{u},\overline{\omega}) = 0$ in Ω, and $g\Big(x,u_\eta,(Du_\eta)_\eta\Big) = 0$ on $\partial\Omega$ implies $g(x,\overline{u},D\overline{u}) = 0$ on $\partial\Omega$. The convergence in $C^1\big(\overline{\Omega}\big)$ of the u_η to some $\overline{u}$ follows trivially by compactness. By the

Intermediate Schauder estimates of [3] for the Dirichlet problem, Equation 2.2 implies that $|u_\eta|^{-(1+\sigma)}_{2+\sigma\alpha;\Omega} < C$ for some constant C, giving for $S \subset\subset \Omega$, $\|u_\eta\|_{2,\sigma\alpha;S} < C_0$, so $\overline{u} - T(\overline{u},\overline{\omega}) = 0$ in Ω. To show that $g(x,\overline{u},D\overline{u}) = 0$ on $\partial\Omega$, it suffices to show that $(Du_\eta)_\eta$ converges to $D\overline{u}$ in $C^0(\partial\Omega)$, given that $u_\eta \to \overline{u}$ in $C^1(\overline{\Omega})$. From the definition of the $(\ )_\eta$ operator, we have (using S_η to denote $S(\eta, Du_\eta)$)

$$\begin{aligned}\left\|(Du_\eta)_\eta - D\overline{u}\right\|_{0;\overline{\Omega}} &\leqslant \|(S_\eta - 1)\,\varphi_\eta * Du_\eta\|_{0;\overline{\Omega}} \\ &\quad + \|\varphi_\eta * Du_\eta - \varphi_\eta * D\overline{u} + \varphi_\eta * D\overline{u} - D\overline{u}\|_{0;\overline{\Omega}} \\ &\leqslant |(S_\eta - 1)|\ C + \|Du_\eta - D\overline{u}\|_{0;\overline{\Omega}} + \|\varphi_\eta * D\overline{u} - D\overline{u}\|_{0;\overline{\Omega}}, \end{aligned} \tag{2.3}$$

where it should be noted that we convert back to the standard mollification process in order to exploit the linearity lacking in the $(\ )_\eta$ operator.

The result then follows directly from the last equation if $S(\eta, Du_\eta) \to 1$ as $\eta \searrow 0$. To show this, we first show that $\|\varphi_\eta * Du_\eta - Du_\eta\|_{0;\overline{\Omega}} \searrow 0$.

$$\begin{aligned}\|\varphi_\eta * Du_\eta - Du_\eta\|_{0;\overline{\Omega}} &\leqslant \|\varphi_\eta * Du_\eta - \varphi_\eta * D\overline{u} + \varphi_\eta * D\overline{u} - D\overline{u} + D\overline{u} - Du_\eta\|_{0;\overline{\Omega}} \\ &\leqslant \|\varphi_\eta * D\overline{u} - D\overline{u}\|_{0;\overline{\Omega}} + 2\,\|D\overline{u} - Du_\eta\|_{0;\overline{\Omega}}, \end{aligned} \tag{2.4}$$

which clearly goes to zero as $\eta \searrow 0$. From (1.10) it is clear that $S(\eta, Du_\eta) \searrow 1$ as $\eta \searrow 0$, ensuring that $g(x,\overline{u},D\overline{u}) = 0$ on $\partial\Omega$. ∎

Theorem 2.3. *If Q and G satisfy (1.5), (1.7) and (1.8), and there exists constants $0 < \sigma$, $C < \infty$ such that (2.2) holds,* then *there exists a classical solution $\overline{u}$ of Equation (1.1) such that $\|\overline{u}\|_{1,\sigma;\overline{\Omega}} \leqslant C$.*

Proof: A simple compactness argument shows that the *a priori* estimate (2.2) allows us to drop the assumption (1.6). The result then follows from the preceding theorems. ∎

It should be noted that the desired convergence results are **not** a trivial consequence of the mollification parameter tending to zero, since the function being mollified (that is, Du_η) depends itself upon the mollification parameter. Examples of the problems possible can be found in [**1**].

