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FREE EXTENSIONS OF CHIRAL POLYTOPES

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ABSTRACT. Abstract polytopes are discrete geometric structures which generalize the classical notion of a convex polytope. Chiral polytopes are those abstract polytopes which have maximal symmetry by rotation, in contrast to the abstract regular polytopes which have maximal symmetry by reflection. Chirality is a fascinating phenomenon which does not occur in the classical theory. The paper proves the following general extension result for chiral polytopes. If $\mathcal{K}$ is a chiral polytope with regular facets $\mathcal{F}$, then among all chiral polytopes with facets $\mathcal{K}$ there is a universal such polytope $\mathcal{P}$, whose group is a certain amalgamated product of the groups of $\mathcal{K}$ and $\mathcal{F}$. Finite extensions are also discussed.

1. **Introduction.** Abstract polytopes generalize the classical notion of a convex polytope to more general combinatorial structures with a distinctive geometric and topological flavor (Grünbaum [8], Danzer-Schulte [5], McMullen [13]). In recent years much work has been done on the classification by topological type of those abstract polytopes which are regular, that is, have maximal symmetry by reflection (*cf.* [14, 15, 16, 29]). Various methods of realizing regular polytopes as faces of polytopes of higher ranks have been described, including also free constructions of such extensions. It is known that among all regular polytopes with a given type $\mathcal{K}$ of facets there is a universal such polytope $\mathcal{P}$; if $\mathcal{F}$ denotes the facet type of $\mathcal{K}$, then its automorphism group is $A(\mathcal{P}) = A(\mathcal{K}) *_{A(\mathcal{F})} (A(\mathcal{F}) \times C_2)$, the free product of the two groups $A(\mathcal{K})$ and $A(\mathcal{F}) \times C_2$ with amalgamation along the two subgroups $A(\mathcal{F})$ (*cf.* [21, 22]). If $\mathcal{K} = \{3\}$ is the triangle, then $\mathcal{P}$ is the famous tessellation $\{3, \infty\}$ of the hyperbolic plane by (tri)asymptotic triangles (with vertices at infinity); see Figure 1 below. Its symmetry group is isomorphic to $\mathrm{PGL}_2(Z)$, which occurs here in the form $S_3 *_{C_2} (C_2 \times C_2)$ (Magnus [10]). In general, the polytope $\mathcal{P}$ is freely generated from its facet type $\mathcal{K}$, just as the hyperbolic tessellation $\{3, \infty\}$ is freely generated from triangles.

Chiral polytopes are abstract polytopes which have maximal symmetry by rotation. Chirality of polytopes is a fascinating phenomenon which does not occur in the classical theory (Coxeter [1]). In rank 3, examples of chiral polytopes are given by the irreflexible maps on surfaces (Coxeter-Moser [4], Coxeter [2]). There are infinitely many irreflexible maps on the 2-torus, but for higher genus their appearance is rather sporadic, with the next examples occurring only for genus 7 (Garbe [6, 7], Sherk [25, 26]).

For higher ranks very little is known about the construction of chiral polytopes and their groups. The basic theory of such polytopes of any rank is discussed in [23]. For

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rank 4 interesting chiral polytopes can be constructed from hyperbolic honeycombs in 3 dimensions using the one-to-one correspondence of isometries in hyperbolic 3-space and complex Möbius transformations ([24], Nostrand [20]); here the automorphism groups are projective linear groups over certain finite rings. All these polytopes are locally spherical or locally toroidal, in the sense that all their facets and vertex-figures are spherical or toroidal, respectively. In fact, we do not know of a finite chiral 4-polytope whose facets or vertex-figures are of higher genus, though they are likely to exist. In ranks 5 or higher it becomes very difficult actually to construct examples. Here the recent results of Wilker [30] on representations of Möbius transformations in higher dimensions by quaternionic (2×2)-matrices suggest that in ranks 5 and 6 examples of chiral polytopes can again be found from hyperbolic honeycombs in 4 and 5 dimensions.

The main purpose of this paper is to describe a free construction for extensions of chiral polytopes. In contrast to regular polytopes, this is done in terms of rotations. It is proved that, if $\mathcal{K}$ is a chiral polytope with regular facets $\mathcal{F}$, then there is again a universal polytope $\mathcal{P}$ with facets isomorphic to $\mathcal{K}$, now with group $A(\mathcal{P}) = A(\mathcal{K}) *_{A^+(\mathcal{F})} A(\mathcal{F})$, the free product of the two groups $A(\mathcal{K})$ and $A(\mathcal{F})$ with amalgamation along the two rotation subgroups $A^+(\mathcal{F})$ of the group of $\mathcal{F}$. Here, our condition on the regularity of the facets $\mathcal{F}$ of $\mathcal{K}$ is necessary in order that any general extension result on chiral polytopes be true; in fact, extending twice would leave us with a chiral polytope with faces $\mathcal{K}$ of co-rank 2, which must always be regular. Again, as for regular polytopes, $\mathcal{P}$ is freely generated from its facet type. In a sense this is an irreflexible version of the free extension for regular polytopes.

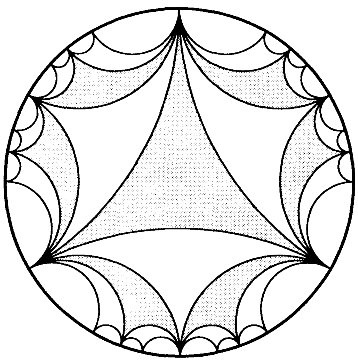

FIGURE 1

2. **Basic notions.** For a detailed discussion of chiral polytopes we refer to [23], and for the theory of regular polytopes to [19]. (Note that in contrast to [23] we now write compositions of maps from left to right.) In this section we briefly outline some definitions and basic results from the theory of abstract polytopes ([5, 15, 23]).

An *(abstract) n-polytope* $\mathcal{P}$ is a partially ordered set with a strictly monotone rank function having range $\{-1, 0, \ldots, n\}$. An element $F \in \mathcal{P}$ with $\mathrm{rank}(F) = j$ is called a *j-face*; typically F_j indicates a j-face. The maximal chains of $\mathcal{P}$ are called *flags*. We require that $\mathcal{P}$ have a smallest (-1)-face F_{-1}, a greatest n-face F_n and that each flag contains exactly $n+2$ faces. Also $\mathcal{P}$ should be *strongly flag-connected*, that is, any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$, such that each Φ_{i-1} and Φ_i are adjacent in the sense that they differ by just one face, and $\Phi \cap \Psi \subset \Phi_i$ for each i. Furthermore, $\mathcal{P}$ is *thin*; that is, whenever $F \leq G$, $\mathrm{rank}(F) = j-1$ and $\mathrm{rank}(G) = j+1$, then there are exactly two j-faces H with $F < H < G$. The latter condition basically says that $\mathcal{P}$ is "topologically real" and shares many combinatorial properties with convex polytopes.

