ON A THEOREM OF HAYMAN CONCERNING QUASI-BOUNDED FUNCTIONS

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1. If $f(z)$ is regular in $|z| < 1$, the expression

$$m(r,f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta \qquad (0 < r < 1)$$

is called the *characteristic* of $f(z)$. This is the notation of Nevanlinna **(4)** for the special case of regular functions; in this note it will not be necessary to discuss meromorphic functions. If $m(r,f)$ is bounded for $0 < r < 1$, then $f(z)$ is called *quasi-bounded* in $|z| < 1$. In particular, every bounded function is quasi-bounded. The class Q of quasi-bounded functions is important because, for instance, a "Fatou theorem" holds for such functions **(4**, p. 134**)**. It is known that $f \in Q$ does not imply $f' \in Q$ in general; the simplest counterexample seems to be that given in **(1)**, §8. Hayman **(1**, Theorem IV**)** found extra hypotheses under which $f \in Q$ implies $f' \in Q$. The purpose of this note is to construct examples which shed some light on the "goodness" of these hypotheses. The method is similar to one used in **(2)**; it is essentially a combination of a construction due to Littlewood **(3**, §8.5**)** with a theorem of Specht **(5)** on conformal mapping.

2. Let D be a domain containing a sequence of open arcs

$$z = e^{i\theta}, \quad \theta_n < \theta < \theta_n + \delta_n, \quad n = 1, 2, 3, \ldots . \tag{2.1}$$

Let $d(\theta)$ denote the distance from $e^{i\theta}$ to the boundary of D, and suppose that there are positive numbers A and B, independent of θ and n, such that

$$d(\theta) > A\{(\theta - \theta_n)(\theta_n + \delta_n - \theta)\}^B \; (\theta_n < \theta < \theta_n + \delta_n,\, n = 1, 2, 3, \ldots).$$

Then, following Hayman **(1)**, we say that D *properly contains* the sequence of arcs (2.1). Hayman has shown that, *if D contains $|z| < 1$ and properly contains a sequence of non-overlapping arcs* (2.1) *such that*

$$\sum_1^\infty \delta_n = 2\pi, \tag{2.2}$$

$$\sum_1^\infty \delta_n \log\frac{1}{\delta_n} < \infty, \tag{2.3}$$

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and if $f(z)$ is regular and bounded in D, then $f'(z)$ is quasi-bounded in $|z| < 1$. This is a very special case of Theorem IV of **(1)**, but it is sufficient for our purpose in this note.

We prove the following theorem, which sets a limit to the extent to which it may be possible to relax the condition (2.3).

THEOREM. *Let $\{\delta_n\}$ be a non-increasing sequence satisfying* (2.2) *and*

$$\limsup_{n\to\infty} \left(\sum_n^\infty \delta_j\right) \log \frac{1}{\delta_n} = \infty, \tag{2.4}$$

and let $\{\theta_n\}$ be such that no two of the arcs (2.1) *overlap. Then there exist a domain D containing $|z| < 1$ and properly containing the sequence of arcs* (2.1), *and a function $f(z)$ regular and bounded in D such that $m(r, f') \to \infty$ as $r \to 1$.*

(2.4) does *not* follow from

$$\sum_1^\infty \delta_n \log \frac{1}{\delta_n} = \infty; \tag{2.5}$$

for example, if

$$\delta_n = \frac{1}{n(\log n)^2}$$

for all large enough n, then (2.4) is false although (2.5) is true. Thus there remains a gap between the positive information given by Hayman's theorem and the negative information which comes from the result proved in this note. I have not found a method of closing this gap. However, our result shows that (2.3) cannot in general be replaced by

$$\sum_1^\infty \delta_n \left(\log \frac{1}{\delta_n}\right)^\alpha < \infty$$

for any $\alpha < 1$, or even by certain rather stronger conditions than this.