We mention in passing a result which uses the definition of the $(\ )_\eta$ operator to derive a restriction on the function $\varphi_\eta * u_\eta - u_\eta$.

Lemma 2.4. *There exists an $\overline{\eta} > 0$ such that for all $0 < \eta < \overline{\eta}$, we have*

$$\|\varphi_\eta * Du_\eta - Du_\eta\|_{0;\Omega} < S_1/2. \tag{2.5}$$

Proof: If the above result does not hold, there exists a sequence of $\eta_i \searrow 0$ such that

$$\|\varphi_\eta * Du_\eta - Du_\eta\|_{0;\Omega} \geqslant S_1/2 \qquad \text{for all } \eta_i \searrow 0. \tag{2.6}$$

It follows from Equation (1.10) that for such values of η, $S(\eta, Du_\eta) = 0$, implying that $(Du_\eta)_\eta \equiv 0$. That implies that u_η is a solution of $Qu_\eta = 0$ in Ω, $u_\eta = q(x,0)$ on $\partial\Omega$, so u_η is in fact independent of η for $\eta_i \searrow 0$. If follows from the convergence properties of the mollifier that $\|\varphi_\eta * Du_\eta - Du_\eta\|_{0;\Omega} \searrow 0$ under such circumstances, contradicting (2.6). The desired result must therefore hold. ∎

3. A PRIORI BOUNDS

In this section we give some partial results on the derivation of the *a priori* bound (2.2). We consider boundary conditions with relatively small nonlinearities, and begin by reformulating the problem (2.1) in the form

$$\begin{aligned} Qu_\eta &= 0 \qquad \text{in } \Omega \\ g(x, u_\eta, Du_\eta) &= \varphi_\eta(x) \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

where
$$\varphi_\eta(\cdot) \equiv q\Big(\cdot, (Du_\eta)_\eta\Big) - q(\cdot, Du_\eta).$$

The results which follow rely heavily on the results of Lieberman [**6**], and give, for a class of PDE and nonlinear boundary condition, the desired *a priori* bound when the relationship between φ_η and u_η is sufficiently strong. Since φ_η involves gradient terms, it is natural to expect that an estimate on $\|\varphi_\eta\|_{\sigma;\partial\Omega}$ will result from a bound on u_η in $C^{1,\sigma}(\overline{\Omega})$. The first result gives the desired *a priori* bound if one can bound $\|\varphi_\eta\|_{\sigma;\partial\Omega}$ in terms of an *a priori* bound on u_η in a Hölder space 'lower' than $C^{1+\sigma}(\overline{\Omega})$ (that is, $C^{1+\sigma-b}(\overline{\Omega})$ where $b > 0$).

THEOREM 3.1. *Consider the differential operators*

$$Q(x, u, Du, D^2u) \overset{\text{def}}{=} a^{ij}(x,u) D_{ij}u + \widetilde{a}(x,u) N(x, Du)$$
$$G(x, u, Du) \overset{\text{def}}{=} u - \vec{\beta}(x,u) \cdot Du - e(x,u) M(x, Du)$$

where $\vec{\beta} \cdot \vec{n} > \chi > 0$ on $\partial\Omega$ and $\mathcal{A} = [a_{ij}]$, $\widetilde{a}$, $\vec{\beta}$, $e \in C^1(\overline{\Omega} \times \mathbb{R})$. We assume $\|M\|_1, \|N\|_1 \leqslant 1$. If there exist constants $0 < b < \sigma < 1$ and constants A_1 and A_2 such that

$$\begin{aligned} |u_\eta|^{(0)}_{1;\Omega} &< A_1 \\ \|u_\eta\|_{\sigma;\Omega} &< A_1 \\ \|\varphi_\eta\|_{\sigma;\partial\Omega} &< A_2 \left(|u_\eta|^{-(1+\sigma-b)}_{2;\Omega} + 1 \right) \end{aligned} \tag{3.2}$$

and $\|e\|_0 < \varepsilon_1$ sufficiently small, then *there exists a constant $\overline{C} > 0$ independent of η such that*

$$|u_\eta|^{-(1+\sigma)}_{2+\rho;\Omega} < \overline{C}, \tag{3.3}$$

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where ρ is such that $b = \rho(2+\rho)^{-1}(1+\sigma)$.