If F and G are faces of $\mathcal{P}$ with $F \leq G$, we shall call $G/F := \{H \mid F \leq H \leq G\}$ a *section* of $\mathcal{P}$. We shall not distinguish between a face F and the section F/F_{-1}, which itself is a polytope with the same rank as F. The faces of rank 0, 1 and $n-1$ are called *vertices, edges* and *facets* respectively. If F is a face, the polytope F_n/F is called the *co-face* of $\mathcal{P}$ at F, or the *vertex-figure* of $\mathcal{P}$ at F if F is a vertex.

A polytope is said to be *regular* if its automorphism group $A(\mathcal{P})$ is transitive on its flags. Let $\Phi := \{F_{-1}, F_0, \ldots, F_n\}$ be a fixed flag, or *base flag*, of $\mathcal{P}$. The group $A(\mathcal{P})$ of a regular polytope $\mathcal{P}$ is generated by the involutions $\rho_o, \ldots, \rho_{n-1}$, where ρ_i is the unique automorphism which fixes all but the i-face of Φ $(i = 0, \ldots, n-1)$. These *distinguished generators* of $A(\mathcal{P})$ satisfy relations of the form

$$(\rho_i\rho_j)^{p_{ij}} = 1 \quad (i,j = 0, \ldots, n-1),$$

where $p_{ii} = 1, p_{ji} = p_{ij} =: p_i$ if $j = i-1$, and $p_{ij} = 2$ otherwise; here, the p_i's are the entries in the (Schläfli) *type* $\{p_1, \ldots, p_{n-1}\}$ of $\mathcal{P}$. The group $A(\mathcal{P})$ and its generators ρ_i satisfy the following *intersection property*:

$$\langle \rho_i \mid i \in I\rangle \cap \langle \rho_i \mid i \in J\rangle = \langle \rho_i \mid i \in I \cap J\rangle \quad \text{for } I, J \subset \{0, \ldots, n-1\}.$$

Conversely, if a group generated by $\rho_0, \ldots, \rho_{n-1}$ satisfies the above relations and this intersection property, then it is the group of a regular polytope. Such groups are called (*string*) *C-groups* to indicate that they are a generalization of Coxeter groups.

For a regular polytope $\mathcal{P}$ the rotations

$$\sigma_j := \rho_j\rho_{j-1} \quad (j = 1, \ldots, n-1)$$

generate the *rotation subgroup* $A^+(\mathcal{P})$ of $A(\mathcal{P})$, which is of index at most 2. These rotations σ_j fix all faces in $\Phi \setminus \{F_{j-1}, F_j\}$ and cyclically permute consecutive j-faces of $\mathcal{P}$ in the section F_{j+1}/F_{j-2} of $\mathcal{P}$ of rank 2. A regular polytope $\mathcal{P}$ is called *directly regular* if $A^+(\mathcal{P})$ has index 2 in $A(\mathcal{P})$. For a regular polytope $\mathcal{P}$, direct regularity is equivalent to orientability of its order complex, the simplicial complex whose simplices are given by the totally ordered subsets of $\mathcal{P}$ not containing F_{-1} and F_n.

Now, let $\mathcal{P}$ be a polytope of rank $n \geq 3$. Then $\mathcal{P}$ is said to be *chiral* if $\mathcal{P}$ is not regular, but if for some base flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ of $\mathcal{P}$ there still exist automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ of $\mathcal{P}$ such that σ_j fixes all faces in $\Phi \setminus \{F_{j-1}, F_j\}$ and cyclically permutes consecutive j-faces of $\mathcal{P}$ in the rank 2 section F_{j+1}/F_{j-2} of $\mathcal{P}$. For a chiral polytope the (orientation of the) σ_j's can be chosen in such a way that, if F'_j denotes the j-face of $\mathcal{P}$ with $F_{j-1} < F'_j < F_{j+1}$ and $F'_j \neq F_j$, then $F'_j\sigma_j = F_j$ (and thus $F_{j-1}\sigma_j = F'_{j-1}$) for $j = 1, \ldots, n-1$. The automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ generate $A(\mathcal{P})$ and satisfy the following relations

$$\sigma_i^{p_i} = 1 \quad (1 \leq i \leq n-1);$$
$$(\sigma_j\sigma_{j-1} \cdot \cdots \cdot \sigma_i)^2 = 1 \quad (1 \leq i < j \leq n-1),$$

with $p_1, \ldots, p_{n-1}$ given by the *type* $\{p_1, \ldots, p_{n-1}\}$ of $\mathcal{P}$. The automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ are called the *distinguished generators* of $A(\mathcal{P})$.

The above local definition of chirality also has an equivalent global counterpart. Given a base flag Φ of $\mathcal{P}$, we call a flag Ψ of $\mathcal{P}$ *even* (*with respect to* Φ) if there exists a finite sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_{2k} = \Psi$ where any two consecutive flags are adjacent (that is, differ in exactly one face). Note that the number of flags in this sequence distinct from Φ is even. A flag of $\mathcal{P}$ is *odd* (*with respect to* Φ) if it is not even. In particular, Φ is even and all flags adjacent to Φ are odd. Then a polytope $\mathcal{P}$ is chiral if and only if its group $A(\mathcal{P})$ has precisely two orbits on the flags and these are the sets of even and odd flags.

The group $A(\mathcal{P})$ of a chiral polytope $\mathcal{P}$ must necessarily satisfy a certain intersection condition, which is more complicated then the corresponding condition for regular polytopes and will be discussed in Lemma 1 in a particularly interesting special case. Then again the converse is true. Namely, if a group A is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$ satisfying the above relations and this intersection condition, then A is the group of a chiral polytope or the rotation group of a directly regular polytope. Often it is hard to decide whether this polytope is chiral or directly regular, but the following criterion is quite helpful. The polytope is directly regular if and only if there exists an involutory group automorphism $\rho: A \longrightarrow A$ such that $\sigma_1 \longmapsto \sigma_1^{-1}$, $\sigma_2 \longmapsto \sigma_2\sigma_1^2$ and $\sigma_j \longmapsto \sigma_j$ for $j = 3, \ldots, n-1$; see [23, Theorem 1], but note that now maps are written from left to right. In this case let $\rho_0 := \rho$ and $\rho_j := \sigma_j\rho_{j-1}, j = 1, \ldots, n-1$. Then $\rho_0, \ldots, \rho_{n-1}$ generate the group $A(\mathcal{P})$, while $\sigma_1, \ldots, \sigma_{n-1}$ generate its rotation subgroup $A^+(\mathcal{P})$.