In the theorem, the sequence of arcs (2.1) may be obtained by putting $\theta_1 = 0$, $\theta_{n+1} = \theta_n + \delta_n$, in which case (2.1) is a sequence of arcs whose end-points have no limit-point except $z = 1$, and whose complement is the sequence $\{e^{i\theta_n}\}$. But it should be noted that a more complicated situation may also occur, in which the arcs (2.1) form the complement of an uncountable set.

Some remarks on the other hypotheses of Hayman's theorem are made in §6.

3. In this section and the next we assume only that (2.1) is a sequence of non-overlapping arcs; the other hypotheses of the theorem are not needed at this stage. However, we suppose, as we may without loss of generality, that $\theta_1 \leqslant \theta_n < \theta_1 + 2\pi$ for every n. Let $c > 0$ be a constant, and for $n = 1, 2, 3, \ldots$ put

$$\psi(\theta) = c(\theta - \theta_n)^3(\theta_n + \delta_n - \theta)^3 \qquad (\theta_n < \theta < \theta_n + \delta_n).$$

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Moreover put $\psi(\theta) = 0$ for all other values of θ in $\theta_1 \leqslant \theta < \theta_1 + 2\pi$, and define $\psi(\theta)$ outside this interval by requiring it to have period 2π. Then we have

LEMMA 1. *There is an absolute constant K with the following properties:*
(i) *for all θ and n,*

$$\psi(\theta) \leqslant Kc|\theta - \theta_n|^3,$$

(ii) *$\psi(\theta)$ is twice continuously differentiable, and for all θ,*

$$|\psi'(\theta)| \leqslant Kc, \qquad |\psi''(\theta)| \leqslant Kc.$$

A full proof of Lemma 1 involves some lengthy repetition, and so we leave some details to the reader. Let the subset S of $(\theta_1, \theta_1 + 2\pi)$ be defined by

$$S = \bigcup_1^\infty (\theta_n, \theta_n + \delta_n),$$

and let T denote the complement of S with respect to the closed interval $(\theta_1, \theta_1 + 2\pi)$. We first prove that, for $\phi \in T$ and all θ,

$$\psi(\theta) \leqslant K_1 c|\theta - \phi|^3, \tag{3.1}$$

where K_1 is an absolute constant. If $\theta \in S$, then $\theta_j < \theta < \theta_j + \delta_j$ for some j, and therefore, since $\phi \in T$,

$$|\theta - \phi| \geqslant \min\,(\theta - \theta_j, \theta_j + \delta_j - \theta).$$

Since

$$\theta - \theta_j < 2\pi, \quad \theta_j + \delta_j - \theta < 2\pi, \tag{3.2}$$

it follows that

$$|\theta - \phi| > \frac{1}{2\pi}(\theta - \theta_j)(\theta_j + \delta_j - \theta).$$

From this and the definition of $\psi(\theta)$ we get (3.1), with a suitable K_1, for all $\theta \in S$; and (3.1) is obvious if $\theta \in T$. Further, a simple argument from the periodicity of $\psi(\theta)$ shows that (3.1) must then be true for all θ. In particular, putting $\phi = \theta_n$, we get (i) with $K = K_1$.

From (3.1) it follows that $\psi'(\phi) = 0$ for all $\phi \in T$. For $\theta_n < \theta < \theta_n + \delta_n$ we have

$$\psi'(\theta) = 3c(\theta - \theta_n)^2(\theta_n + \delta_n - \theta)^2(2\theta_n + \delta_n - 2\theta).$$

We can now carry out an argument similar to that used above, with $\psi'(\theta)$ instead of $\psi(\theta)$, to find that

$$|\psi'(\theta)| \leqslant K_2 c|\theta - \phi|^2 \quad (\phi \in T, \text{ all } \theta),$$

where K_2 is an absolute constant. It follows easily that $\psi'(\theta)$ is everywhere continuous and satisfies $|\psi'(\theta)| \leqslant 4\pi^2 K_2 c$. Thus the part of (ii) which refers to $\psi'(\theta)$ is proved with $K = 4\pi^2 K_2$.

The required properties of $\psi''(\theta)$ are proved similarly. Lemma 1 then follows, for some sufficiently large absolute constant K.

