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DOUBLING CONSTRUCTIONS IN LATTICE THEORY

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ABSTRACT. This paper examines the simultaneous doubling of multiple intervals of a lattice in great detail. In the case of a finite set of W-failure intervals, it is shown that there in a unique smallest lattice mapping homomorphically onto the original lattice, in which the set of W-failures is removed. A nice description of this new lattice is given. This technique is used to show that every lattice that is a bounded homomorphic image of a free lattice has a projective cover. It is also used to give a sufficient condition for a fintely presented lattice to be weakly atomic and shows that the problem of which finitely presented lattices are finite is closely related to the problem of characterizing those finite lattices with a finite W-cover.

0. **Introduction.** The doubling of intervals and pseudo-intervals has proven to be a valuable construction tool in the solution of several problems in Lattice Theory—see the references for some of the applications. In one such application, [2], the construction's ability to "repair" W-failures is used to show that all finitely generated free lattices are weakly atomic. The method of proof required firstly, the simultaneous repairing of all W-failures in a lattice, and secondly, the iteration of this repair process to produce an inverse limit that satisfied Whitman's condition. The stated result then followed by initializing the repair process to the lattice $FD(3)$ and invoking an equivalence proved in McKenzie [12]. One should also note that a syntactical proof of weak atomicity was eventually found by Freese and Nation in [8].

Attempts to apply the above methods from [2] to solve other problems in Lattice Theory have been thwarted particularly at the first stage of this construction. The simultaneous "repair" of all W-failures in a given lattice, L, was obtained by taking the (generalized) pullback of all individual repair epimorphisms $L[I] \longrightarrow L$. This produced an oversized prehomomorphic image of L that lacked a decent description as a lattice manufactured from L and the family of all W-failure intervals on L. The lack of a lattice-like description limited further manipulations to the generality of Category Theory. The fact that certain sublattices of this large lattice would serve just as well showed that it also contained many redundancies.

In this paper, we examine the multiple doubling of intervals—actually of convex sets—in great detail. In § 2, we present a complete lattice description of the simultaneous doubling of a family of convex sets on a given ordered set that generalizes the original interval construction introduced in [1]. In § 3, we restrict our attention to doubling finite

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sets of W-failure intervals on a lattice and show that there is a unique smallest sublattice of the full pullback that maps by restriction onto the original lattice. This sublattice provides then a canonical W-failure repairing and is described by the induced binary relation of "precedence" defined on intervals. In § 4, we consider (inverse) limits of this canonical repairing and show that every bounded lattice has a projective cover (in the variety of all lattices).

In § 5, we apply the earlier results to the construction of finitely presented lattices. It is shown that, starting with a certain model of a partial lattice, **P**, $FL(\mathbf{P})$ is constructed by iterating the canonical repairing of "Dean-failure" intervals and taking the appropriate sublattice of the inverse limit. Thus finitely presented lattices are produced in the same way that $FL(3)$ was produced from $FD(3)$ in [2], and projective covers of bounded lattices in § 4. This shows that the existence of finite projective covers of bounded lattices and the finiteness of finitely presented lattices are identical problems intimately connected with the doubling construction.

Finally in § 6, we show that finitely generated free distributive lattices can be produced by the multiple doubling of intervals from certain canonical models. This provides a new way of counting the elements of such lattices that might be of some future use in solving Dedekind's problem.

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1. **Preliminaries.** In McKenzie [12], many important concepts for the study of lattices were introduced. A lattice homomorphism, $\varphi\colon L \longrightarrow M$, is called *lower bounded* if for every $m \in M$, $\{x \in L : \varphi x \geq m\}$ is either empty or contains a smallest member. That φ is *upper bounded* is defined dually, and φ is *bounded* if it is both lower and upper bounded.

When $\varphi\colon L \longrightarrow M$ is an epimorphism, i.e. surjective, then φ is lower [resp. upper] bounded if and only if there exists a join [resp. meet] monomorphism, $\alpha\colon M \longrightarrow L$ [$\beta\colon M \longrightarrow L$] satisfying $\alpha y \leq x$ iff $y \leq \varphi x$ [resp. $x \leq \beta y$ iff $\varphi x \leq y$] for all $x \in L$ and all $y \in M$. A map $\alpha\colon M \longrightarrow L$ [resp. $\beta\colon M \longrightarrow L$] that satisfies the above property is called a *left* [resp. *right*] *adjoint of* φ.[1]

A *finite* lattice, L, is called *lower* [resp. *upper*] *bounded* if any epimorphism, $FL(n) \longrightarrow L$ is lower [resp. upper] bounded.

The interval construction, $L[I]$, works just as well by replacing the interval, I, of a lattice, L with an aribtrary convex set. One can also replace the lattice, L, by an arbitrary ordered set, P. The following is just a slight generalization of the original definitions and results of [1].

DEFINITION. Let $(P, \leq)$ be an ordered set and $C \subseteq P$ a convex subset of P. The ordered set, $P[C]$, produced by *doubling the convex subset* C, is given by:

1. $P[C] := (P \setminus C) \cup (C \times \mathbf{2})$;

[1] This reverses the role of α and β used by McKenzie [12] and Freese and Nation [8].

2. A relation, $\leq_{P[C]}$, on $P[C]$ defined by: for any $x, y \in P \setminus C$ and any $(z, i), (t, j) \in C \times \mathbf{2}$,
   a. $x \leq_{P[C]} y$ iff $x \leq_P y$;
   b. $x \leq_{P[C]} (t, j)$ iff $x \leq_P t$;
   c. $(z, i) \leq_{P[C]} y$ iff $z \leq_P y$;
   d. $(z, i) \leq_{P[C]} (t, j)$ iff $z \leq_P x$ and $i \leq_{\mathbf{2}} j$;
3. Functions, $\kappa : P[C] \longrightarrow P$, and $\alpha, \beta : P \longrightarrow P[C]$ defined by taking the (disjoint) union of the identity function, $1_{P \setminus C}$, with the projection, $C \times \mathbf{2} \longrightarrow C$, and the two embeddings, $C \longrightarrow C \times \mathbf{2}$, given by $x \longmapsto (x, 0)$ and $x \longmapsto (x, 1)$, respectively.

THEOREM [1]. *Let C be a convex subset of an ordered set, $(P, \leq)$. Then $(P[C], \leq_{P[C]})$ is an ordered set, and the mappings $\kappa : P[C] \longrightarrow P$ and $\alpha, \beta : P \longrightarrow P[C]$, are order homomorphisms. Moreover, α [resp. β] is the (injective) left [resp. right] adjoint of the surjection, κ. Thus α and κ preserve all existing joins, while κ and β preserve all existing meets.*

The connections between lower and/or upper bounded lattices and the doubling, in lattices, of particular classes of convex sets appeared in [3]. We first need the relevant definitions.

DEFINITION. Let L be a lattice, and C, a convex subset of L. C is called a *lower pseudo-interval* if there exist $u, v_1, \ldots, v_n \in L$ such that $u \leq v_i$ for all i and C is the (finite) union of the intervals, $[u, v_i]$. An *upper pseudo-interval* is defined dually.

Clearly a convex set is an interval if and only if it is both a lower and an upper pseudo-interval. If L is a finite lattice, then a convex $C \subseteq L$ has a least [resp. greatest] element if and only if C is $\wedge$-[resp., $\vee$-] closed. Thus lower [resp. upper] pseudo-intervals are precisely the convex $\wedge$-[resp. $\vee$-] subsemilattices.

THEOREM [3]. *The class of all lower bounded (finite) lattices is the smallest isomorphism closed class,* **LB**, *satisfying the properties:*
1. $\mathbf{1} \in \mathbf{LB}$, *where* $\mathbf{1}$ *is the singleton lattice;*
2. $L \in \mathbf{LB}$ *and* $C \subseteq L$ *a lower pseudo-interval implies* $L[C] \in \mathbf{LB}$.

We leave to the reader the proper formulations of the above theorem for upper bounded lattices with upper pseudo-intervals, and bounded lattices with intervals.

2. **The doubling construction.**

In this section we extend the definition of doubling a single convex set to handle the simultaneous doubling of an arbitrary collection of convex sets. We show that this multi-doubling is the pullback of the family of canonical epimorphisms determined by individual doublings.

A collection of convex sets from a given ordered set, say P, can be dealt with in two natural ways: as a subset of $\mathbf{Cvx}(P)$, the system of all convex subsets, or as an indexed family of convex subsets, i.e. a function with codomain, $\mathbf{Cvx}(P)$. It will be convenient for our purposes to always work with the indexed family representation. Therefore if we

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speak of the *set* of convex subsets, $\mathbf{C} = (C_\sigma : \sigma \in \Sigma)$, we are implicitly assuming that the indexing function, $\sigma \longmapsto C_\sigma$, is injective.