Each chiral polytope $\mathcal{P}$ occurs in two *enantiomorphic forms*, in a sense in a *right* and a *left* version ([24]). In terms of groups and generators, these can be represented by two distinct systems of generators for the automorphism group, one defined with respect to the base flag Φ of $\mathcal{P}$, and the other with respect to the flag Φ^0 of $\mathcal{P}$ which is adjacent to Φ and differs from Φ in its vertex. More precisely, these systems of generators of $A(\mathcal{P})$ are $\{\sigma_1, \ldots, \sigma_{n-1}\}$ and $\{\sigma_1^{-1}, \sigma_2\sigma_1^2, \sigma_3, \ldots, \sigma_{n-1}\}$. Note that for a directly regular polytope $\mathcal{P}$ the corresponding systems are equivalent under conjugation by the "reflection" ρ_0 $(= \rho)$; that is, there is no distinction between a left and right version of $\mathcal{P}$ or, equivalently, the two enantiomorphic forms are the same.

An *oriented chiral* or *oriented directly regular* polytope is a chiral or directly regular polytope together with a distinguished enantiomorphic form; in the chiral case there are two "orientations", in the directly regular case only one. In this paper, if enantiomorphism is not important for what is under discussion, or is understood, we drop the qualification "oriented" and simply talk about chiral or directly regular polytopes. However, enantiomorphism is important in the following definition of a class of chiral polytopes.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two n-polytopes. Recall that, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are regular, then $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ denotes the class of all regular $(n+1)$-polytopes $\mathcal{P}$ with facets isomorphic to $\mathcal{P}_1$ and vertex-figures isomorphic to $\mathcal{P}_2$. Each non-empty class $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ contains a *universal* member denoted by $\{\mathcal{P}_1, \mathcal{P}_2\}$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are directly regular, then so is $\{\mathcal{P}_1, \mathcal{P}_2\}$. For chiral polytopes the definition of classes is more subtle and involves taking care of the two enantiomorphic forms in which the polytopes can occur. More precisely, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are oriented chiral or directly regular polytopes, then $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$ denotes the class of all oriented chiral $(n+1)$-polytopes $\mathcal{P}$ with (oriented) facets isomorphic to $\mathcal{P}_1$ and (oriented) vertex-figures isomorphic to $\mathcal{P}_2$. Again, if $\mathcal{P}_1$ or $\mathcal{P}_2$ is chiral and the class $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$ is non-empty, then it also contains a universal member denoted by $\{\mathcal{P}_1, \mathcal{P}_2\}^{\mathrm{ch}}$. Note that if the orientations of both $\mathcal{P}_1$ and $\mathcal{P}_2$ get changed, then the orientations of all members $\mathcal{P}$ in the class get changed, and hence that of $\{\mathcal{P}_1, \mathcal{P}_2\}^{\mathrm{ch}}$. However, the classes seem to be unrelated if the orientation of only one polytope is changed.

In our applications we use the intersection condition for a group A only under an additional assumption on a certain subgroup of A. In this case it takes the following simpler form ([23], Lemma 10).

LEMMA 1. *Let $n \geq 4$ and $A = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ be a group whose generators σ_i satisfy the above relations. Assume that the subgroup $\langle \sigma_1, \ldots, \sigma_{n-2} \rangle$ of A is the group of a chiral $(n-1)$-polytope or the rotation group of a directly regular $(n-1)$-polytope. Also, suppose that the following intersection condition holds:*

$$\langle \sigma_1, \ldots, \sigma_{n-2} \rangle \cap \langle \sigma_i, \ldots, \sigma_{n-1} \rangle = \langle \sigma_i, \ldots, \sigma_{n-2} \rangle \quad \text{for } i = 2, \ldots, n-1.$$

Then A itself is the group of a chiral n-polytope or the rotation group of a directly regular n-polytope.

For the purposes of this paper it will be convenient to introduce a new set of generators for the group $A(\mathcal{P})$ of a chiral polytope $\mathcal{P}$:

$$\tau_j := \sigma_j \sigma_{j-1} \cdot \cdots \cdot \sigma_1, \quad (j \geq 1).$$

Then $\tau_1 = \sigma_1$, and τ_j for $j > 1$ is an involution. We also define $\tau_0 := 1$. It follows that $A(\mathcal{P}) = \langle \tau_1, \ldots, \tau_{n-1} \rangle$ and the τ_j's satisfy the following relations

$$\begin{aligned} (\tau_j \tau_{j-1}^{-1})^{p_j} &= 1 \quad (1 \leq j \leq n-1), \\ \tau_j^2 &= 1 \quad (2 \leq j \leq n-1), \\ (\tau_j \tau_i^{-1})^2 &= 1 \quad (1 \leq i < j-1 \leq n-2). \end{aligned} \tag{1}$$

Furthermore we remark that all sections of a chiral polytope must be directly regular or chiral polytopes. In particular, the $(n-2)$-faces and the co-faces at edges are directly regular (*cf.* [23, pp. 498–499]).

3. **Free extensions.** Let $\mathcal{K}$ be an (oriented) chiral polytope of rank n ($n \geq 3$) and type $\{p_1, \ldots, p_{n-1}\}$, and let $\sigma_1, \ldots, \sigma_{n-1}$ be the distinguished generators of $A(\mathcal{K})$ (defining the orientation of $\mathcal{K}$). We shall also assume, for the reasons explained in the Introduction, that the facets of $\mathcal{K}$ are (directly) regular. Let $\mathcal{F}$ be a facet of $\mathcal{K}$ with $A^+(\mathcal{F}) = \langle \sigma_1, \ldots, \sigma_{n-2} \rangle$ and $A(\mathcal{F}) = \langle \rho_0, \ldots, \rho_{n-2} \rangle$, where $\sigma_i = \rho_i \rho_{i-1}$ for $i = 1, \ldots, n-2$. Then $A(\mathcal{F}) = A^+(\mathcal{F}) \ltimes C_2$.

We shall now construct (oriented) chiral polytopes of rank $n + 1$ whose (oriented) facets are isomorphic to $\mathcal{K}$. In particular, we shall describe a free construction which gives a universal polytope among all such polytopes. The following lemma motivates our approach. Its proof is straightforward.

LEMMA 2. *Let $\mathcal{L}$ be an (oriented) chiral polytope of rank $n + 1$ and type $\{p_1, \ldots, p_{n-1}, \infty\}$ with (oriented) facets $\mathcal{K}$. Let $A(\mathcal{L}) = \langle \tau_1, \ldots, \tau_{n-1}, \tau_n \rangle$.*

(a) Then, in addition to (1) the τ_j's also satisfy the relations

$$\tau_n^2 = 1, \tag{2}$$
$$(\tau_n \tau_i^{-1})^2 = 1 \quad (1 \leq i \leq n-2).$$

(b) τ_n induces the automorphism ρ_0 on $\mathcal{F}$.

Lemma 2 suggests considering the group $A = \langle \tau_1, \ldots, \tau_n \rangle$ with the set of relations given by all the defining relations for $A(\mathcal{K})$ (including those of (1)) and by the relations (2). Then by construction, $A(\mathcal{L})$ of Lemma 2 is a quotient of A, so that A is the natural candidate for the group of the universal polytope with facet type $\mathcal{K}$. To prove that this polytope really exists, and has certain properties, the key step is now to identify A as a suitable free product with amalgamation and then to use general properties of such free products.