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Let the domain D be the interior of the simple closed contour $r = \exp \psi(\theta)$ in the $z = re^{i\theta}$-plane. Then we have the following lemma.

LEMMA 2. *D contains $|z| < 1$ and properly contains the sequence of arcs* (2.1).

It is obvious that D contains $|z| < 1$. Let $d\ (\theta, h)$ denote the distance between $e^{i\theta}$ and $\exp\{\psi(\theta + h) + i(\theta + h)\}$. To complete the proof of Lemma 2 it is plainly enough to show that there is a number $A > 0$, independent of θ, h, and n, such that

$$d(\theta, h) > A(\theta - \theta_n)^3(\theta_n + \delta_n - \theta)^3 \tag{3.3}$$

for all θ and h satisfying

$$\theta_n < \theta < \theta_n + \delta_n, \quad \theta_n < \theta + h < \theta_n + \delta_n. \tag{3.4}$$

By periodicity we may suppose

$$|h| \leqslant \pi. \tag{3.5}$$

When (3.4) is true, we have

$$\{d(\theta, h)\}^2 = \{\exp \psi(\theta + h) - 1\}^2 + 4 \sin^2\tfrac{1}{2}h \exp \psi(\theta + h).$$

Thus by (3.5),

$$d(\theta, h) > 2|\sin \tfrac{1}{2} h| \geqslant 2\pi^{-1}|h|,$$

and if

$$|h| \geqslant (4\pi)^{-1}(\theta - \theta_n)(\theta_n + \delta_n - \theta), \tag{3.6}$$

then (3.3) follows from this with a suitable absolute constant A, say $A = A_0$.
On the other hand, if (3.6) is false, then by (3.2)

$$|h| < \tfrac{1}{2}(\theta - \theta_n), \quad |h| < \tfrac{1}{2}(\theta_n + \delta_n - \theta).$$

Therefore

$$\begin{aligned} d(\theta, h) &\geqslant \exp \psi(\theta + h) - 1 > \psi(\theta + h) \\ &\geqslant c(\theta - \theta_n - |h|)^3(\theta_n + \delta_n - \theta - |h|)^3 \\ &> 2^{-6}c(\theta - \theta_n)^3(\theta_n + \delta_n - \theta)^3, \end{aligned}$$

and this is (3.3) with $A = 2^{-6}c$. Thus in any case (3.3) is true with $A = \min(A_0, 2^{-6}c)$, and this proves Lemma 2.

4. We continue in the notation of §3. We now suppose that the constant c, occurring in the definition of $\psi(\theta)$, is chosen so small that

$$1 \leqslant \exp \psi(\theta) < \frac{81}{80}, \tag{4.1}$$

$$|\psi'(\theta)| < \frac{1}{80}, \tag{4.2}$$

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$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\psi'(\theta)-\psi'(\theta_0)}{\sin\frac{1}{2}(\theta-\theta_0)}\right| d\theta < \frac{1}{80}, \tag{4.3}$$

$$\exp\psi(\theta) < 2-r \qquad \text{for } |\theta-\theta_n| < 1-r \qquad (n = 1, 2, 3, \ldots), \tag{4.4}$$

the integral in (4.3) being uniformly convergent with respect to θ_0 for $0 \leqslant \theta_0 \leqslant 2\pi$. The possibility of such a choice of c is easily seen from Lemma 1.

Let $w = w(z)$ map D one-one and conformally onto $|w| < 1$, with $w(0) = 0$ and $w'(0) > 0$. We then have the following lemma, due to Specht (**5**, Theorem III).

LEMMA 3. (4.1), (4.2), *and* (4.3) *together imply that* $w'(z)$ *exists* for all* z *in the closure of* D *and satisfies*

$$|w'(z) - 1| < \tfrac{1}{2}. \tag{4.5}$$

Lemma 3 follows at once from Specht's result if, in that result, we interchange the z- and w-planes, write $\exp\psi(\theta)$ instead of $\rho(\theta)$, and put $\epsilon = \delta = 1/80$. It should be noted that Specht actually gives a bound for $|z'(w) - 1|$, where $z(w)$ is the inverse function of $w(z)$; but from this one may readily derive (4.5), which is more suitable for our purpose.