PROOF: Let Λ denote an upper bound on $\|\mathcal{A}\|_{0,1} + \|\widetilde{a}\|_{0,1} + \left\|\vec{\beta}\right\|_{0,1} + \|e\|_{0,1}$, and let m, n be two distinct values of η. The function

$$Y = u_m - u_n,$$

is a solution of the linear problem

$$\begin{aligned} a^{ij}(x,u_m)D_{ij}Y &= f && \text{in } \Omega \\ Y - \vec{\beta}(x,u_m)\cdot DY &= \psi && \text{on } \partial\Omega, \end{aligned} \tag{3.4}$$

where

(3.5)

$$\begin{aligned} f &\overset{\text{def}}{=} \left[a^{ij}(x,u_n) - a^{ij}(x,u_m)\right] D_{ij}u_n + \left[\widetilde{a}(x,u_n) - \widetilde{a}(x,u_m)\right] N(x,Du_n) \\ &\quad + \widetilde{a}(x,u_m)\left[N(x,Du_n) - N(x,Du_m)\right] \\ \psi &\overset{\text{def}}{=} \varphi_m - \varphi_n + \left[e(x,u_m) - e(x,u_n)\right] M(x,Du_n) + \left[M(x,Du_m) - M(x,Du_n)\right] e(x,u_m) \\ &\quad + Du_n \cdot \left[\beta(x,u_m) - \beta(x,u_n)\right]. \end{aligned}$$

Under the above assumptions, the intermediate Schauder estimates of [**6**] apply, giving

$$|Y|_{2+\rho;\Omega}^{-(1+\sigma)} \leqslant C\left(|f|_{\rho}^{(1-\sigma)} + \|\psi\|_{\sigma;\partial\Omega}\right), \tag{3.6}$$

where $C = C(\Lambda, A_1, \sigma, \rho, \Omega)$. Now

$$\begin{aligned} |f|_{\rho;\Omega}^{(1-\sigma)} \leqslant &\left| \left[a^{ij}(x,u_n) - a^{ij}(x,u_m)\right] D_{ij}u_n \right|_{\rho;\Omega}^{(1-\sigma)} \\ &+ \left| \left[\widetilde{a}(x,u_n) - \widetilde{a}(x,u_m)\right] N(x,Du_n) \right|_{\rho;\Omega}^{(1-\sigma)} \\ &+ \left| \widetilde{a}(x,u_m)\left[N(x,Du_n) - N(x,Du_m)\right] \right|_{\rho;\Omega}^{(1-\sigma)}. \end{aligned} \tag{3.7}$$

By a simple result in [**3**] we have, (since $b < \rho$)

(3.8)

$$\begin{aligned} \left| \left[a^{ij}(x,u_n) - a^{ij}(x,u_m)\right] D_{ij}u_n \right|_{\rho;\Omega}^{(1-\sigma)} &\leqslant \left| a^{ij}(x,u_n) - a^{ij}(x,u_m) \right|_{0;\Omega} \left| D_{ij}u_n \right|_{\rho;\Omega}^{(1-\sigma)} \\ &\quad + \left|a^{ij}(x,u_n) - a^{ij}(x,u_m)\right|_{\rho;\Omega}^{(-b)} \left| D_{ij}u_n \right|_{0;\Omega}^{(1-\sigma+b)} \\ &\leqslant \varepsilon(m,n)K_1 + C_0(\Lambda, A_1)K_2, \end{aligned}$$

where

$$\begin{aligned} K_1 &\overset{\text{def}}{=} \max\left\{|u_n|_{2+\rho;\Omega}^{-(1+\sigma)}, |u_m|_{2+\rho;\Omega}^{-(1+\sigma)}\right\} \\ K_2 &\overset{\text{def}}{=} \max\left\{|u_n|_{2;\Omega}^{-(1+\sigma-b)}, |u_m|_{2;\Omega}^{-(1+\sigma-b)}\right\}, \end{aligned} \tag{3.9}$$