Let

$$A_1 := A(\mathcal{K}) = \langle \tau_1, \ldots, \tau_{n-1} \rangle,$$
$$A_2 := A(\mathcal{F}) = A^+(\mathcal{F}) \ltimes C_2 = \langle \hat{\tau}_1, \ldots, \hat{\tau}_{n-1} \rangle \ltimes \langle \rho_0 \rangle,$$

and

$$A := A_1 *_{A^+(\mathcal{F})} A_2,$$

the free product of A_1 and A_2 with amalgamation of the two subgroups $A^+(\mathcal{F})$ of A_1 and A_2. Then A is the group with generators

$$\tau_1, \ldots, \tau_{n-1}, \hat{\tau}_1, \ldots, \hat{\tau}_{n-2}, \rho_0,$$

and the following defining relations: all the defining relations for $A(\mathcal{K})$, all the defining relations for $A(\mathcal{F})$, and

$$\tau_j = \hat{\tau}_j \quad (j = 2, \ldots, n-2).$$

Let $\tau_n := \rho_0$, $\sigma_n := \tau_n \tau_{n-1}^{-1} = \tau_n \tau_{n-1}$. Before proceeding we require

LEMMA 3. *(a) A is generated by $\tau_1, \ldots, \tau_n$, and is defined by all the relations for $A(\mathcal{K})$ (including those of (1)) and the relations (2).*

(b) $\langle \tau_1, \ldots, \tau_{n-1} \rangle \cong A_1 = A(\mathcal{K})$.

(c) $\langle \tau_1, \ldots, \tau_{n-2}, \tau_n \rangle \cong A_2 = A(\mathcal{F})$.

PROOF. To prove (a), rewrite the defining relations of A in terms of $\tau_1, \ldots, \tau_n$, using the relations $\tau_j = \hat{\tau}_j$. Parts (b) and (c) follow from general properties of free products with amalgamation (more precisely, from (3) below).

From now on we identify A_1 with $\langle \tau_1, \ldots, \tau_{n-1} \rangle$, A_2 with $\langle \tau_1, \ldots, \tau_{n-2}, \tau_n \rangle$, and consequently $A^+(\mathcal{F})$ with $\langle \tau_1, \ldots, \tau_{n-2} \rangle$. Then $A_1 \cap A_2 = A^+(\mathcal{F})$. We refer to A_1 and A_2 as *factors*. A sequence $\alpha_1, \ldots, \alpha_m$ (of *length* m) in A is called *reduced* if and only if the following conditions are satisfied:

(i) α_i is in one of the factors A_1 or A_2 for $i = 1, \ldots, m$;

(ii) α_i, α_{i+1} are in different factors A_1 or A_2 for $i = 1, \ldots, m-1$;

(iii) $\alpha_i \notin A^+(\mathcal{F})$ for $i = 1, \ldots, m$ if $m > 1$;

(iv) $\alpha_1 \neq 1$ if $m = 1$.

Below we use the following important fact known as the normal form theorem for free products with amalgamations (Lyndon-Schupp [9]):

(3) If $\alpha_1, \ldots, \alpha_m$ is a reduced sequence in A, then $\alpha_1 \alpha_2 \cdots \alpha_m \neq 1$ in A.

Note that if $\alpha_1, \ldots, \alpha_m$ is a reduced sequence, then $\alpha_1 \alpha_2 \cdots \alpha_m \notin A_1$ and $\alpha_1 \alpha_2 \cdots \alpha_m \notin A_2$ if $m \geq 2$.

The following lemma is used in the proof of the intersection condition.

LEMMA 4. *Let $2 \leq i \leq n-1$ and $\varphi \in \langle \sigma_i, \ldots, \sigma_{n-1} \rangle$.*

(a) If $\tau_{n-1}\varphi \in A^+(\mathcal{F})$, then $\sigma_{n-1}^{-1}\varphi \in \langle \sigma_i, \ldots, \sigma_{n-2} \rangle$.

(b) If $\tau_{n-1}\varphi\tau_{n-1} \in A^+(\mathcal{F})$, then $\sigma_{n-1}^{-1}\varphi\sigma_{n-1} \in \langle \sigma_i, \ldots, \sigma_{n-2} \rangle$.

PROOF. Let $\{F_{-1}, F_0, \ldots, F_n\}$ be the base flag of $\mathcal{K}$. If $\tau_{n-1}\varphi \in A^+(\mathcal{F})$, then $F_{n-1}\tau_{n-1}\varphi = F_{n-1}$ and thus $F_{n-1}\varphi^{-1} = F_{n-1}\tau_{n-1}$, the $(n-1)$-face F'_{n-1} of $\mathcal{K}$. Hence $F_{n-1} = F'_{n-1}\sigma_{n-1} = (F_{n-1}\varphi^{-1})\sigma_{n-1} = F_{n-1}(\varphi^{-1}\sigma_{n-1})$. But $F_j\varphi = F_j$ for $j = 0, \ldots, i-2$, so that also $F_j(\varphi^{-1}\sigma_{n-1}) = F_j$ for $j = 0, \ldots, i-2$. It follows that $\varphi^{-1}\sigma_{n-1} \in \langle \sigma_i, \ldots, \sigma_{n-2} \rangle$, the stabilizer of $F_0, \ldots, F_{i-2}$ and F_{n-1} in $A(\mathcal{K})$. This proves (a).

If $\tau_{n-1}\varphi\tau_{n-1} \in A^+(\mathcal{F})$, then $F'_{n-1}\varphi = F_{n-1}\tau_{n-1}\varphi = F_{n-1}\tau_{n-1} = F'_{n-1}$. But $F'_{n-1}\sigma_{n-1} = F_{n-1}$, so that $F_{n-1}\sigma_{n-1}^{-1}\varphi\sigma_{n-1} = F_{n-1}$. Also, for $j = 0, \ldots, i-2$, $F_j\sigma_{n-1}^{-1}\varphi\sigma_{n-1} = F_j$ because $F_j\sigma_{n-1} = F_j = F_j\varphi$. It follows that $\sigma_{n-1}^{-1}\varphi\sigma_{n-1} \in \langle \sigma_i, \ldots, \sigma_{n-2} \rangle$. This proves (b).