From Lemma 3 and (4.4) we now deduce

LEMMA 4. *For* $|\theta - \theta_n| < 1 - r$ *we have*

$$\tfrac{1}{2}(1-r) < 1 - |w(re^{i\theta})| < 3(1-r).$$

First let $z(w)$ be the inverse function of $w(z)$, as above. If $re^{i\theta} = z(\rho e^{i\varphi})$, then

$$1 - r \leqslant |z(e^{i\varphi})| - |z(\rho e^{i\varphi})| \leqslant |z(e^{i\varphi}) - z(\rho e^{i\varphi})|.$$

This is less than $2(1-\rho)$ by the first mean-value theorem, since $|z'(w)| < 2$ by (4.5). Thus for all θ,

$$1 - |w(re^{i\theta})| = 1 - \rho > \tfrac{1}{2}(1-r).$$

On the other hand, we have

$$\begin{aligned}1 - |w(re^{i\theta})| &= |w[\exp\{\psi(\theta) + i\theta\}]| - |w(re^{i\theta})|\\ &\leqslant |w[\exp\{\psi(\theta) + i\theta\}] - w(re^{i\theta})|\\ &< \frac{3}{2}\{\exp\psi(\theta) - r\},\end{aligned}$$

by the first mean-value theorem, since $|w'(z)| < 3/2$ by (4.5). Hence by (4.4), for $|\theta - \theta_n| < 1 - r$,

*If z is on the boundary of D, then by $w'(z)$ we mean $\lim\,\{w(t) - w(z)\}/(t - z)$ as $t \to z$ with t in the closure of D.

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$$1 - |w(re^{i\theta})| < \frac{3}{2}(2 - r - r) = 3(1 - r).$$

This proves Lemma 4.

5. In this section we make use of all the hypotheses of the theorem. Let D be as in §3, so that, in view of Lemma 2, the proof of the theorem will be complete when we have shown that a function $f(z)$ with the required properties exists.

Choose a sequence of integers $\{m_k\}$ such that, as $k \to \infty$,

$$\frac{\delta_{m_k}}{\delta_{m_k+1}} \to \infty, \tag{5.1}$$

$$\left(\sum_{m_k}^{\infty} \delta_j\right) \log \frac{1}{\delta_{m_k}} \to \infty. \tag{5.2}$$

Such a choice is possible by (2.4). Put

$$n_k = \left[\frac{1}{\delta_{m_k}}\right] \tag{5.3}$$

where the square brackets denote the integer part, and let

$$f(z) = \sum_{j=1}^{\infty} j^{-2}\{w(z)\}^{n_j}, \tag{5.4}$$

where $w(z)$ is as in §4. Since $w(z)$ is regular in D and $|w(z)| < 1$ there, it follows that $f(z)$ is regular and bounded in D.

Let

$$r_k = 1 - \delta_{m_k}, \tag{5.5}$$

and denote by E_k the union of the set of arcs

$$z = r_k e^{i\theta}, \quad |\theta - \theta_n| < \delta_{m_k} \qquad (n = 1, 2, 3, \ldots).$$

When $z \in E_k$ we have, by Lemma 4 and (5.5),

$$\tfrac{1}{2}\delta_{m_k} < 1 - |w(z)| < 3\delta_{m_k}.$$

Hence by (5.4), when $z \in E_k$,

$$\begin{aligned}\left|\frac{w(z)f'(z)}{w'(z)}\right| &= \left|\sum_{j=1}^{\infty} j^{-2}n_j\{w(z)\}^{n_j}\right| \\ &\geqslant k^{-2}n_k|w(z)|^{n_k} - \sum_{1}^{k-1} j^{-2}n_j|w(z)|^{n_j} - \sum_{k+1}^{\infty} j^{-2}n_j|w(z)|^{n_j} \\ &> k^{-2}n_k(1 - 3\,\delta_{m_k})^{n_k} - \sum_{1}^{k-1} j^{-2}n_j - \sum_{k+1}^{\infty} j^{-2}n_j(1 - \tfrac{1}{2}\,\delta_{m_k})^{n_k},\end{aligned}$$

that is,

$$\left|\frac{w(z)f'(z)}{w'(z)}\right| > k^{-2}n_k(1 - 3\,\delta_{m_k})^{n_k} - \textstyle\sum_1 - \sum_2, \tag{5.6}$$

say.