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and $\varepsilon(m,n) = \varepsilon(m,n,\Lambda,A_1)$ is such that $\varepsilon(m,n) \searrow 0$ as $m,n \searrow 0$, and C_0 depends only on Λ and A_1. This choice of intermediate norm is used since

$$|D_{ij}u|^{(1-\sigma)}_{\rho;\Omega} \leqslant |u|^{-(1+\sigma)}_{2+\rho;\Omega}. \tag{3.10}$$

The result concerning K_2 follows similarly. Similar arguments give

$$|f|^{(1-\sigma)}_{\rho;\Omega} \leqslant \varepsilon(m,n)K_1 + C_3K_2. \tag{3.11}$$

By assumption, we have

$$\|\varphi_m - \varphi_n\|_{\sigma;\partial\Omega} \leqslant A_2\left(|u_m|^{-(1+\sigma-b)}_{2;\Omega} + |u_n|^{-(1+\sigma-b)}_{2;\Omega} + 2\right) \leqslant 2A_2\,(K_2+1), \tag{3.12}$$

so arguments similar to those used above give

$$\|\psi\|_\sigma \leqslant [\varepsilon_1 + \varepsilon(m,n)]\,K_1 + C(K_2+1). \tag{3.13}$$

It follows from (3.6),(3.11), and (3.13) that

$$|Y|^{-(1+\sigma)}_{2+\rho;\Omega} \leqslant C\Big([\varepsilon_1 + \varepsilon(m,n)]K_1 + C_6(K_2+1)\Big). \tag{3.14}$$

Noting that

$$\Big(-(1+\sigma-b),2\Big) = t_0\,(0,0) + (1-t_0)\Big(-(1+\sigma),2+\rho\Big), \tag{3.15}$$

for $t_0 \stackrel{\text{def}}{=} \rho\,(2+\rho)^{-1}$, it follows from the interpolation inequalities for intermediate norms that for any $\tau > 0$ there exists a constant $C(\tau) < \infty$, determined only by τ,ρ,σ and Ω such that

$$K_2 < C(\tau)\,\|u\|_{0;\overline{\Omega}} + \tau\,K_1.$$

Using this in (3.14), we obtain for sufficiently small τ

$$|Y|^{-(1+\sigma)}_{2+\rho;\Omega} \leqslant \big(\varepsilon(m,n) + B\big)K_1 + C_7, \tag{3.16}$$

where $B < 1$ and $C_7 = C_7(\Lambda, A_1, A_2, \sigma, \rho, \Omega)$.

The desired *a priori* bound then follows as in [5]. $\square$

REMARK. The assumption that there exists a constant A_1 such that $|u_\eta|^{(0)}_{1;\Omega} < A_1$ follows from the structure conditions (1.7) by the interior estimates of [**8**].

The next result is similar, but avoids the problem of deriving a bound on $\|\varphi_\eta\|_{\sigma;\partial\Omega}$ from an estimate on u_η in a space of the form $C^{1+\sigma-b}(\overline{\Omega})$, using instead the more natural space $C^{1,\sigma}(\overline{\Omega})$. The result, however, requires that the constant relating the two be sufficiently small.

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THEOREM 3.2. *Consider the differential operators*

$$Q(x,u,Du,D^2u)\overset{\text{def}}{=}a^{ij}(x,u)D_{ij}u+\widetilde{a}(x,u)N(x,Du)$$
$$G(x,u,Du)\overset{\text{def}}{=}u-\vec{\beta}(x,u)\cdot Du-e(x,u)M(x,Du),$$

as in the previous result. If there exist constants $0<b<\rho<\sigma<\alpha\leqslant 1$ *such that* $b=\rho(2+\rho)^{-1}(1+\sigma)$, *and constants* $D_0<1$, A_1 *and* A_2 *such that*

$$\begin{aligned} &|u_\eta|^{(0)}_{1;\Omega}<A_1\\ (3.17)\qquad &|u_\eta|_{\sigma;\Omega}<A_1\\ &\|\varphi_\eta\|_{\sigma;\partial\Omega}<D_0C^{-1}\,|u_\eta|^{-(1+\sigma)}_{2;\Omega}+A_2,\end{aligned}$$