THEOREM 1. *A is the group of an (oriented) chiral polytope $\mathcal{P}$ of type $\{p_1, \ldots, p_{n-1}, \infty\}$ with (oriented) facets isomorphic to $\mathcal{K}$.*

PROOF. We use Lemma 1. In order that A be the group of a chiral polytope it remains to verify the intersection condition

(4) $$\langle \sigma_i, \ldots, \sigma_n \rangle \cap \langle \sigma_1, \ldots, \sigma_{n-1} \rangle = \langle \sigma_i, \ldots, \sigma_{n-1} \rangle \quad (i = 2, \ldots, n).$$

First, let $i \le n-1$ and $\varphi \in \langle \sigma_i, \ldots, \sigma_n \rangle$. Then φ can be written as

$$\varphi = \varphi_0 \sigma_n^{\varepsilon_1} \varphi_1 \sigma_n^{\varepsilon_2} \cdot \cdots \cdot \sigma_n^{\varepsilon_{k-1}} \varphi_{k-1} \sigma_n^{\varepsilon_k} \varphi_k \tag{5}$$

with $\varepsilon_s = \pm 1$ and $\varphi_s \in \langle \sigma_i, \ldots, \sigma_{n-1} \rangle$ (possibly $\varphi_s = 1$) for all s. Let $k(\varphi)$ be the minimum k, such that φ can be written as in (5). If $k(\varphi) = 0$, then $\varphi \in \langle \sigma_i, \ldots, \sigma_{n-1} \rangle$. We need to prove that $\varphi \notin \langle \sigma_1, \ldots, \sigma_{n-1} \rangle = A_1$ if $k(\varphi) \ge 1$. We use induction on $k := k(\varphi)$. Without loss of generality we may assume that $\varphi_0 = 1 = \varphi_k$. Then the case $k = 1$ is trivial because $\sigma_n \notin A_1$.

Let $k = 2$ and φ be as in (5). Assume $(\varepsilon_1, \varepsilon_2) = (+1, +1)$. Then

$$\varphi = \sigma_n \varphi_1 \sigma_n = \tau_n \tau_{n-1} \varphi_1 \tau_n \tau_{n-1}.$$

However, if $\tau_{n-1}\varphi_1 \notin A^+(\mathcal{F})$, then $\tau_n, \tau_{n-1}\varphi_1, \tau_n, \tau_{n-1}$ is a reduced sequence of length 4 and thus $\varphi \notin A_1$. On the other hand we obviously cannot have $\tau_{n-1}\varphi_1 \in A^+(\mathcal{F})$. In fact, if $\tau_{n-1}\varphi_1 \in A^+(\mathcal{F})$, then Lemma 4(a) implies $\varphi_1 \in \sigma_{n-1}\langle \sigma_i, \ldots, \sigma_{n-2} \rangle$, $\varphi_1 = \sigma_{n-1}\gamma$ (say) with $\gamma \in \langle \sigma_i, \ldots, \sigma_{n-2} \rangle$. It follows that

$$\begin{aligned}
\varphi &= \tau_n \tau_{n-1} \sigma_{n-1} \gamma \tau_n \tau_{n-1} \\
&= \tau_n \tau_{n-2}^{-1} \gamma \tau_n \tau_{n-1} \\
&= \tau_{n-2}(\tau_n \gamma \tau_n) \tau_{n-1} \in \tau_{n-2} \tau_n A^+(\mathcal{F}) \tau_n \tau_{n-1} = \tau_{n-2} A^+(\mathcal{F}) \tau_{n-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \subseteq A(\mathcal{K}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle;
\end{aligned}$$

here we used the fact that $(\tau_n \tau_{n-2}^{-1})^2 = 1$. Now consider how φ acts on F_j with $j \le i-2$. For $j \ge 1$ the elements τ_{n-2}, $\tau_n \gamma \tau_n (= \rho_0 \gamma \rho_0)$ and τ_{n-1} fix F_j, and so does φ. For $j = 0$ we have $F_0 \tau_{n-2} = F_0' = F_0 \tau_{n-1}$ and $F_0' \tau_n \gamma \tau_n (= F_0' \rho_0 \gamma \rho_0) = F_0'$, so that again φ fixes $F_j (= F_0)$. Consequently, $\varphi \in \langle \sigma_i, \ldots, \sigma_{n-1} \rangle$ and thus $k(\varphi) = 0$, contradicting $k(\varphi) = 2$.

Now assume $(\varepsilon_1, \varepsilon_2) = (+1, -1)$. Then

$$\varphi = \sigma_n \varphi_1 \sigma_n^{-1} = \tau_n(\tau_{n-1}\varphi_1\tau_{n-1})\tau_n.$$

As before, if $\tau_{n-1}\varphi_1\tau_{n-1} \notin A^+(\mathcal{F})$, then $\tau_n, \tau_{n-1}\varphi_1\tau_{n-1}, \tau_n$ is a reduced sequence of length 3 and thus $\varphi \notin A_1$. Again we cannot have $\tau_{n-1}\varphi_1\tau_{n-1} \in A^+(\mathcal{F})$. In fact, if $\tau_{n-1}\varphi_1\tau_{n-1} \in A^+(\mathcal{F})$, then Lemma 4(b) implies $\varphi_1 \in \sigma_{n-1}\langle \sigma_i, \ldots, \sigma_{n-2} \rangle \sigma_{n-1}^{-1}$, $\varphi_1 = \sigma_{n-1}\gamma\sigma_{n-1}^{-1}$ (say) with $\gamma \in \langle \sigma_i, \ldots, \sigma_{n-2} \rangle$. Then

$$\begin{aligned}
\varphi &= \tau_n(\tau_{n-1}\sigma_n)\gamma\sigma_n^{-1}\tau_{n-1}\tau_n \\
&= \tau_n \tau_{n-2}^{-1} \gamma \tau_{n-2} \tau_n \\
&= \tau_{n-2}(\tau_n \gamma \tau_n)\tau_{n-2}^{-1} \in \tau_{n-2} A^+(\mathcal{F}) \tau_{n-2}^{-1} \subseteq A(\mathcal{K}).
\end{aligned}$$

Considering again the action on the faces F_j, we observe as above that φ fixes $F_0, \ldots, F_{i-2}$ and thus $\varphi \in \langle \sigma_i, \ldots, \sigma_{n-1} \rangle$, contradicting $k = 2$ again.

The possibility of the signature $(-1, -1)$ can be refuted by applying the above arguments for the case $(+1, +1)$ to φ^{-1}. In particular, if $(\varepsilon_1, \varepsilon_2) = (-1, -1)$, we cannot have $\varphi_1\tau_{n-1} \in A^+(\mathcal{F})$ because of the minimality of k. It remains to discuss the signature $(-1,+1)$.

If $(\varepsilon_1, \varepsilon_2) = (-1,+1)$, we have

$$\varphi = \sigma_n^{-1}\varphi_1\sigma_n = \tau_{n-1}(\tau_n\varphi_1\tau_n)\tau_{n-1}.$$

Now, if $\varphi_1 \notin A^+(\mathcal{F})$, then τ_{n-1}, τ_n, φ_1, τ_n, τ_{n-1} is a reduced sequence of length 5, so that $\varphi \notin A_1$. But we cannot have $\varphi_1 \in A^+(\mathcal{F})$, because of the minimality of k. In fact, if $\varphi_1 \in A^+(\mathcal{F})$, then so is $\tau_n\varphi_1\tau_n (= \rho_0\varphi_1\rho_0)$, implying $\tau_{n-1}(\tau_n\varphi_1\tau_n)\tau_{n-1} (= \varphi) \in A(\mathcal{K})$. As above, τ_{n-1} and $\tau_n\varphi_1\tau_n$ fix $F_1, \ldots, F_{i-2}$, and so does φ. Also, $F_0\tau_{n-1} = F_0' = F_0'\tau_n\varphi_1\tau_n$, so that φ also fixes F_0. It follows that $\varphi \in \langle \sigma_i, \ldots, \sigma_{n-1}\rangle$, contradicting $k = 2$. This settles the case $k = 2$.