DEFINITION. Let P be an ordered set, and $\mathbf{C} = (C_\lambda : \lambda \in \Lambda)$ be a family of convex subsets of P. We define the *doubling of the family*, $\mathbf{C} = (C_\lambda : \lambda \in \Lambda)$, *of convex subsets of* P by:

1. For $x \in P$, $\Lambda(x) = \Lambda x := \{\lambda \in \Lambda : x \in C_\lambda\}$.
2. $P[\mathbf{C}] = P[C_\lambda : \lambda \in \Lambda] := \bigcup\big\{\{x\} \times \mathbf{2}^{\Lambda x} : x \in P\big\}$.
3. For $\langle x, X\rangle, \langle y, Y\rangle \in P[\mathbf{C}]$, $\langle x, X\rangle \leq \langle y, Y\rangle :\Leftrightarrow x \leq y$ and $\Lambda x \cap \Lambda y \subseteq X \Rightarrow Y$, where $\Rightarrow$ is the Boolean operation of implication, i.e. $X \Rightarrow Y = (\Lambda \setminus X) \cup Y$.
4. Functions, $\kappa : P[\mathbf{C}] \longrightarrow P$ and $\alpha, \beta : P \longrightarrow P[\mathbf{C}]$ defined by $\kappa\langle x, X\rangle = x, \alpha x = \langle x, \emptyset\rangle$ and $\beta x = \langle x, \Lambda x\rangle$ respectively.

THEOREM. *Let P be an ordered set, and $\mathbf{C} = (C_\lambda : \lambda \in \Lambda)$ be a family of convex subsets of P. Then $(P[\mathbf{C}], \leq)$ is an ordered set, and the mappings, $\kappa : P[\mathbf{C}] \longrightarrow P$ and $\alpha, \beta : P \longrightarrow P[\mathbf{C}]$, are order homomorphisms. Moreover, α [resp. β] is the (injective) left [resp. right] adjoint of the surjection, κ. Thus α and κ preserve all existing joins while κ and β preserve all existing meets.*

PROOF. That $\leq$ is reflexive is trivial. Anti-symmetry follows easily from the observation that $\Lambda x \cap \Lambda y \subseteq X \Rightarrow Y$ if and only if $\Lambda y \cap X \subseteq Y$. To show transitivity, we need the following claim:

$$x \leq y \leq z \Rightarrow \Lambda x \cap \Lambda z \subseteq \Lambda y$$

Now if $\langle x, X\rangle \leq \langle y, Y\rangle \leq \langle z, Z\rangle$, we have $\Lambda x \cap \Lambda z = \Lambda x \cap \Lambda y \cap \Lambda z$. Therefore $\Lambda x \cap \Lambda z \subseteq (X \Rightarrow Y) \cap (Y \Rightarrow Z) \subseteq X \Rightarrow Z$, as desired.

To show the claim, we need only note that for any convex set C_α containing x and z, we have $y \in C_\alpha$ for all $y \in x \backslash z$. That is $\alpha \in \Lambda x \cap \Lambda z$ implies $\alpha \in \Lambda y$. ∎

COROLLARY. *If $|\mathbf{C}| = 1$, i.e. $\Lambda = \{1\}$ and $\mathbf{C} = \{C\}$, then $P[\mathbf{C}] \cong P[C]$.*

PROOF. Since $\Lambda = \{1\}$, $\mathbf{2}^\Lambda = \{\emptyset, \Lambda\}$ and for any $\langle x, X\rangle$, $\emptyset$ and Λ are the only possible values of $X \subseteq \Lambda x \subseteq \Lambda$. Define $\varphi : P[\mathbf{C}] \longrightarrow P[C]$ by

$$\varphi\langle x, X\rangle = \begin{cases} x, & x \notin C \\ (x, 0), & x \in C \text{ and } X = \emptyset \\ (x, 1), & x \in C \text{ and } X = \Lambda. \end{cases}$$

That φ is an order isomorphism is an easy exercise. ∎

Since the dual of an ordered set [and any convex subset] is again such, we can also consider the doubling of convex sets associated with the dual structure, $P^\partial = (P, \geq)$. Note that as sets, $P[\mathbf{C}] = P^\partial[\mathbf{C}]$, and the order relation on $P^\partial[\mathbf{C}]$, which we write imprecisely as $\leq^\partial$, is given by

$$\langle x, X\rangle \leq^\partial \langle y, Y\rangle \Leftrightarrow y \leq x \text{ [in } P] \text{ and } \Lambda x \cap \Lambda y \subseteq X \Rightarrow Y.$$

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LEMMA. *The function $\partial\colon P[\mathbf{C}] \longrightarrow P[\mathbf{C}]$ defined by $\partial\langle x, X\rangle := \langle x, \Lambda x \setminus X\rangle$ provides a dual order isomorphism between $(P[\mathbf{C}], \leq)$ and $(P^{\partial}[\mathbf{C}], \leq^{\partial})$.*

PROOF. For $\langle x, X\rangle \in (P[\mathbf{C}] = (P^{\partial}[\mathbf{C}], \partial^2\langle x, X\rangle = \partial\langle x, \Lambda x \setminus X\rangle = \langle x, \Lambda x \setminus (\Lambda x \setminus X)\rangle = \langle x, X\rangle$. Therefore $\partial^2 = 1_{(P[\mathbf{C}])}$.

For $\langle x, X\rangle, \langle y, Y\rangle \in (P[\mathbf{C}], \partial\langle y, Y\rangle \leq^2 \partial\langle x, X\rangle$ iff $\langle y, \Lambda y \setminus Y\rangle \leq^{\partial} \langle x, \Lambda x \setminus X\rangle$ iff $x \leq y$ and $\Lambda x \cap \Lambda y \subseteq (\Lambda y \setminus Y) \Rightarrow (\Lambda x \setminus X) = X \Rightarrow Y$ iff $\langle x, X\rangle \leq \langle y, Y\rangle$. ∎

The duality above allows results concerning joins and meets in $P[\mathbf{C}]$ to come in dual pairs.

THEOREM. *Let P be an ordered set, and $\mathbf{C} = (C_\lambda : \lambda \in \Lambda)$ be a family of convex subsets of P. For any family, $\langle a_i, A_i\rangle_{i\in I}$, in $P[\mathbf{C}]$, its supremum exists in $P[\mathbf{C}]$ if and only if the supremum of $(a_i)_{i\in I}$ exists in P. In this case, we have*

$$\bigvee\langle a_i, A_i\rangle = \langle \bigvee a_i, \Lambda(\bigvee a_i) \cap \bigcup A_i\rangle.$$

PROOF. Let $\langle a, A\rangle = \vee\langle a_i, A_i\rangle$. by applying κ, which preserves existing joins, we get $a = \kappa\langle a, A\rangle = \kappa(\vee\langle a_i, A_i\rangle) = \vee\kappa\langle a_i, A_i\rangle = \vee a_i$.

Conversely, assume $a = \vee a_i$ exists in P and define $A := \Lambda a \cap \cup A_i$. Now $\langle x, X\rangle$ is an upper bound of $\langle a_i, A_i\rangle_{i\in I}$ if and only if for all $i \in I$, $a_i \leq x$ and $\Lambda x \cap \Lambda a_i \subseteq A_i \Rightarrow X$ if and only if a $\leq x$ and for all $i \in I$, $\Lambda x \cap \Lambda a_i \subseteq A_i \Rightarrow X$. Now $a_i \leq a \leq x$ implies $\Lambda x \cap \Lambda a \cap \Lambda a_i = \Lambda x \cap \Lambda a_i$. Elementary reworking of the last equivalence makes $\langle x, X\rangle$ an upper bound of $\langle a_i, A_i\rangle_{i\in I}$ if and only if $\langle a, A\rangle \leq \langle x, X\rangle$. ∎

COROLLARY. *Let P be an ordered set, and $\mathbf{C} = (C_\lambda : \lambda \in \Lambda)$ be a family of convex subsets of P. For any family, $\langle a_i, A_i\rangle_{i\in I}$, in $P[\mathbf{C}]$, the infimum exists in $P[\mathbf{C}]$ if and only if the infimum of $(a_i)_{i\in I}$ exists in P. In this case, we have*

$$\bigwedge\langle a_i, A_i\rangle = \langle \bigwedge a_i, \Lambda(\bigwedge a_i) \cap \bigcap(\Lambda a_i \Rightarrow A_i)\rangle.$$

PROOF. We need only apply duality and calculate $\wedge\langle a_i, A_i\rangle = \partial(\vee\partial\langle a_i, A_i\rangle)$ to obtain the meet formula. ∎

COROLLARY. *If P is a (complete) lattice, then so is $P[\mathbf{C}]$. In this case, $\kappa\colon P[\mathbf{C}] \longrightarrow P$ is a (complete) lattice homomorphism and $\alpha, \beta\colon P \longrightarrow P[\mathbf{C}]$ is a (complete) join [resp. meet] semilattice monomorphism.*

If $\mathbf{C} = (C_\lambda : \lambda \in \Lambda)$ is a family of convex subsets of an ordered set, P, then for any subset of the indices, say $\Xi \subseteq \Lambda$, we have the subfamily,

$$\mathbf{C} \mid \Xi := (C_\xi : \xi \in \Xi),$$

obtained by restriction, and a canonical function,

$$\kappa_{\Lambda,\Xi}\colon P[\mathbf{C}] \longrightarrow P[\mathbf{C} \mid \Xi],$$

called the (Λ, Ξ)-*projection* defined by $\langle x, X\rangle \longrightarrow \langle x, X \cap \Xi\rangle$. If $\Sigma \subseteq \Xi \subseteq \Lambda$, we easily have $\kappa_{\Xi,\Sigma} \circ \kappa_{\Lambda,\Xi} = \kappa_{\Lambda,\Sigma}$, and that $\kappa_{\Xi,\Xi}$ is the identity on $P[\mathbf{C} \mid \Xi]$. Note that for $\Xi = \emptyset$, $P[\mathbf{C} \mid \emptyset] = P$ and the $(\Lambda, \emptyset)$-projection is the previously defined $\kappa\colon P[\mathbf{C}] \longrightarrow P$.

