ASYMMETRIC AND SYMMETRIC GRAPHS

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An (n, q) graph consists of n nodes and q edges, i.e. q distinct unordered pairs of different nodes, so that there are no loops or multiple edges. We write T for the number of unlabelled (n, q) graphs and F for the number of labelled (n, q) graphs. We say that a labelled graph is *symmetric* if there is a nonidentical permutation of its nodes which leaves the graph unaltered. We write r for the order of the automorphism group of the graph, i.e. the group of all those permutations of the nodes which leave the graph unaltered; we say that the graph is of symmetry order r. A graph which is not symmetric is called *asymmetric* and, for such a graph, obviously $r = 1$. We say that an unlabelled graph is symmetric or asymmetric according as the graph obtained by labelling its nodes is symmetric or asymmetric.

We write $N = n(n-1)/2$ and $B(h, k) = h!/\{k!(h-k)!\}$. Then $0 \leqq q \leqq N$ and $F = B(N, q)$. If an (n, q) graph is symmetric of order r, then so is its complement, i.e. the $(n, N-q)$ graph which has just those edges that the original graph lacks. Hence we can take $0 \leqq q \leqq (N/2)$ without loss of generality. We write $\mu = (2q - n \log n)/n$. We write C for a positive number, not always the same at each occurrence, independent of n and q. The notations $O(\ )$ and $o(\ )$ refer to the passage of n to infinity and each constant implied is a C. If we say that " almost all " graphs of a particular class have property P, we mean that the ratio of the number of those which lack the property to the number of those which have the property tends to 0 as $n \to \infty$. All our statements carry the implied condition that $n > C$.

Erdös and Renyi [1] considered labelled asymmetric graphs and, amongst other results, showed that almost all labelled graphs on n nodes are asymmetric. They announced the further result that, if $\mu \to \infty$, then almost all labelled (n, q) graphs are asymmetric.

We write $T(r)$ for the number of unlabelled (n, q) graphs of symmetry order r and $F(r)$ for the corresponding number of labelled graphs. Clearly

$$n!\,T(r) = rF(r). \tag{1}$$

We write $T^{(a)}$ (resp. $T^{(s)}$) for the number of unlabelled asymmetric (resp. symmetric) (n, q) graphs and $F^{(a)}$ (resp. $F^{(s)}$) for the numbers of labelled asymmetric (resp. symmetric) (n, q) graphs, so that

$$F^{(a)} = F(1), \quad F^{(s)} = \sum_{r=2}^{n!} F(r), \quad T^{(a)} = T(1), \quad T^{(s)} = \sum_{r=2}^{n!} T(r).$$

Let F_π be the number of labelled (n, q) graphs which are invariant under the permutation π of the n labelled nodes. The identity permutation is I, so that $F_I = F$. By the Polya–Burnside Counting Theorem [4], we have

$$n!\,T = \sum_{\pi} F_\pi = F + S \quad \Big(S = \sum_{\pi \neq I} F_\pi\Big),$$

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where the first sum is taken over all the $n!$ possible permutations π of the n labelled nodes. We have

$$F = \sum_{r=1}^{n!} F(r), \quad T = \sum_{r=1}^{n!} T(r)$$

and so, by (1),

$$S = \sum_{r=2}^{n!} (r-1) F(r) = n! \sum_{r=2}^{n!} (r-1) T(r)/r. \tag{2}$$

We require two lemmas.

LEMMA 1. *If* $\mu \to \infty$, *then* $S = o(F)$.

LEMMA 2. *If* $\mu \leqq 0$, *then* $F = o(S)$.

I proved Lemma 1 in [6] (in fact, I showed that $S = o(F)$ if and only if $\mu \to \infty$). I prove Lemma 2 later in the present paper.

THEOREM 1. *If* $\mu \to \infty$, *then almost all labelled* (n, q) *graphs are asymmetric; i.e.*,

$$F^{(s)} = o(F^{(a)}). \tag{3}$$

This is the theorem announced by Erdös and Renyi [1]. It follows at once from Lemma 1, since we have

$$F^{(s)} = \sum_{r=2}^{n!} F(r) \leqq \sum_{r=2}^{n!} (r-1) F(r)$$

$$= S = o(F) = o(F^{(a)} + F^{(s)}).$$

THEOREM 2. *If* $\mu \to \infty$, *then almost all unlabelled* (n, q) *graphs are asymmetric; i.e.*,

$$T^{(s)} = o(T^{(a)}). \tag{4}$$

We have

$$n!\, T^{(s)} = n! \sum_{r=2}^{n!} T(r) \leqq 2(n!) \sum_{r=2}^{n!} (r-1) T(r)/r$$

$$= 2S = o(F) = n!\, o(T)$$

and so (4). Since

$$F^{(a)}/F^{(s)} \geqq 2T^{(a)}/T^{(s)} \tag{5}$$

by (1), Theorem 2 implies Theorem 1.

THEOREM 3. *If* $\mu \leqq 0$, *then almost all unlabelled* (n, q) *graphs are symmetric; i.e.*,

$$T^{(a)} = o(T^{(s)}).$$

We have

$$n!\, T^{(a)} = n!\, T(1) = F(1) \leqq F = o(S)$$

by Lemma 2 and

$$S \leqq n! \sum_{r=2}^{n} T(r) = n!\, T^{(s)}.$$

by (2). The theorem follows.