Now, let $k \geq 3$ and φ as in (5). Using $\sigma_n = \tau_n\tau_{n-1}$ we can write φ as a product of terms in A_1 or A_2, with consecutive terms in different factors, such that all A_2-terms are τ_n and all A_1 terms are of the form

$$\varphi_s, \quad \tau_{n-1}\varphi_s, \quad \varphi_s\tau_{n-1} \quad \text{or} \quad \tau_{n-1}\varphi_s\tau_{n-1}. \tag{6}$$

Now, this gives a reduced sequence if and only if each A_1-term is not in $A^+(\mathcal{F})$. However, the terms in (6) cannot be in $A^+(\mathcal{F})$ because the above arguments for $k = 2$ applied to the terms $\sigma_n^{\varepsilon_s}\varphi_s\sigma_n^{\varepsilon_{s+1}}$ would again contradict the minimality of k. It follows that φ is the product of terms in a reduced sequence of length at least k. Therefore, $\varphi \notin A_1$. This completes the proof of (4) for $i \leq n-1$.

If $i = n$, then (4) is equivalent to $\sigma_n^j \notin A_1$ for each $j \neq 0$. But $\sigma_n^j = (\tau_n\tau_{n-1})^j$ gives a reduced sequence of length $2j$, so that the proof of the intersection condition is now complete.

Finally, the last entry in the Schläfli symbol is ∞ because σ_n does not have finite order.

REMARK 1. Our construction of the polytopes $\mathcal{P}$ of Theorem 1 with facets of type $\mathcal{K}$ also carries over to the case where $\mathcal{K}$ is directly regular. If $\mathcal{K}$ is directly regular, then so is $\mathcal{F}$, and the rotation subgroups $A^+(\mathcal{K})$ of $A(\mathcal{K})$ and $A^+(\mathcal{F})$ of $A(\mathcal{F})$ have index 2. The polytope $\mathcal{P}$ is now constructed from the group $A := A^+(\mathcal{K}) *_{A^+(\mathcal{F})} A(\mathcal{F})$. However, now $\mathcal{P}$ is directly regular and A is the rotation group $A^+(\mathcal{P})$.

In fact, since $\mathcal{K}$ is directly regular, there is an involution $\alpha_0 \in A(\mathcal{K})$ with $\alpha_0 \notin A^+(\mathcal{K})$ which fixes all faces of the base flag of $\mathcal{K}$ except for the vertex. Conjugation by α_0 induces an automorphism α on $A^+(\mathcal{K})$, and also an automorphism β on $A(\mathcal{F})$ which maps $A^+(\mathcal{F})$ onto itself. Since the two automorphisms α and β have the same effect on $A^+(\mathcal{F})$, they can be extended to the amalgamated product A by an automorphism γ (*cf.* [11, p. 207]). But

$$\sigma_j\gamma = \sigma_j\alpha = \alpha_0\sigma_j\alpha_0 = \begin{cases} \sigma_1^{-1} & \text{if } j = 1, \\ \sigma_2\sigma_1^2 & \text{if } j = 2, \\ \sigma_j & \text{if } 3 \leq j \leq n-1, \end{cases}$$

and

$$\sigma_n\gamma = (\tau_n\tau_{n-1})\gamma = (\tau_n\gamma)(\tau_{n-1}\gamma) = (\alpha_0\rho_0\alpha_0)\tau_{n-1} = \rho_0\tau_{n-1} = \tau_n\tau_{n-1} = \sigma_n.$$

The regularity of $\mathcal{P}$ now follows from what was said in Section 2 ([23, Theorem 1]). To check that $\mathcal{P}$ is indeed directly regular, note that γ cannot be realized by conjugation with an involution in A; the proof again uses the normal form theorem for A.

We remark that the construction for directly regular polytopes produces the same polytope as in [21, Theorem 2] and [22, Theorem 2].

THEOREM 2. *Let $\mathcal{K}$ be an (oriented) chiral n-polytope of type $\{p_1,\ldots,p_{n-1}\}$ with directly regular facets $\mathcal{F}$ and with (oriented) vertex-figures $\mathcal{K}_0$. Then there exists an (oriented) chiral $(n+1)$-polytope $\mathcal{P}$ with the following properties:*

(a) $\mathcal{P}$ has (oriented) facets isomorphic to $\mathcal{K}$.

(b) $\mathcal{P}$ is universal among all (oriented) chiral $(n+1)$-polytopes with (oriented) facets isomorphic to $\mathcal{K}$; that is, any other such polytope is a quotient of $\mathcal{P}$.

(c) $A(\mathcal{P}) \cong A(\mathcal{K}) *_{A^+(\mathcal{F})} A(\mathcal{F})$, the free product of $A(\mathcal{K})$ and $A(\mathcal{F})$ with amalgamation of the two subgroups isomorphic to $A^+(\mathcal{F})$.

(d) $\mathcal{P}$ is of type $\{p_1,\ldots,p_{n-1},\infty\}$.

(e) $\mathcal{P} = \{\mathcal{K},\mathcal{P}_0\}^{\mathrm{ch}}$, the universal (oriented) chiral $(n+1)$-polytope with (oriented) facets isomorphic to $\mathcal{K}$ and with (oriented) vertex-figures isomorphic to the n-polytope $\mathcal{P}_0$ which is constructed from $\mathcal{K}_0$ in the same way as $\mathcal{P}$ from $\mathcal{K}$.

PROOF. Let $\mathcal{P}$ be the polytope from Theorem 1. It remains to prove parts (b) and (e). For the proof of (b) let $\mathcal{L}$ be any chiral $(n+1)$-polytope with facets isomorphic to $\mathcal{K}$, and let $A(\mathcal{L}) = \langle\alpha_1,\ldots,\alpha_n\rangle$, with $\alpha_1,\ldots,\alpha_n$ the distinguished generators of $A(\mathcal{L})$. Then $\alpha_1,\ldots,\alpha_n$ satisfy all defining relations for $A = A(\mathcal{P})$ (expressed in terms of $\sigma_1,\ldots,\sigma_n$), so that the mapping $\sigma_i \longmapsto \alpha_i$ $(i = 1,\ldots,n)$ defines a surjective homomorphism $\kappa: A(\mathcal{P}) \longrightarrow A(\mathcal{L})$. But κ induces a covering $\hat{\kappa}: \mathcal{P} \longrightarrow \mathcal{L}$; that is, $\hat{\kappa}$ is surjective, rank preserving and incidence preserving, and identifies faces of $\mathcal{P}$ modulo $\ker(\kappa)$ (*cf.* [17]). Hence $\mathcal{L}$ is a quotient of $\mathcal{P}$, proving (b).