LEMMA. *For any $\Xi \subseteq \Lambda$, the (Λ,Ξ)-projection $\kappa_{\Lambda,\Xi}\colon P[\mathbf{C}] \longrightarrow P[\mathbf{C} \mid \Xi]$ is a surjective order homomorphism with left and right adjoints,*

$$\alpha_{\Xi,\Lambda}, \beta_{\xi,\Lambda}\colon P[\mathbf{C} \mid \Xi] \longrightarrow P[\mathbf{C}]$$

defined by $\langle x, X\rangle \longmapsto \langle x, X\rangle$ and $\langle x, X\rangle \longmapsto \langle x, \Lambda x \cap (\Xi \Rightarrow X)$ respectively.

PROOF. One need only note that, for $X \subseteq \Lambda x$, $\Xi x = \Lambda x \cap \Xi$ and $(\Lambda x \setminus X) \cap \Xi = \Xi x \setminus X$. Now the join formula and the meet formula, via the duality lemma, are easily deduced for the projection. ∎

By our previous remarks, any suitably closed system of (Ξ, Σ)-projections will form a commutative diagram. We are particularly interested in the system,

$$(P[\mathbf{C}] \longrightarrow P[\mathbf{C} \mid \Xi] \longrightarrow P)_{\Xi \in \pi},$$

where π is a partition of Λ.

LEMMA. *If π is a partition of Λ, the above diagram is a (generalized) pullback.*

PROOF. Set-theoretically, a typical member in the (generalized) pullback of $(P[\mathbf{C} \mid \Xi] \longrightarrow P)_{\Xi\in\pi}$ is of the form $(\langle x_\Xi, X_\Xi\rangle)_{\Xi\in\pi}$ where $x_\Xi = x_{\Xi'}$, for all $\Xi, \Xi' \in \pi$ and $X_\Xi \subseteq \Xi_x = \Lambda x \cap \Xi$. Since π is a partition of Λ, we have for any $x \in P$ and $X, X' \subseteq \Lambda x$ that $X = X'$ if and only if for all $\Xi \in \pi$, $X \cap \Xi = X' \cap \Xi$. Therefore the elements of the pullback can be represented uniquely by the elements of $P[\mathbf{C}]$. ∎

COROLLARY. *$P[\mathbf{C}]$ is the [generalized] pullback of the family. $(P[\mathbf{C}_\lambda] \longrightarrow P)_{\lambda\in\Lambda}$.*

PROOF. Let π be the identity partition of Λ. ∎

3. **Lattices and W-failure intervals.** As mentioned in the introduction, several important applications of the (old) interval construction have dealt with particular cases of the general construction, when the underlying ordered set is a (finite) lattice, and when the family of convex subsets is a (finite) set of intervals, especially W-failure intervals. In these applications, the main concern was to "repair" the W-failures of a given (finite) lattice by the construction of a suitable preimage (finite) lattice.

The known method, [2], of repairing the W-failures of a (finite) lattice, L, is to let $\mathbf{I}$ be the set of all W-failure intervals of L and produce $\kappa\colon L[\mathbf{I}] \longrightarrow L$. In general, $L[\mathbf{I}]$ is extremely superfluous in that $L[\mathbf{I}]$ will contain many sublattices that map via κ onto L and any of these would do just as well. The main result of this section is to show that, when L is finite, and $\mathbf{I}$ is a (finite) set of W-failure intervals, $L[\mathbf{I}]$ contains a unique least sublattice, $L_{\ll}[\mathbf{I}]$, with this property. This sublattice is described by defining a relation, $\ll$, on $\mathbf{I}$ and considering only those pairs, $\langle x, X\rangle \in L[\mathbf{I}]$, where X is suitably $\ll$-closed.

Recall that a *cover* is a surjective homomorphism, $f\colon L \longrightarrow M$, such that, for any $g\colon X \longrightarrow L$, g is an epimorphism whenever $f \circ g$ is. Equivalently, L contains no proper sublattice with M as its direct image under f. An interval, $u \backslash v = \{x \in L : u \leq x \leq v\}$, in a lattice, L, is called a *W-failure interval* if there exist finite $X, Y \subseteq L$ with $\bigwedge X = u$, $\bigvee Y = v$, and $u \backslash v \cap (X \cup Y) = \emptyset$.

LEMMA. *Let L be a lattice and $I = u\backslash v$ an interval in L; then the canonical κ: $L[I] \longrightarrow L$ is a cover if and only if I is a W-failure interval.*

PROOF. Let I be a W-failure interval on L, and S be a sublattice of $L[I]$ with $\kappa[S] = L$. Therefore there exist finite $X, Y \subseteq L\backslash I$ with $\bigwedge X = u$, $\bigvee Y = v$, and $X \cap \downarrow v = Y \cap \uparrow u = \emptyset$. Now for each $p \in L \setminus I$, we have $\Lambda p = \emptyset$ and $\alpha p = \beta p = \langle p, \emptyset\rangle$. Since $\kappa[S] = L$, we also obtain $\langle p, \emptyset\rangle \in S$ for all $p \in L\setminus I$. Therefore, $\langle u, \Lambda\rangle = \bigwedge \beta[X]$ and $\langle v, \emptyset\rangle = \bigvee \alpha[Y]$ also belong to S. Again since $\kappa[S] = L$ we have for each $q \in I$, $\langle q, Q\rangle \in S$ for some $Q \in \{\emptyset, \Lambda\}$. Since $\langle q, \Lambda\rangle = \langle q, Q\rangle \vee \langle u, \Lambda\rangle$ and $\langle q, \emptyset\rangle = \langle q, Q\rangle \wedge \langle v, \emptyset\rangle$, these elements also lie in S. Thus $S = L[I]$ and κ is a cover.

Conversely, assume κ is a cover. Since $\kappa\big[\alpha[L]\big] = \kappa\big[\beta[L]\big] = L$, neither of these proper subsets can be sublattices. Therefore $\alpha[L]$ is not meet-closed and $\beta[L]$ is not join-closed.

Take finite $X \subseteq L$ such that $\bigwedge \alpha[X] = \langle r, \Lambda\rangle$ where $r = \bigwedge X$. Now $\langle u, \Lambda\rangle \leq \langle x, \emptyset\rangle$ if and only if $x \in (L \setminus I) \cap \uparrow u$. Therefore the meet closure of $\alpha[L]$ is given by

$$M := \alpha[L] \cup \{\langle r, \Lambda\rangle : \text{ for some } X \subseteq (L \setminus I) \cap \uparrow u \text{ with } r = \bigwedge X \leq v\},$$

and the join closure of this set is contained in

$$S := \alpha[L] \cup \{\langle p, \Lambda\rangle : \text{ for some } X \subseteq (L \setminus I) \cap \uparrow u \text{ with } v \geq p \geq \bigwedge X\}.$$

Easy calculations show that S is indeed a sublattice of L and, since κ is a cover, $S = L$. Therefore $u = \bigwedge X$ for some $X \subseteq (L \setminus I) \cap \uparrow u$.

Similarly, by working with $\beta[L]$, we produce a $Y \subseteq (L \setminus I) \cap \downarrow v$ with $\bigvee Y = v$. Therefore I is a W-failure interval. ∎

THEOREM. *Let $\mathbf{I} = (I_\lambda : \lambda \in \Lambda)$ be a* finite *set of W-failure intervals in a lattice, L. For any sublattice, $S \subseteq L[\mathbf{I}]$, the following are equivalent:*

(1) $\kappa[S] = L$;

(2) $\alpha[L] \subseteq S$;

(3) $\beta[L] \subseteq S$.

In particular, there exists a smallest sublattice of $L[\mathbf{I}]$, namely $T := \langle\alpha[L]\rangle = \langle\beta[L]\rangle$ that maps via κ onto L.

PROOF. Clearly (2) and (3) individually imply (1). The proofs of (1)⇒(2) and (1)⇒(3) are by induction on $|\Lambda|$. If $|\Lambda| = 1$, these results hold by the previous lemma.

Let $\Lambda = \Xi \cup \{\infty\}$, $\mathbf{J} = (I_\lambda : \lambda \in \Xi)$ and $\mathbf{I} = \mathbf{J} \cup (I_\infty)$, and set $P := L[\mathbf{I}]$ and $Q := L[\mathbf{J}] = L[\mathbf{I} \mid \Xi]$. Then the canonical projections,

$$\begin{aligned} &\kappa_{\Xi,\emptyset} \circ \kappa_{\Lambda,\Xi} : P \longrightarrow Q \longrightarrow L \\ &= \kappa_{\{\infty\},\emptyset} \circ \kappa_{\Lambda,\{\infty\}} : P \longrightarrow L[I_\infty] \longrightarrow L \end{aligned}$$

are the two edges of a pullback square. Let S be a sublattice of P which maps onto L. Since $\kappa_{\{\infty\},\emptyset}$ is a cover, $\kappa_{\Lambda,\{\infty\}}[S] = L[I_\infty]$.