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THEOREM 4. *If $\mu \leqq 0$, then, for any fixed R, almost all unlabelled (n, q) graphs are of symmetry order greater than R.*

For, by Lemma 2,

$$n! \sum_{r=1}^{R} T(r) \leqq R \sum_{r=1}^{R} n! \, T(r)/r = R \sum_{r=1}^{R} F(r)$$

$$\leqq RF = o(S) = o(n! \, T).$$

THEOREM 5. *If $\mu \to -\infty$ as $n \to \infty$, then almost all labelled (n, q) graphs are symmetric; i.e., $F^{(a)} = o(F^{(s)})$.*

I conjecture that the conditions in Theorems 3 and 5 are necessary as well as sufficient but I am unable to prove this. What I can prove however is the following theorem, which shows that the conditions of Theorems 1 and 2 are necessary as well as sufficient.

THEOREM 6. *If μ is bounded above as $n \to \infty$, then $F^{(s)} \neq o(F^{(a)})$ and $T^{(s)} \neq o(T^{(a)})$.*

Before proving Theorem 5 it is convenient to prove Lemma 2, since a subsidiary lemma is needed to prove both.

Proof of Lemma 2. Lemma 2 can be deduced from an asymptotic approximation to T which I announced in [7] and indeed the result can be seen to be true under the slightly wider condition that $\overline{\lim}\, \mu \leqq 0$. Hence Theorem 3 is true under this wider condition. But the calculations leading to this approximation are very much more elaborate than the proof of Lemma 2 which I give here.

Let p be the number of nodes unchanged by the permutation π. Then an (n, q) graph composed of any (p, q) graph on these p nodes and the other $n-p$ isolated nodes is invariant under π. Hence

$$F_\pi \geqq F(p, q) = B(P, q),$$

where $P = p(p-1)/2$. The p unchanged nodes may be chosen in $B(n, p)$ ways and, when these are chosen, there are $H_1(n-p)$ ways of permuting the remaining $n-p$ nodes, where $H_1(n)$ is Euler's rencontre number, i.e. the number of ways of permuting n different objects so that none remains unmoved. Hence there are just

$$B(n, p) \, H_1(n-p)$$

different π which leave just p nodes unchanged.

We have then

$$S \geqq \sum_{p=0}^{n-2} H_1(n-p) \, B(n, p) \, B(P, q).$$

It was proved by Euler [3, 5] that

$$H_1(n) = (n-1)\{H_1(n-1) + H_1(n-2)\}$$

and, from this, we can prove by induction on n that

$$H_1(n) \geqq C(n!) \quad (n \geqq 2).$$

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Hence, if we write $t = n-p$ and

$$\Omega_t = B(P, q)/p!$$

we have

$$S/F > C \sum_{t=2}^{n} \Omega_t/\Omega_0.$$

We write $j = [n^{1/2}/\log n]$. A little calculation suffices to deduce the following lemma from Stirling's Theorem and the Second Mean Value Theorem.

LEMMA 3. *If $\mu < C$ and $0 \leqq t \leqq j$, then*

$$\Omega_t \sim \Omega_0 e^{-\mu t}$$

as $n \to \infty$.

If $\mu \leqq 0$, we deduce that

$$S/F \geqq C \sum_{t=2}^{j} e^{-\mu t} \to \infty$$

as $n \to \infty$. This is Lemma 2.

Proof of Theorem 5. We write $L = L(n, q)$ for the number of labelled (n, q) graphs which contain at least 2 isolated nodes and so are necessarily symmetric. Again $f = f(n, q)$ is the number of connected labelled (n, q) graphs. Clearly

$$F^{(s)} \geqq L(n, q). \tag{6}$$

Next

$$L(n, q) \geqq \sum_{t=2}^{n} B(n, t) f(n-t, q),$$

since the typical term on the right enumerates the number of labelled (n, q) graphs that consist of t isolated nodes and a connected $(n-t, q)$ graph.

Erdös and Renyi [2] showed that, for bounded μ, we have

$$f(n, q)/F(n, q) \sim \exp(-e^{-\mu}).$$

Hence, if $2 \leqq t \leqq j$, we have

$$f(n-t, q) \geqq F(n-t, q)\{\exp(-e^{-\mu'}) + o(1)\},$$
$$\geqq F(n-t, q)\{\exp(-e^{-\mu}) + o(1)\},$$

where

$$\mu' = \{2q/(n-t)\} - \log(n-t) > \mu$$

if $t > 0$. Hence

$$L \geqq FE\{\exp(-e^{-\mu}) + o(1)\},$$

where

$$E = \sum_{t=2}^{j} B(n, t) B(P, q)/B(N, q) \sim \sum_{t=2}^{j} e^{-\mu t}/t!,$$

by Lemma 3. Hence

$$E \sim \sum_{t=2}^{\infty} e^{-\mu t}/t! = \exp(e^{-\mu}) - 1 - e^{-\mu}$$

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and so

$$L/F \geqq \{1-(1+e^{-\mu})\exp(-e^{-\mu})\}\{1+o(1)\}.$$

This is true for bounded μ. But it is easy to show that L/F, the proportion of labelled (n, q) graphs which contain at least two isolated nodes, decreases (at least non-strictly) as q increases for fixed n. Hence, if $\mu \to -\infty$ as $n \to \infty$, we have $L/F \to 1$. Hence, by (6), $F^{(s)}/F \to 1$ and this is Theorem 5.

Again, if $\mu \to c$, a fixed finite number, as $n \to \infty$, we see that

$$1-(1+e^{-\mu})\exp(-e^{-\mu}) \to 1-(1+e^{-c})/\exp(e^{-c}) > 0.$$

Hence $F^{(s)}/F$ does not tend to zero, nor, by (5) does $T^{(s)}/T^{(a)}$. Hence Theorem 6.

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