To prove (e) rewrite the defining relations for $A = A(\mathcal{P})$ with generators $\tau_1,\ldots,\tau_{n-1}$, $\hat{\tau}_1,\ldots,\hat{\tau}_{n-2},\rho_0$ in terms of the generators $\sigma_1,\ldots,\sigma_n$. Then the following relations give a set of defining relations for A: all defining relations for $A(\mathcal{K})$ in terms of $\sigma_1,\ldots,\sigma_{n-1}$; $\tau_n^2 = 1$, with $\tau_n = \sigma_n\sigma_{n-1}\cdot\cdots\cdot\sigma_1$; and the relations expressing the action of $\tau_n (= \rho_0)$ on $A^+(\mathcal{F})$, that is,

$$\tau_n\sigma_j\tau_n = \begin{cases} \sigma_1^{-1} & \text{if } j = 1,\\ \sigma_2\sigma_1^2 & \text{if } j = 2,\\ \sigma_j & \text{if } 3 \leq j \leq n-2.\end{cases} \tag{7}$$

Now, the facets of the vertex-figures $\hat{\mathcal{P}}_0$ of $\mathcal{P}$ are isomorphic to $\mathcal{K}_0$. Hence, by part (b), $\hat{\mathcal{P}}_0$ is a quotient of $\mathcal{P}_0$. It follows that, if the universal polytope $\{\mathcal{K},\mathcal{P}_0\}^{\mathrm{ch}}$ exists, then $\mathcal{P}$ must be a quotient of it. In any case, if $\tilde{A} := \langle\tilde{\sigma}_1,\ldots,\tilde{\sigma}_n\rangle$ is the group which would define $\{\mathcal{K},\mathcal{P}_0\}^{\mathrm{ch}}$ (regardless of existence or non-existence) in terms of generators and relations, then $A(\mathcal{P})$ is a quotient of $\tilde{A}$ (under the mapping $\tilde{\sigma}_i \longmapsto \sigma_i$).

To see that the converse is also true, consider the following set of defining relations for $\tilde{A}$: all defining relations for $A(\mathcal{K})$ in terms of $\tilde{\sigma}_1,\ldots,\tilde{\sigma}_{n-1}$; $\tilde{\tau}_{n,2}^2 = 1$, with $\tilde{\tau}_{n,2} :=$

$\tilde\sigma_n\tilde\sigma_{n-1}\cdot\cdots\cdot\tilde\sigma_2$; the relations expressing the action of $\tilde\tau_{n,2}$ on the rotation subgroup $A^+(\mathcal{F}_0)$ of the group of the vertex-figure $\mathcal{F}_0$ of $\mathcal{F}$, that is,

$$\tilde\tau_{n,2}\tilde\sigma_j\tilde\tau_{n,2}=\begin{cases}\tilde\sigma_2^{-1} & \text{if } j=2,\\ \tilde\sigma_3\tilde\sigma_2^2 & \text{if } j=3,\\ \tilde\sigma_j & \text{if } 4\le j\le n-2;\end{cases}\tag{8}$$

and last, the extra relation $\tilde\tau_{n,1}^2=1$, with $\tilde\tau_{n,1}:=\tilde\sigma_n\tilde\sigma_{n-1}\cdot\cdots\cdot\tilde\sigma_1$. We need to check that $\tilde\sigma_1,\ldots,\tilde\sigma_n$ satisfy all the defining relations for $A(\mathcal{P})$ in terms of $\sigma_1,\ldots,\sigma_n$. The only problem is (7). But the relations (7) follow from (8); to give an example,

$$\begin{aligned}\tilde\tau_{n,1}\tilde\sigma_3\tilde\tau_{n,1}&=\tilde\tau_{n,2}\tilde\sigma_1\tilde\sigma_3\tilde\tau_{n,2}\tilde\sigma_1=\tilde\sigma_1^{-1}\tilde\tau_{n,2}\tilde\sigma_3\tilde\tau_{n,2}\tilde\sigma_1=\tilde\sigma_1^{-1}\tilde\sigma_3\tilde\sigma_2^2\tilde\sigma_1\\&=(\tilde\sigma_1^{-1}\tilde\sigma_3\tilde\sigma_2)(\tilde\sigma_2\tilde\sigma_1)=\tilde\sigma_3(\tilde\sigma_2\tilde\sigma_1)(\tilde\sigma_2\tilde\sigma_1)=\tilde\sigma_3,\end{aligned}$$

as required. Hence, $\sigma_i\longmapsto\tilde\sigma_i$ defines a homomorphism of $A(\mathcal{P})$ onto $\tilde A$, which is in fact an isomorphism. Therefore, $\{\mathcal{K},\mathcal{P}_0\}^{\mathrm{ch}}$ exists, has group $\tilde A$, and is isomorphic to $\mathcal{P}$. This completes the proof.

Note that in Theorem 2 the vertex-figure $\mathcal{K}_0$ of $\mathcal{K}$ may be directly regular, so that in part (e) the Remark 1 applies to give a directly regular vertex-figure $\mathcal{P}_0$ of $\mathcal{P}$. We also remark that an analogue of Theorem 2(e) can be proved for the polytope constructed in [22, Theorem 2].

The lowest rank to which the above theorems apply is $n=3$, because all 2-polytopes are regular. If $\mathcal{K}$ is a chiral 3-polytope of type $\{p_1,p_2\}$, then its facets are p_1-gons and hence directly regular. Typical examples are the irreflexible maps on surfaces, for instance the toroidal maps $\{4,4\}_{(b,c)}$, $\{3,6\}_{(b,c)}$ and $\{6,3\}_{(b,c)}$, with $bc(b-c)\neq 0$ (*cf.* Coxeter-Moser [4]). These give chiral 4-polytopes of types $\{4,4,\infty\}$, $\{3,6,\infty\}$ and $\{6,3,\infty\}$; more precisely, we obtain the universal polytopes $\big\{\{4,4\}_{(b,c)},\{4,\infty\}\big\}^{\mathrm{ch}}$, $\big\{\{3,6\}_{(b,c)},\{6,\infty\}\big\}^{\mathrm{ch}}$ and $\big\{\{6,3\}_{(b,c)},\{3,\infty\}\big\}^{\mathrm{ch}}$. To give an example with facets of higher genus, if $\mathcal{K}$ is the chiral map of genus 7 and type $\{9,6\}$ mentioned in Garbe [7], then $\mathcal{P}=\{\mathcal{K},\{6,\infty\}\}^{\mathrm{ch}}$ has facets of genus 7.

There are also examples of higher ranks. For instance, Nostrand [20] uses the honeycombs $\{4,3,6\}$, $\{5,3,6\}$ and $\{5,3,5\}$ in hyperbolic 3-space to construct infinite series of finite chiral 4-polytopes of the same types whose facets are cubes or dodecahedra; the groups are certain projective linear groups over finite fields. Now the extensions are of type $\{4,3,6,\infty\}$, $\{5,3,6,\infty\}$ and $\{5,3,5,\infty\}$, respectively. Similar examples of types $\{3,4,4,\infty\}$ and $\{3,3,6,\infty\}$ can also be derived from the polytopes in [24].