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Define $T := \kappa_{\Lambda,\Xi}[S]$. Then $\kappa_{\Xi,\emptyset}[T] = L$. Since $|\mathbf{J}| < |\mathbf{I}|$, we have, by induction, that T contains both $\alpha_{\emptyset,\Xi}[L]$ and $\beta_{\emptyset,\Xi}[L]$. To complete the proofs, take $x \in L$.

(1)⇒(2): Since $\alpha_{\emptyset,\Xi}(x) = \langle x, \emptyset \rangle \in T$, there exists $A \subseteq \Lambda x$ with $\langle x, A \rangle \in S$ and $A \cap \Xi = \emptyset$. Therefore $A \subseteq \{\infty\}$. If $A = \emptyset$, we are done; therefore assume $A = \{\infty\}$ and hence $\infty \in \Lambda x$. Since $\langle x, \emptyset \rangle \in L[I_\infty]$, there is a $B \subseteq \Lambda x$ with $\langle x, B \rangle \in S$ and $B \cap \{\infty\} = \emptyset$. Therefore $\alpha_{\emptyset,\Lambda}(x) = \langle x, \emptyset \rangle = \langle x, A \rangle \wedge \langle x, B \rangle \in S$.

(1)⇒(3): Since $\beta_{\emptyset,\Xi}(x) = \langle x, \Xi x \rangle \in T$, there exists $A \subseteq \Lambda x$ with $\langle x, A \rangle \in S$ and $A \cap \Xi = \Xi x$. If $\Lambda x = \Xi x$ we are done; therefore assume $\infty \in \Lambda x$. This implies that $\langle x, \{\infty\} \rangle \in L[I_\infty]$ and hence there exists $B \subseteq \Lambda x$ with $\langle x, B \rangle \in S$ and $B \cap \{\infty\} = \{\infty\}$. Therefore $\beta_{\emptyset,\Lambda}(x) = \langle x, \Lambda x \rangle = \langle x, A \rangle \vee \langle x, B \rangle \in S$. ∎

Now that the existence of T is known, we need a suitable description.

DEFINITION. Let $I = a \backslash b$ and $J = c \backslash d$ be two intervals in a lattice L. We say that I *precedes* J [or J *succeeds* I] if $a \leq c \leq b \leq d$. This is denoted by $I \ll J$. If $(I_\lambda : \lambda \in \Lambda)$ is a family of intervals in a lattice L, we define $\ll$ on the index set Λ by $\lambda \ll \mu$ if and only if $I_\lambda \ll I_\mu$. A *$\ll$-ideal of* Λ is a subset, $X \subseteq \Lambda$, satisfying $\alpha \ll \beta \in X$ implies $\alpha \in X$. We let $\mathrm{Id}(\Lambda, \ll)$ be the set of all $\ll$-ideals of Λ.

The concept of a *$\ll$-filter* and $\mathrm{Fil}(\Lambda, \ll)$ is defined dually. As with order ideals and filters, it is trivial to show that the systems of $\ll$-ideals and filters are closed under unions and intersections, hence are distributive lattices. Moreover, the complement of a $\ll$-ideal [resp. filter] is a $\ll$-filter [resp. ideal].

LEMMA. *Let* $\mathbf{I} = (I_\lambda : \lambda \in \Lambda)$ *be a* set *of intervals on a lattice L; then $\ll$ is a reflexive, acyclic relation on* Λ.

PROOF. $\ll$ is clearly reflexive. Take $\alpha_1, \ldots, \alpha_n \in \Lambda$ with $\alpha_1 \ll \cdots \ll \alpha_n \ll \alpha_1$. Then we have $u_1 \leq \cdots \leq u_n \leq u_1$ and $v_1 \leq \cdots \leq v_n \leq v_1$. Thus the intervals are equal and since $\mathbf{I}$ is a set, $\alpha_1 = \cdots = \alpha_n$. ∎

THEOREM. *Let* $\mathbf{I} = (I_\lambda : \lambda \in \Lambda)$ *be a set of intervals on a lattice L; then*

$$L[\mathbf{I}, \ll] := \{\langle x, X \rangle \in L[\mathbf{I}] : X \in \mathrm{Id}(\Lambda x, \ll)\}$$

is a sublattice of $L[\mathbf{I}]$.

PROOF. Let $\langle a, A \rangle = \bigvee \langle a_i, A_i \rangle$ where A_i is a $\ll$-ideal of Λa_i for each index i, and $I_\alpha = u_\alpha \backslash v_\alpha$ for each $\alpha \in \Lambda$. Take $\beta \in A$ and $\alpha \in \Lambda a$ with $\alpha \ll \beta$. From the definition of A, there exists an index, i, with $\beta \in A_i$. The above relations give us $u_\alpha \leq u_\beta \leq a_i \leq a \leq v_\alpha \leq v_\beta$. Therefore $\alpha \in \Lambda a_i$ and, since $A_i \in \mathrm{Id}(\Lambda a_i, \ll)$, $\alpha \in \Lambda a \cap A_i \subseteq A$. Thus $A \in \mathrm{Id}(\Lambda a, \ll)$ and $\langle a, A \rangle \in L[\mathbf{I}, \ll]$.

To show closure with respect to meets, we use duality. With respect to L^∂, we have, for intervals I and J, $I \ll^\partial J$ if and only if $J \ll I$. Thus, $\ll^\partial$-ideals are just $\ll$-filters. Therefore, the duality, $\partial : L[\mathbf{I}] \longrightarrow L^\partial[\mathbf{I}]$ provides a duality between $L[\mathbf{I}, \ll]$ and $L^\partial[\mathbf{I}, \ll^\partial]$. ∎

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COROLLARY. *Let* $\mathbf{I} = (I_\lambda : \lambda \in \Lambda)$ *be a set of W-failure intervals on a finite lattice* L*; then* $T \subseteq L[\mathbf{I}, \lll]$.

PROOF. Both $\emptyset$ and Λx are $\lll$-ideals of Λx. Thus both $\alpha[L]$ and $\beta[L]$ are subsets of $L[\mathbf{I}, \lll]$, a sublattice of $L[\mathbf{I}]$. ∎

We now wish to show that the above inclusion is actually equality. For any $X \subseteq \Lambda$, we let $\downarrow X$ denote the $\lll$*-ideal closure of* X *in* $(\Lambda, \lll)$. Thus for any $\mu \in \Lambda$, $\downarrow\mu$ is the smallest (or principal) $\lll$-ideal of $(\Lambda, \lll)$ containing μ. When working with a restricted version of $\lll$, say in $(\Lambda x, \lll)$ we use $\downarrow_x$ to denote the *closure operation of* $(\Lambda x, \lll)$. Thus if $\mu \in \Lambda x$, $\downarrow_x\mu := \Lambda x \cap \downarrow\mu$ is the smallest (or principal) $\lll$-ideal of $(\Lambda x, \lll)$ containing μ. Note that, for a set of intervals, $\lll$ is usually not transitive. Thus computing a principal $\lll$-ideal in Λ would normally require the calculation of transitive chains. When we restrict ourselves to some Λx, this problem disappears.

LEMMA. *For any* $x \in L$, $(\Lambda x, \lll)$ *is an ordered set.*

PROOF. Take $\lambda, \mu, \nu \in \Lambda x$ with $\lambda \lll \mu \lll \nu$. Then $u_\lambda \leq u_\nu$ and $v_\lambda \leq v_\nu$ follow easily. Since $\lambda, \nu \in \Lambda x$, the third condition, $u_\nu \leq v_\lambda$ is trivially true. ∎

COROLLARY. *For any* $x \in L$ *and any* $X \subset \Lambda x$, $\downarrow_x X = \{\lambda \in \Lambda x : \lambda \lll \mu$ *for some* $\mu \in X\}$.

THEOREM. *Let* $\mathbf{I} = (I_\lambda : \lambda \in \Lambda)$ *be a set of W-failure intervals on a finite lattice L, and T be the smallest sublattice of* $L[\mathbf{I}]$ *mapping onto L. Then* $T = L[\mathbf{I}, \lll]$.

PROOF. We need only show that, for every $x \in L$ and $\mu \in \Lambda x$, $\langle x, \downarrow_x\mu\rangle \in T$. We claim that

$$\langle x, \downarrow_x\mu\rangle = \big(\langle x, \emptyset\rangle \vee \langle u_\mu, \Lambda u_\mu\rangle\big) \wedge \bigwedge\{\langle x, \Lambda x\rangle \wedge \langle v_\lambda, \emptyset\rangle : \lambda \in \Lambda x \text{ and } v_\lambda \not\leq v_\mu\}.$$

This is in T since $\alpha[L]$ and $\beta[L]$ are subsets of T.

Let

$$\langle x, X\rangle := \langle x, \emptyset\rangle \vee \langle u_\mu \Lambda u_\mu\rangle = \langle x, \Lambda x \cap \Lambda u_\mu\rangle$$

and for any $\lambda \in \Lambda x$,

$$\langle x, Y_\lambda\rangle := \langle x, \Lambda x\rangle \wedge \langle v_\lambda, \emptyset\rangle = \langle x, \Lambda x \setminus \Lambda v_\lambda\rangle.$$

Since $\mu \in \Lambda v_\lambda$ if and only if $v_\lambda \leq v_\mu$, we have $\mu \in X \cap \bigcap\{Y_\lambda : v_\lambda \not\leq v_\mu\}$. This provides the desired conclusion in one direction. On the other hand, X forces all its λ to satisfy $u_\lambda \leq u_\mu$ and $\bigcap\{Y_\lambda : v_\lambda \not\leq v_\mu\}$ throws out all λ with $v_\lambda \not\leq v_\mu$. Therefore the equality is proven. ∎

COROLLARY. $T = L[\mathbf{I}]$ *if and only if* $\lll$ *is the equality relation on* Λ.