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By (5.3),

$$(5.7) \qquad (1 - 3\,\delta_{m_k})^{n_k} \to e^{-3} \qquad (k \to \infty).$$

Moreover, by (5.1) and (5.3),

$$(5.8) \qquad \sum\nolimits_1 = O(1)(k-1)^{-2}n_{k-1} = o(k^{-2}n_k).$$

Further, the simple inequality

$$(1-x)^n < (nx)^{-2},$$

which holds for $0 < x < 1$ and $n > 0$, gives

$$(1 - \tfrac{1}{2}\,\delta_{m_k})^{n_j} < 4\,n_j^{-2}\delta_{m_k}^{-2} = O(n_j^{-2}n_k^2),$$

by (5.3). Hence by (5.1) and (5.3),

$$\sum_2 = O(n_k^2)\sum_{k+1}^{\infty} j^{-2}n_j^{-1}$$
$$= O(k^{-2}n_k^2 n_{k+1}^{-1}),$$

and so

$$(5.9) \qquad \sum\nolimits_2 = o(k^{-2}n_k).$$

Combining (5.6), (5.7), (5.8), and (5.9), we get

$$\left|\frac{w(z)f'(z)}{w'(z)}\right| > 2\eta k^{-2}n_k,$$

for all large enough k, where $\eta > 0$ is a constant. But by (4.5) and the fact that $|w(z)| < 1$ we have $|w'(z)/w(z)| > \frac{1}{2}$, and so

$$(5.10) \qquad |f'(z)| > \eta k^{-2}n_k \qquad (z \in E_k)$$

for all large enough k.

By the monotonicity of $\{\delta_n\}$ and the definition of E_k, the union of the sequence of *non-overlapping* arcs

$$z = r_k e^{i\theta},\ \theta_n < \theta < \theta_n + \delta_n,\ n = m_k, m_k + 1, m_k + 2, \ldots,$$

is contained in E_k. Hence the angular measure of E_k is at least

$$\sum_{m_k}^{\infty} \delta_j.$$

But

$$m(r_k, f') \geqslant \frac{1}{2\pi}\int \log^+|\ldots|f'(r_k e^{i\theta})|\,d\theta,$$

the integral being taken over the set of θ in $(-\pi, \pi)$ such that $r_k e^{i\theta} \in E_k$. Hence by (5.10),

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$$m(r_k, f') > \frac{1}{2\pi}\left(\sum_{j=m_k}^{\infty} \delta_j\right) \log (\eta k^{-2} n_k),$$

for all large enough k. It is easy to deduce from this, with (5.1), (5.2), and (5.3), that $m(r_k, f') \to \infty$ as $k \to \infty$. Since $m(r, f')$ is an increasing function of r (**4**, p. 8), it follows that $m(r, f') \to \infty$ as $r \to 1$. This proves the theorem.

6. It is easy to see from Theorem III of (**2**) that the condition (2.2) is essential for the truth of Hayman's theorem, and that some "order-of-contact condition," of the kind occurring in the definition of "properly contains," is also necessary. It is possible to construct examples giving more complete information about these hypotheses, but this does not seem worthwhile.

Professor Hayman has sent me the following remark. "As Professor A. A. Goldberg has kindly pointed out to me, Theorem III of (**1**) is in fact contained in a more general result of R. Nevanlinna (*Acta Soc. Sci. Fennicae*, *50*, no. 12 (1925)), and this paper is also in other ways related to (**1**)."

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