In general it is very difficult to actually construct chiral polytopes of higher ranks. In fact, if a group A is already known to be the group of a chiral polytope or the rotation group of a directly regular polytope $\mathcal{P}$, then we must ensure that $\mathcal{P}$ really occurs in two (distinct) enantiomorphic forms. This is a condition on not being able to map the two corresponding systems of generators of $A=A(\mathcal{P})$ onto each other by an involutory automorphism of A. This can also be put in another, equivalent way as follows.

Let $\mathcal{P}$ be of type $\{p_1,\ldots,p_{n-1}\}$, and let $[p_1,\ldots,p_{n-1}]^+$ denote the rotation group of the Coxeter group $[p_1,\ldots,p_{n-1}]$ whose diagram is the string diagram with n nodes and

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$n-1$ branches marked $p_1,\ldots,p_{n-1}$, respectively. Now, if A is given as a quotient of $[p_1,\ldots,p_{n-1}]^+$ by the normal subgroup N (say), then $\mathcal{P}$ is chiral if and only if N is not normal in $[p_1,\ldots,p_{n-1}]$ (that is, N is not invariant under conjugation by any generating reflection of $[p_1,\ldots,p_{n-1}]$).

Examples of such subgroups N which are normal in $[p_1,\ldots,p_{n-1}]^+$ but not in $[p_1,\ldots,p_{n-1}]$ are easy to find if $n=3$; for instance, take a suitable translation subgroup in the symmetry group $[4,4]$ of the euclidean square tessellation. However, this method of construction breaks down for euclidean reflection groups in higher dimensions; in fact, it was proved in [16] that there are no chiral toroids of rank 4 or higher. This fact explains the exclusion of euclidean reflection groups in the following conjectures.

CONJECTURE 1. *Let $p_1,\ldots,p_{n-1}$ be integers ≥ 3 such that the Coxeter group $[p_1,\ldots,p_{n-1}]$ is neither spherical nor euclidean. Then there are infinitely many finite chiral n-polytopes of type $\{p_1,\ldots,p_{n-1}\}$.*

CONJECTURE 2. *Let $\mathcal{K}$ be a chiral n-polytope of type $\{p_1,\ldots,p_{n-1}\}$ with directly regular facets, and let $q\geq 3$. Then there exists a chiral $(n+1)$-polytope $\mathcal{P}$ of type $\{p_1,\ldots,p_{n-1},q\}$ whose facets are isomorphic to $\mathcal{K}$.*

Concluding we use the concept of residual finiteness of groups to give a nonconstructive approach to the above conjectures and to related problems of a similar nature. This generalizes analogous results of Vince [27] on combinatorial maps and of McMullen and Schulte [18] on polytopes which are regular.

A group U is called *residually finite* if and only if for each finite subset T of $U\setminus\{1\}$ there exists a homomorphism f of U onto a finite group such that $\varphi f\neq 1$ for all $\varphi\in T$. It is well-known that the finitely generated linear groups are examples of such groups (Malcev [12], Wehrfritz [28]).

THEOREM 3. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be finite (oriented) chiral or directly regular n-polytopes but not both directly regular, and let $\langle\mathcal{P}_1,\mathcal{P}_2\rangle^{\mathrm{ch}}\neq\emptyset$. Let $\mathcal{P}$ be an infinite (oriented) chiral $(n+1)$-polytope in $\langle\mathcal{P}_1,\mathcal{P}_2\rangle^{\mathrm{ch}}$ whose group $A(\mathcal{P})$ is residually finite. Then $\langle\mathcal{P}_1,\mathcal{P}_2\rangle^{\mathrm{ch}}$ contains infinitely many (oriented) chiral $(n+1)$-polytopes which are finite and are covered by $\mathcal{P}$.*

PROOF. The proof follows the same pattern as that of Theorem 1 in [18], so we shall only sketch it here. Let $A(\mathcal{P})=\langle\sigma_1,\ldots,\sigma_n\rangle$, and let $G_1:=\langle\sigma_1,\ldots,\sigma_{n-1}\rangle$ and $G_2:=\langle\sigma_2,\ldots,\sigma_n\rangle$ be the (rotation) groups of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then $T_1:=(G_1G_2)\setminus\{1\}$ is a finite subset of $A(\mathcal{P})$. Since $A(\mathcal{P})$ is residually finite, there exists a surjective homomorphism $f_1\colon A(\mathcal{P})\longrightarrow A_1$ with A_1 a finite group such that $\varphi f_1\neq 1$ for all $\varphi\in T_1$. But $G_1\setminus\{1\},G_2\setminus\{1\}\subset T_1$, and so the restrictions of f_1 to G_1 and G_2 are isomorphisms. Now apply Lemma 1 to $A_1=\langle\sigma_1f_1,\ldots,\sigma_nf_1\rangle$, using the fact that its subgroup $\langle\sigma_1f_1,\ldots,\sigma_{n-1}f_1\rangle$ is known to be isomorphic to the group G_1 of $\mathcal{P}_1$. On the other hand, the subgroup $\langle\sigma_2f_1,\ldots,\sigma_nf_1\rangle\cong G_2$ also satisfies the intersection property, and therefore the conditions of Lemma 1 can be reduced to the simple condition

$$G_1f_1\cap G_2f_1=(G_1\cap G_2)f_1.$$

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But now, if $\varphi_1 \in G_1$, $\varphi_2 \in G_2$ and $\varphi_1 f_1 = \varphi_2 f_1$, then $\varphi_2^{-1}\varphi_1 \in G_2G_1 \cap \ker(f_1)$, so that by the construction of T_1 we must have $\varphi_1 = \varphi_2 \in G_1 \cap G_2$; the other inclusion is trivial.

It follows that A_1 is the group of a finite chiral polytope in $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$; here, the chirality follows from that of $\mathcal{P}_1$ or $\mathcal{P}_2$. By construction the polytope is covered by $\mathcal{P}$. The further proof now proceeds as in [18] by suitably constructing larger subsets T_i of $A(\mathcal{P})$ and corresponding homomorphisms $f_i \colon A(\mathcal{P}) \longrightarrow A_i$ with A_i a finite group.

It would be interesting to know to what extent Theorem 3 carries over to the case where $\mathcal{P}_1$ and $\mathcal{P}_2$ are both directly regular. The non-existence of chiral toroids of rank 4 or higher implies that it is not sufficient to simply assume $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle \neq \emptyset$. In other words, we need at least one chiral polytope $\mathcal{P}$ to start with or, equivalently, $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}} \neq \emptyset$. However, it is not at all clear how to adjust the proof of Theorem 3 to obtain chirality of the polytopes if this chirality is not already given locally by the facets or vertex-figures.

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