PROOF. For each $x \in L$, $\mathbf{2}^{\Lambda x} = \mathrm{Id}(\Lambda x, \lll)$ if and only if $(\Lambda x, \lll)$ is an antichain. ∎

We close this section by noting that both $L[\mathbf{I}]$ and $L[\mathbf{I}, \lll]$ are (generalized) gluings in the sense of Herrmann and Day [5] with L as the skeleton and $L(x) = \mathbf{2}^{\Lambda x}$ [resp. $\mathrm{Id}(\Lambda x, \lll)$] as the blocks. As such, these lattices can sometimes be more easily constructed by the gluing approach.

4. **Inverse limits of bounded covers.** In the last section, we constructed from a given finite lattice, L, a canonical bounded cover, $L' \longrightarrow L$, which "repaired" a finite set of W-failures in L. This construction can be of course iterated, and, depending on the choices of W-failure intervals, different finite and infinite limits can be produced that are to various degrees free of W-failures. In this section we present general results on the inverse limit of sequences of bounded covers, together with a specific application that ensures the existence of projective covers for all (finite) bounded lattices. This latter result generalizes the same result for finite distributive lattices given in Day, Gaskill, and Poguntke [4].

Let $(\pi_{n+1,n}: L_{n+1} \longrightarrow L_n)_{n\in\mathbb{N}}$ be a sequence of bounded covers with lower and upper bound functions, $\alpha_{n,n+1}$ and $\beta_{n,n+1}: L_n \longrightarrow L_{n+1}$, respectively. By [even empty] composition, we produce for every $n \geq m$ in $\mathbb{N}$, a bounded cover, $\pi_{nm}: L_n \longrightarrow L_m$, with associated lower and upper bound functions, α_{mn} and $\beta_{mn}: L_m \longrightarrow L_n$, respectively. If $L_\infty = \varprojlim(\pi_{n,m} : n \geq m)$, then we can represent L_∞ by

$$L_\infty = \{\mathbf{x} \in \Pi(L_n : n \in \mathbb{N}) : \pi_{n+1,n}(x_{n+1}) = x_n\}.$$

The projection maps, $\pi_{\infty n}: L_\infty \longrightarrow L_n$, are also bounded by $\alpha_{n\infty}$ and $\beta_{n\infty}$ where, for $a \in L_n$,

$$\alpha_{n\infty}(a)_m = \begin{cases} \alpha_{nm}(a), & n \leq m \\ \pi_{nm}(a), & n \geq m, \end{cases}$$

and $\beta_{n\infty}$ is defined similarly.

LEMMA. *For a sublattice $S \subseteq L_\infty$, the following are equivalent:*

(1) There exists $n \in \mathbb{N}$ such that $\pi_{\infty n}[S] = L_n$;

(2) For all $n \in \mathbb{N}$ we have $\pi_{\infty n}[S] = L_n$.

PROOF. Clearly (2) implies (1), so assume $\pi_{\infty n}[S] = L_n$ for some $n \in \mathbb{N}$. Since for all $m \leq n$, π_{nm} is surjective, $\pi_{\infty m}[S] = L_m$. Since for all $m \geq n$, $\pi_{m,n}$ is a cover and $\pi_{mn}\big[\pi_{\infty m}[S]\big] = \pi_{\infty n}[S] = L_n$, $\pi_{\infty m}[S] = L_m$. Therefore (2) holds. ∎

LEMMA. *Let S be a sublattice of L_∞ satisfying $\pi_{\infty n}[S] = L_n$, for some $n \in \mathbb{N}$,*

(1) $\pi_{\infty n}|_S$ is lower bounded if and only if $\alpha_{n\infty}[L_n] \subseteq S$;

(2) $\pi_{\infty n}|_S$ is upper bounded if and only if $\beta_{n\infty}[L_n] \subseteq S$.

PROOF. We show (1) and leave (2) for the reader. We may also assume without loss of generality that $n = 0$. The condition is clearly sufficient, so let $\rho := \pi_{\infty 0}|_S$ be lower bounded by $\sigma: L_0 \longrightarrow S$. Since $S \subseteq L_\infty$, we have $\alpha_{0\infty}(x) \leq \sigma(x)$ for all $x \in L_0$. By the last lemma, we have $\pi_{\infty m}[S] = L_m$ for all $m \in \mathbb{N}$. Therefore, for any $x \in L_0$ and for each $m \in \mathbb{N}$, there exists $\mathbf{s}_m \in S$ such that $\pi_{\infty m}(\mathbf{s}_m) = \alpha_{0m}(x)$. Since $\pi_{\infty 0}(\mathbf{s}_m) = x$, we must have $\sigma(x) \leq \mathbf{s}_m$ for all $m \in \mathbb{N}$. Therefore $\sigma(x)_m \leq \alpha_{0\infty}(x)_m$ for all $m \in \mathbb{N}$ and $\sigma = \alpha_{0\infty}$. ∎

COROLLARY. *$\langle \alpha_{0\infty}[L] \rangle$ and $\langle \beta_{0\infty}[L] \rangle$ are the least sublattices of L_∞ that are lower, resp. upper, bounded preimages of L_0.*

There seems to be little more that one can say about inverse limits of bounded covers in this general setting.

DEFINITION. Let L be a lattice. The *Whitman-dilation of* L is the lattice, $L^{\#} := L[\mathbf{I}, \ll]$, where $\mathbf{I}$ is the set of all W-failure intervals on L. The *W-dilation series for* L is the sequence of bounded covers canonically determined by $L_0 = L$, and $L_{n+1} = L_n^{\#}$.

LEMMA [2]. *The inverse limit of the W-dilation series for a finite lattice satisfies* (W).

Again, there is very little one can say about sublattices of L that are preimages of L save that they do indeed satisfy (W). If we further restrict ourselves to bounded lattices, we can extend a result of [4]. Recall that a *projective cover* in lattices of a lattice L is a cover, $P \longrightarrow L$, where P is a projective lattice. When a projective cover $\varphi: P \longrightarrow L$ exists, it is unique up to "isomorphism over L" in that, if $\psi: Q \longrightarrow L$ is a projective cover, there is an isomorphism $\eta: P \longrightarrow Q$ such that $\psi \circ \eta = \varphi$.

THEOREM. *Every bounded [finite] lattice has a projective cover in the variety of all lattices.*

PROOF. Let L be a bounded lattice and L_∞ the limit of its W-dilation series. Since L is finite, any sublattice of L_∞ that maps onto $L[= L_0]$ contains a finitely generated sublattice that does likewise. Since L is bounded, any homomorphism from a fintely generated lattice onto L is bounded. Therefore $T := \langle \alpha_{0\infty}[L] \rangle = \langle \beta_{0\infty}[L] \rangle$ is the smallest sublattice of L_∞ that maps onto L by $\pi_{\infty 0}$.

Let $\rho := \pi_{\infty 0}|_T: T \longrightarrow L$. We have that ρ is a bounded cover of L and that T is finitely generated and satisfies (W). Since L is bounded, T is projective via Kostinsky [10]. ∎

It is an open problem to decide which other lattices have projective covers. If L has a projective cover, then the fact that the maps in the W-dilation series of L are covers endows the series members with extra approximation properties. Let $\varphi_0: P \longrightarrow L = L_0$ be the projective cover of L. Since each $\pi_{n+1,n}: L_{n+1} \longrightarrow L_n$ is surjective, there exists a lifting $\varphi_{n+1}: P \longrightarrow L_{n+1}$ with $\pi_{n+1,n} \circ \varphi_{n+1} = \varphi_n$ for each $n \in \mathbb{N}$. Since each $\pi_{n+1,n}: L_{n+1} \longrightarrow L_n$ is a cover, each φ_n is surjective. Therefore, even without passing to the limit, the W-dilation members provide better and better finite approximations of any existing projective cover of L. We will return to this approximation idea in the next section on finitely presented lattices.

If some member of the W-dilation series satisfies (W), be L bounded or not, then the sequence terminates and a finite W-cover is produced. In [4], it is shown that, for a finite distributive lattice, the existence of such a finite W-cover is determined by the exclusion of 6 forbidden sublattices. While such a list seems impossible in general—or even if L is bounded—there might be some solution hidden in the first few terms of the W-dilation series.

PROBLEM. Is there a bounding formula, $n = n(L)$, such that a finite lattice L has a finite W-cover (or projective cover) if and only if its nth W-dilation satisfies (W)?

5. **Finitely presented lattices.** In this section, we examine finitely presented lattices from a model theoretical view and use the inverse limit of doubling constructions to approximate and produce them. As is well-known (c.f. Freese [7], or Ježek and Slavík [9]), every finitely presented lattice, $FL(X,R)$, is obtainable as the lattice freely generated by a finite ordered set with certain declared joins and meets, say $\mathbf{P} = (P, \leq, \sup, \inf)$. Here, sup and inf are partial functions from the system of all finite subsets of P into P satisfying for all X in the appropriate domain, $\sup X = \bigvee_{(P,\leq)} X$ and $\inf X = \bigwedge_{(P,\leq)} X$ respectively. We denote this lattice by $FL(\mathbf{P}) = FL(P, \leq, \sup, \inf)$.