where C *is the constant from (3.6), and* $\|e\|_0<\varepsilon_1(D_0)$ *is sufficiently small,* then *there exists a constant* $\overline{C}>0$ *independent of* η *such that*

$$|u_\eta|^{-(1+\sigma)}_{2+\rho;\Omega}<\overline{C}. \tag{3.18}$$

PROOF: The proof proceeds as before, using the constant b and K_1,K_2, except that (3.12) is replaced with

$$\|\varphi_m-\varphi_n\|\,F_{\sigma;\partial\Omega}\leqslant D_0C^{-1}\,\left(|u_n|^{-(1+\sigma)}_{2;\Omega}+|u_m|^{-(1+\sigma)}_{2;\Omega}\right)+2A_2, \tag{3.19}$$

giving in place of (3.14),

$$\begin{aligned}|Y|^{-(1+\sigma)}_{2+\rho;\Omega}&\leqslant C\left(\left[\varepsilon_1+\varepsilon(m,n)\right]K_1+C_4\left(K_2+1\right)+2A_2\right)\\ &\qquad+D_0\left(|u_n|^{-(1+\sigma)}_{2;\Omega}+|u_m|^{-(1+\sigma)}_{2;\Omega}\right).\end{aligned} \tag{3.20}$$

Let $\delta_1>0$ be sufficiently small that $D_0<(1-\delta_1)^2(1+\delta_1)^{-1}$. We claim now that $|u_\eta|^{-(1+\sigma)}_{2+\rho;\Omega}$ is bounded independently of η. If not, by freezing n and allowing m to increase, we may choose m,n such that $|u_n|^{-(1+\sigma)}_{2+\rho;\Omega}<\delta_1\,|u_m|^{-(1+\sigma)}_{2+\rho;\Omega}$, giving

$$(1-\delta_1)\,|u_m|^{-(1+\sigma)}_{2+\rho;\Omega}<|Y|^{-(1+\sigma)}_{2+\rho};\Omega<(1+\delta_1)\;|u_m|^{-(1+\sigma)}_{2+\rho;\Omega}.$$

This gives, redefining C_4,

(3.21)

$$\begin{aligned}|Y|^{-(1+\sigma)}_{2+\rho;\Omega}&\leqslant C([\varepsilon_1+\varepsilon(m,n)]K_1+C_4(K_2+1))+\frac{(1-\delta_1)^2}{(1+\delta_1)}\left(|u_n|^{-(1+\sigma)}_{2+\rho;\Omega}+|u_m|^{-(1+\sigma)}_{2+\rho;\Omega}\right)\\ &\leqslant C([\varepsilon_1+\varepsilon(m,n)]K_1+C_4(K_2+1))+(1-\delta_1)^2\;|u_m|^{-(1+\sigma)}_{2+\rho;\Omega}\\ &\leqslant C([\varepsilon_1+\varepsilon(m,n)]K_1+C_4(K_2+1))+(1-\delta_1)\;|Y|^{-(1+\sigma)}_{2+\rho},\end{aligned}$$

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giving

$$|Y|^{-(1+\sigma)}_{2+\rho;\Omega} \leqslant (\delta_1)^{-1}\ C([\varepsilon_1 + \varepsilon(m,n)]K_1 + C_4(K_2+1)) \tag{3.22}$$
$$\leqslant [\varepsilon_1 + \varepsilon(m,n)]K_1 + C_5(K_2+1),$$

where $C_5 = C_5(\Lambda, A_1, A_2, \sigma, \rho, \Omega)$ and ε_1 and $\varepsilon(m,n)$ are rescaled in the obvious way.

Again we have

$$|Y|^{-(1+\sigma)}_{2+\rho;\Omega} \leqslant [\varepsilon_1 + \varepsilon(m,n)]K_1 + C_5(K_2+1) \tag{3.23}$$
$$\leqslant (\varepsilon(m,n) + B)K_1 + C_6,$$

where $C_6 = C_6(\Lambda, A_1, A_2, \sigma, \rho, \Omega)$ and $B < 1$.

As before, this suffices to give the desired *a priori* estimate. $\square$

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