Let $\mathbf{P} = (P, \leq, \sup, \inf)$ be such a partial lattice which we fix for the remainder of this section. A *model of* $\mathbf{P}$ [or a $\mathbf{P}$-lattice] is a lattice, L, together with an order preserving function, $\varphi: P \longrightarrow L$ that preserves the joins and meets declared by sup and inf; i.e. for X in the domain of sup, $\varphi(\sup X) = \bigvee_{(L,\leq)} \varphi[X]$ and dually. We let Mod($\mathbf{P}$) be the class of all $\mathbf{P}$-models. $\mathbf{P}$-homomorphisms, $f: (L, \varphi) \longrightarrow (M, \psi)$, are lattice homomorphisms, $f: L \longrightarrow M$ satisfying $f \circ \varphi = \psi$. Note that Mod($\mathbf{P}$) can be considered as a variety of algebras by extending the type of lattices to include constants for all $p \in P$ and adding, as identities, all the relations induced by the order relation and the declared sup's and inf's.

There are two canonical members of Mod($\mathbf{P}$): $\big(\mathrm{Id}(\mathbf{P}), \downarrow\big)$, the system of all sup-ideals of $(P, \leq, \sup)$, together with the principal ideal map, $x \longmapsto \downarrow_P x := \{y \in P : y \leq X\}$; and Fil($\mathbf{P}$), the system of all inf-filters on $(P, \leq, \inf)$ ordered by reverse set inclusion, together with the principal filter map $x \longmapsto \uparrow_P x := \{y \in P : y \geq x\}$. By adroitly adjoining the bound elements, 0 and 1, as required, these lattices can be seen to be the join- [resp. meet-] semilattice freely generated by the appropriate partial semilattice. This last observation produces, by elementary categorical methods, the natural lower and upper bounded lattice epimorphisms, $\downarrow: FL(\mathbf{P}) \longrightarrow \mathrm{Id}(\mathbf{P})$ and $\uparrow: FL(\mathbf{P}) \longrightarrow \mathrm{Fil}(\mathbf{P})$, respectively. The lower bound map for $\downarrow$ is $I \longmapsto \bigvee I$, and the upper bound for $\uparrow$ is $F \longmapsto \bigwedge F$. Dean's solution to the word problem for $FL(\mathbf{P})$, [6], now can take the following form.

THEOREM [6]. *For $(L, \varphi) \in$ Mod($\mathbf{P}$), $L \cong FL(\mathbf{P})$ if and only if $L = \langle \varphi P \rangle$ and for all $p, q \in P$ and $a, b, c, d \in L$,*

D1 $\varphi p \leq \varphi q \Leftrightarrow p \leq q$;

D2 $\varphi p \leq c \vee d \Leftrightarrow p \in \mathrm{Id}(\varphi^{-1}[\downarrow c] \cup \varphi^{-1}[\downarrow d])$;

D3 $a \wedge b \leq \varphi q \Leftrightarrow q \in \mathrm{Fil}(\varphi^{-1}[\uparrow a] \cup \varphi^{-1}[\uparrow b])$; *and*

D4 $a \wedge b \leq c \vee d \Leftrightarrow (a \wedge b) \backslash (c \vee d) \cap (\{a, b, c, d\} \cup \varphi P) \neq \emptyset$.

Here, Id *and* Fil *are the appropriate closure operators on* Id($\mathbf{P}$) *and* Fil($\mathbf{P}$) *respectively.*

Dean's fourth condition, as listed above, indicates the type of W-failure intervals that are of interest when working in some Mod($\mathbf{P}$). For $(L, \varphi) \in$ Mod($\mathbf{P}$) and convex $C \subseteq L$, C is called $\mathbf{P}$-*disjoint* if $\varphi[P] \cap C = \emptyset$.

LEMMA. *For* $(L, \varphi) \in \mathrm{Mod}(\mathbf{P})$ *and* $\mathbf{P}$*-disjoint interval* $C \subset L$*, there exists a unique* $\psi : P \longrightarrow L[C]$ *such that*

(1) $(L[C], \psi) \in \mathrm{Mod}(\mathbf{P})$*,*

(2) $\kappa : (L[C], \psi) \longrightarrow (L, \varphi)$*, and*

(3) $\kappa \circ \psi = \varphi$*.*

PROOF. Since C is $\mathbf{P}$-disjoint, $\alpha\big(\varphi(p)\big) = \beta\big(\varphi(p)\big)$ for all $p \in P$. Therefore we define $\psi := \alpha \circ \varphi = \beta \circ \varphi$. Clearly, $\kappa \circ \psi = \varphi$ and ψ is uniquely defined.

Now for any $X \in \mathrm{dom(sup)}$, $\bigvee_{L[C]} \psi X = \bigvee_{L[C]} \alpha[\varphi X] = \alpha(\bigvee_L \varphi X) = \alpha\big(\varphi(\sup X)\big) = \psi(\sup X)$. Similarly, using $\psi = \beta \circ \varphi$, inf's are preserved. ∎

We now can generalize [1] to provide a semantic justification of Dean's (fourth) condition.

COROLLARY. *Any projective objective,* $(L, \varphi) \in \mathrm{Mod}(\mathbf{P})$*, satisfies Dean's (fourth) condition:*

$$a \wedge b \leq c \vee d \textit{ implies } (a \wedge b)\backslash(c \vee d) \cap (\{a, b, c, d\} \cup \varphi[P]) \neq \emptyset.$$

PROOF. Assume the condition fails for some $a, b, c, d \in L$, and let $I := (a \wedge b)\backslash(c \wedge d)$. Then I is a $\mathbf{P}$-disjoint interval and by the lemma, $(L[I], \psi) \in \mathrm{Mod}(\mathbf{P})$. As in [1], it is impossible to construct a section $\mu : L \longrightarrow L[I]$ of κ. ∎

We now are in the position to repair Dean-failure intervals via inverse limits of certain canonical doubling constructions. We require a minor modification to the W-dilation series defined previously.

DEFINITION. Let $\mathbf{P} = (P, \leq, \sup, \inf)$ be a partial lattice, and take $(L, \varphi) \in \mathrm{Mod}(\mathbf{P})$. The *Dean-dilation of* (L, φ) is the pair $(L^\#, \varphi^\#) \in \mathrm{Mod}(\mathbf{P})$ where $L^\# := L[\mathbf{I}, \ll]$, $\mathbf{I}$ is the set of all $\mathbf{P}$-disjoint W-failure intervals on L, and $\varphi^\# : P \longrightarrow L^\#$ is the canoncial function determined by the pullback. The *D-dilation series for* (L, φ) is the sequence of bounded covers canonically determined by $(L_0, \varphi_0) = (L, \varphi)$, and $(L_{n+1}, \varphi_{n+1}) = (L_n^\#, \varphi_n^\#)$.

THEOREM. *The inverse limit of the D-dilation series for a finite* $(L, \varphi) \in \mathrm{Mod}(\mathbf{P})$ *satisfies Dean's (fourth) condition.*

PROOF. If L_∞ is the inverse limit of the Dean dilation series, then there exists a unique $\varphi_\infty : P \longrightarrow L_\infty$ with $\pi_{\infty n} \circ \varphi_\infty = \varphi_n$ for all $n \in \mathbb{N}$. Standard limit calculation gives us $(L_\infty, \varphi_\infty) \in \mathrm{Mod}(\mathbf{P})$. For convenience, we write $\varphi_\infty p = \mathbf{p}$ for all $p \in P$.

Now take $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in L_\infty$ with $\mathbf{u} := \mathbf{a} \wedge \mathbf{b} \leq \mathbf{v} := \mathbf{c} \vee \mathbf{d}$ and $\mathbf{u}\backslash\mathbf{v} \cap (\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\} \cup \varphi_\infty P) = \emptyset$. Thus there exists indices $i, j, k, l, \big(m(p), p \in P\big)$ in $\mathbb{N}$ such that $\mathbf{a}_i \not\leq \mathbf{v}_i$, $\mathbf{b}_j \not\leq \mathbf{v}_j$, $\mathbf{u}_k \not\leq \mathbf{c}_k$, $\mathbf{u}_l \not\leq \mathbf{d}_l$, and $\mathbf{p}_{m(p)} \notin \mathbf{u}_{m(p)}\backslash\mathbf{v}_{m(p)}$ for all $p \in P$. Since $|P| < \infty$, the maximum of all these indices exists and these inequalities hold for any $n \in \mathbb{N}$ greater than or equal to this maximum. Since $\mathbf{u} \leq \mathbf{v}$ is assumed, $\mathbf{u}_n\backslash\mathbf{v}_n$ is a P-disjoint W-failure interval in L_n for all such $n \in \mathbb{N}$.

Let $\mathbf{u}_n\backslash\mathbf{v}_n = I_\lambda \in \mathbf{I}$ where $L_{n+1} = L_n[\mathbf{I}, \ll]$. Thus we have a factorization,

$$L_n[\mathbf{I}, \ll] \longrightarrow \mathbf{L}_n[\mathbf{I}_\lambda] \longrightarrow \mathbf{L}_n.$$

Now $\mathbf{u}_{n+1} = \mathbf{a}_{n+1} \wedge \mathbf{b}_{n+1} \geq \alpha_{n,n+1}(\mathbf{a}_n) \wedge \alpha_{n,n+1}(\mathbf{b}_n)$, and dually, $\mathbf{v}_{n+1} \leq \beta_{n,n+1}(\mathbf{c}_n) \vee \beta_{n,n+1}(\mathbf{d}_n)$. The right sides of these inequalities map via $L_{n+1} \longrightarrow L_n[I_n]$ to $\langle \mathbf{u}_n, \{\lambda\} \rangle$ and $\langle \mathbf{v}_n, \emptyset \rangle$ respectively and this contradicts $\mathbf{u}_{n+1} \leq \mathbf{v}_{n+1}$. ∎

We now consider conditions D1 through D3. For any **P**-model, $\varphi: P \longrightarrow L$, we obtain a function, $f_{(L,\varphi)}: L \longrightarrow \mathrm{Id}(\mathbf{P})$, defined by $f_{(L,\varphi)}(c) := \varphi^{-1}[\downarrow_L c] = \{p \in P : \varphi p \leq c\}$. Easy calculations show that $f_{(L,\varphi)}$ is a meet-semilattice homomorphism. Dually, there is an induced $g_{(L,\varphi)}: L \longrightarrow \mathrm{Fil}(\mathbf{P})$ by $g_{(L,\varphi)}(c) := \varphi^{-1}[\uparrow_L c]$ which is a join-semilattice homomorphism.

LEMMA. *For $(L,\varphi) \in \mathrm{Mod}(\mathbf{P})$,*

*(1) $(L,\varphi) \models$ D1 & D2 if and only if $f_{(L,\varphi)}: (L,\varphi) \longrightarrow (\mathrm{Id}(\mathbf{P}), \downarrow_P)$ is a **P**-morphism;*

*(2) $(L,\varphi) \models$ D1 & D3 if and only if $g_{(L,\varphi)}: (L,\varphi) \longrightarrow (\mathrm{Fil}(\mathbf{P}), \uparrow_P)$ is a **P**-morphism;*

(3) $(L,\varphi) \models$ D1 & D2, and $h: (M,\psi) \longrightarrow (L,\varphi)$ lower bounded by $\sigma: L \longrightarrow M$ satisfying $\sigma \circ \varphi = \psi$ imply $(M,\psi) \models$ D1 & D2;

(4) $(L,\varphi) \models$ D1 & D3, and $h: (M,\psi) \longrightarrow (L,\varphi)$ upper bounded by $\pi: L \longrightarrow M$ satisfying $\pi \circ \varphi = \psi$ imply $(M,\psi) \models$ D1 & D3;

PROOF. Re (1): For $p, q \in P$, $\varphi p \leq \varphi q$ if and only if $p \in f_{(L,\varphi)}(\varphi q)$. Thus D1 is equivalent to the Mod(**P**) morphism property, $f_{(L,\varphi)} \circ \varphi = \downarrow_P$. D2 is the join preserving property, thinly disguised.

Re (3): It is enough to show, under the assumptions, that $f_{(M,\psi)} = f_{(L,\varphi)} \circ h$, i.e. $\psi p \leq x \Leftrightarrow \varphi p \leq hx$ for all $p \in P$ and $x \in M$. This is trivial given the properties of the lower bound map, α. ∎

The above lemma suggests the following definition.

DEFINITION. Let $\mathbf{P} = (P, \leq, \sup, \inf)$ be a partial lattice; the *partial completion of* **P** is the "P-diagonal" sublattice of Id(**P**) × Fil(**P**) generated by the image of P under $x \longmapsto (\downarrow x, \uparrow x)$. We denote this lattice by $PC(\mathbf{P})$, and use $\delta: P \longrightarrow PC(\mathbf{P})$ for the embedding.

LEMMA. *For any partial lattice, **P**, $(PC(\mathbf{P}), \delta) \models$ D1, D2 and D3.*

PROOF. We must show that the first and second projections of $PC(\mathbf{P})$ onto Id(**P**) and Fil(**P**) are the induced $f = f_{(PC(\mathbf{P}),\delta)}$ and $g = g_{(PC(\mathbf{P}),\delta)}$ respectively. Take $p \in P$ and $(I,F) \in PC(\mathbf{P})$. Now

$$p \in f(I,F) \Leftrightarrow (\downarrow p, \uparrow p) \leq (I,F) \Leftrightarrow p \in I \text{ and } F \subseteq \uparrow p.$$

Since $PC(\mathbf{P}) = \langle \delta P \rangle$, we have $F \subseteq U(I)$ for all $(I,F) \in PC(\mathbf{P})$, where $U(I)$ is the set of all upper bounds of I. Thus $p \in I$ and $(I,F) \in PC(\mathbf{P})$ imply $F \subseteq U(I) \subseteq U(p) = \uparrow p$. This shows that $f = \pi_1$ as desired. ∎

We now can construct $FL(\mathbf{P})$ by the Dean-dilation series for $L_0 = PC(\mathbf{P})$.

THEOREM. *$FL(\mathbf{P})$ is the sublattice of L_∞ generated by $\varphi_\infty P$.*

PROOF. Let $T := \langle \varphi_\infty P \rangle \subseteq L_\infty$. Then T satisfies D4 since L_∞ does and we need only check D1 through D3. The above lemma states that L_0 satisfies these properties, and

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the previous lemma (3) & (4) provide the induction steps to show that all L_n satisfy D1 through D3. The passage of these properties to the inverse limit, and hence T, is easily left to the reader. ■

Thus, $PC(\mathbf{P})$ is the "first approximation" of $FL(\mathbf{P})$, and each Dean-dilation provides an improved approximation. This gives yet another proof of:

COROLLARY. *Finitely presented lattices are residually finite.*

There are two properties of some interest for finitely presented lattices: finiteness and weak atomicity. Finitely generated free lattices are weakly atomic [2], while Freese [7] shows that not all finitely presented lattices need be. Our general results in §4 on limits of bounded covers produced the following.

THEOREM. *For any partial lattice* $\mathbf{P} = (P, \leq, \sup, \inf)$ *with Dean-dilation series* (L_n) *of* $PC(\mathbf{P})$*, the following are equivalent:*

(1) $FL(\mathbf{P}) \longrightarrow PC(\mathbf{P})$ *is bounded;*

(2) $FL(\mathbf{P}) \longrightarrow \mathrm{Id}(\mathbf{P})$ *is upper and* $FL(\mathbf{P}) \longrightarrow \mathrm{Fil}(\mathbf{P})$ *is lower bounded;*

(3) For some $n \in \mathbb{N}$, $FL(\mathbf{P}) \longrightarrow L_n$ *is bounded;*

(4) $\alpha_{0\infty}[PC(\mathbf{P})]$ *and* $\beta_{0\infty}[PC(\mathbf{P})] \subseteq FL(\mathbf{P})$.

COROLLARY. *If* $\mathbf{P}$ *is finite and* $FL(\mathbf{P}) \longrightarrow PC(\mathbf{P})$ *is bounded, then* $FL(\mathbf{P})$ *is weakly atomic.*

PROOF. Suppose that $a < b$ in $FL(\mathbf{P})$. Then, for some n, the homomorphism $\pi_{\infty n}$: $FL(\mathbf{P}) \longrightarrow L_n$ satisfies $\pi_{\infty n}(a) < \pi_{\infty n}(b)$. Since L_n is finite, there are elements x and y with $\pi_{\infty n}(a) \leq x \prec y \leq \pi_{\infty n}(b)$. Let $c = b \wedge \beta_{n\infty}(x)$ and $d = c \vee \alpha_{n\infty}(y)$. Then $\pi_{\infty n}(c) = x$ and $\pi_{\infty n}(d) = y$ so if $c \leq e \leq d$ then $\pi_{\infty n}(e)$ is either x or y. If $\pi_{\infty n}(e) = x$ then $c \leq e \leq \beta_{n\infty}(x) \wedge b = c$ and similarly, if $\pi_{\infty n}(e) = y$, then $e = d$. Thus $a \leq c \prec d \leq d$, proving that $FL(\mathbf{P})$ is weakly atomic. ■

In unpublished work, Freese has constructed a counter-example which shows that the converse of the above corollary is false.

PROBLEM. Is there a characterization of finite finitely presented lattices[2]?

Ježek and Slavík [9] have answered this question when the partial lattice is join- or meet-trivial, e.g. $\mathbf{P} = (P, \leq, \sup, \emptyset)$. A solution following our $\mathrm{Mod}(\mathbf{P})$ approach requires finding a member of the Dean-dilation series for $PC(\mathbf{P})$ that satisfies Dean's (fourth) condition. Thus this problem is identical to finding finite W-covers [resp. projective covers] of finite [bounded] lattices, in that one examines finite lattices of the form, $L[\mathbf{I} \ll]$ in search of canonical patterns that force infinite W-failure generation. Preliminary (unpublished) studies by Slavík and by this author leads one to be reasonably optimistic about the feasibility of this approach.

[2] V. Slavík has recently announced that, if $|FL(\mathbf{P})| > 86 \cdot |PC(\mathbf{P})|$, then $FL(\mathbf{P})$ is infinite, thus effectively solving this problem. His proof makes use of the results of this section.

6. **Counting free distributive lattices.** It is an open problem, despite the result in Kisielewicz [11], to determine the cardinalities of finitely generated free distributive lattices. In this section, we construct $FD(n)$ using the doubling construction, and examine the counting formulae determined by this representation. These formulae do not advance the state of the art to any great degree but may prove valuable in future investigations. We refer the reader to Quackenbush [13] for a recent survey of this problem.

For $n \in \mathbb{N}$, with $n \geq 3$ (to avoid degenerate cases), let $\mathbf{n}_= := (\mathbf{n}, =, \emptyset, \emptyset)$ be the partial lattice given by the n-element antichain. Then $\mathrm{Id}(\mathbf{n}_=) \cong \mathrm{Fil}(\mathbf{n}_=) \cong \mathbf{2}^n$, and $PC(\mathbf{n}_=)$ is the distributive lattice obtained by gluing these two copies of the power set on top of each other and adding n generators between coatoms of the bottom power set and their bijectively associated atoms in the upper power set. Since $PC(n) := PC(\mathbf{n}_=)$ is distributive, we have a canonical factorization,

$$FL(n) \longrightarrow FD(n) \longrightarrow PC(n),$$

where both homomorphisms are bounded covers. Moreover $FD(n)$ is produced from $PC(n)$ by doubling those W-failure intervals that keep the resultant lattice distributive. The relevant observation from [4] is:

LEMMA. *Let L be a distributive lattice and take $I = u \backslash v$ in L. $L[I]$ is again distributive if and only if $L = \uparrow u \cup \downarrow v$.*

Note that the doubling of such an interval as in the lemma produces a join-prime element and recall that any join-prime element in a finitely generated lattice is a meet of generators. These observations provide the following result.

THEOREM. *For $n \in \mathbb{N}$ with $n \geq 3$, $FD(n) \cong L[\mathbf{I}, \ll]$ where $L = PC(n)$ and $\mathbf{I}$ is all intervals of the form, $I_Y = \bigwedge\{x_i : i \notin Y\} \backslash \bigvee\{x_j : j \in Y\}$, $Y \subseteq \mathbf{n}$ with $2 \leq |Y| \leq n-2$.*

PROOF. By the preceding lemma, the lattice $L[\mathbf{I}, \ll]$ is distributive. It is also generated by the n-element set, $X = \{\langle x_i, \emptyset\rangle : i = 1, \dots n\}$ that satisfies for any $Y, Z \subseteq X$, $\bigwedge Y \leq \bigvee Z$ if and only if $Y \cap Z \neq \emptyset$. This describes $FD(n)$. ∎

COROLLARY. $|FD(n)| = \sum\{\,|\mathrm{Id}(\Lambda x, \ll)| : x \in PC(n)\}$.

PROOF. Elements of $L[\mathbf{I}, \ll]$ are of the form, $\langle x, X\rangle$, $x \in L$ and $X \in \mathrm{Id}(\Lambda x, \ll)$. ∎

For more detailed analysis, we need an explicit description of $PC(n)$ as a sublattice of $\mathbf{2}^n \times \mathbf{2}^n$, $[= \mathrm{Id}(\mathbf{n}_=) \times \mathrm{Fil}(\mathbf{n}_=)]$. For $i \in \mathbf{n} = \{0, \dots, n-1\}$, let

$$x_i = (\{i\}, \mathbf{n} \setminus \{i\})$$

be the generators of $PC(n)$. Thus we replace the filter, $\uparrow\{i\} = \{i\}$, in the second component by its complement. For any $A \subseteq \mathbf{n}$, we let

$$u_A = (\emptyset, A),$$
$$v_A = (A, \mathbf{n}),$$

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Then

$$PC(n) := \{x_i : i < n\} \cup \{u_A, v_A : A \subseteq \mathbf{n}\}.$$

LEMMA. *The following properties hold in $PC(n)$:*
(1) $u_A = \bigwedge\{x_i : i \notin A\}$, *for* $A \subseteq \mathbf{n}$ *with* $|A| \leq n-2$;
(2) $v_A = \bigvee\{x_i : i \in A\}$, *for* $A \subseteq \mathbf{n}$ *with* $|A| \geq 2$;
(3) $m := u_{\mathbf{n}} = v_\emptyset$;
(4) $u_A \leq u_B \Leftrightarrow v_A \leq v_B \Leftrightarrow A \subseteq B$;
(5) $u_A \leq v_B$, *for all* A *and* B;
(6) $u_A \backslash v_A \ll u_B \backslash v_B \Leftrightarrow A \subseteq B$, *for* $2 \leq |A|, |B| \leq n-2$.

The W-failure [and $\mathbf{n}_=$-disjoint] intervals we need to repair have the form, $I_A = u_A \backslash v_A$. These are conveniently indexed by

$$\Lambda := \{A \in \mathbf{2}^n : 2 \leq |A| \leq n-2\}.$$

By (6) above, the precedence relation on Λ is set-inclusion:

$$A \ll B \Leftrightarrow A \subseteq B.$$

LEMMA. *For any* A, $B \subseteq \mathbf{n}$,
(1) $A \subseteq B \Leftrightarrow \Lambda(u_A) \subseteq \Lambda(u_B) \Leftrightarrow \Lambda(v_B) \subseteq \Lambda(v_A)$;
(2) $S \in \Lambda(u_A) \Leftrightarrow \neg S \in \Lambda(v_{\neg A})$;
(3) $\big(\mathrm{Id}\big(\Lambda(u_A), \ll\big), \subseteq\big)^\partial \cong \big(\mathrm{Id}\big(\Lambda(v_{\neg A}), \ll\big), \subseteq\big)$.

PROOF. Only the last statement requires some deduction. From (2), we have $\big(\Lambda(u_A), \ll\big) \cong \big(\Lambda(v_{\neg A}), \ll\big)^\partial$. Now, $\big(\mathrm{Id}\big(\Lambda(v_{\neg A}), \ll\big), \subseteq\big) \cong \big(\mathrm{Fil}\big(\Lambda(v_{\neg A}, \ll\big), \supseteq\big) = \big(\mathrm{Id}\big(\Lambda(v_{\neg A}), \ll\big)^\partial, \supseteq\big) \cong \big(\mathrm{Id}\big(\Lambda(u_A), \ll\big), \supseteq\big) = \big(\mathrm{Id}\big(\Lambda(u_A), \ll\big), \subseteq\big)^\partial$. ∎

A precise formula can now be given for $|FD(n)|$ by running over just the generators and the u_A's of $PC(n)$, doubling the latter contributions to cover the v_B's. For $n \in \mathbb{N}$, $n \geq 3$ and $0 \leq k \leq n$, we define

$$p(k,n) := |\,\mathrm{Id}(\{X \subseteq \mathbf{k} : 2 \leq |X| \leq n-2\})|.$$

THEOREM. $|FD(n)| = n + p(n,n) + 2\sum_{i=0}^{n-1}\binom{n}{i}p(i,n)$.

PROOF. There are n generators that are not in any interval, $m = u_n = v_\emptyset$ which is contained in every interval, and twice the count for each u_A with $0 \leq |A| < n$. There are of course, $\binom{n}{i}$ subsets of cardinality i. ∎

The above formula represents only a modest improvement on the canonical version, $|FD(n)| + 2 = |\,\mathrm{Id}(\mathbf{2}^n, \subseteq)|$. At present, the author knows of no way to further refine the numbers, $p(k,n)$. For example, the calculation of $p(5,5) = 5232$ seems at least as complicated as computing $|FD(5)| = 7579$, even though $p(4,5) = 113$ can easily be done by hand.

Another avenue for exploration might be the decomposition of Λ into $n-3$ antichains

$$\Lambda_k := \{S \in \Lambda : |S| = k\},$$

for $2 \leq k \leq n-2$. We let $\mathbf{I}_k := \{I_\alpha \in \mathbf{I} : \alpha \in \Lambda_k\}$ and define

$$D_n(k) = PC(n)[\mathbf{I}_k].$$

Since $\mathbf{I}_k$ is an antichain, the full pullback provides a bounded cover factorization, $FD(n) \longrightarrow D_n(k) \longrightarrow PC(n)$.

LEMMA. *For $A \subseteq \mathbf{n}$ and $2 \leq k \leq n-2$,*
(1) $|\Lambda_k(u_A)| = \binom{|A|}{k}$;
(2) $|\Lambda_k(v_A)| = \binom{n-|A|}{n-k}$.

THEOREM. *For any $n \in \mathbb{N}$ with $n \geq 3$ and $k \in [2, n-2]$, $D_n(k)$ is a homomorphic image of $FD(n)$ of cardinality*

$$|D_n(k)| = n + 2 + 2^{\binom{n}{k}} + \sum_{i=1}^{n-1} \binom{n}{i} 2^{\binom{i}{k}} + \sum_{i=1}^{n-1} \binom{n}{i} 2^{\binom{n-i}{n-k}}.$$

In the above formula, we define $\binom{i}{k} = 0$ if $i < k$.

COROLLARY. $|FD(n)| \geq |D_n(k)|$ *for each $k \in [2, n-2]$.*

While these lower bounds of $|FD(n)|$ are not too interesting, $|D_5(3)| = 1571$, the partition of Λ into the Λ_k's does provide a chain of $n-3$ consecutive pullbacks that produce $FD(n)$ from $PC(n)$. We merely "repair"one k-level at a time, reinterpretting the remaining Λ_j's in the new pullback. It is unclear though whether this approach could supply any enlightenment to the main counting problem.

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