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# LIMITS OF PURE STATES

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## 1. Introduction

In [7, Section 5], Glimm showed that if $\phi$ and $\psi$ are inequivalent pure states of a liminal $C^*$-algebra $A$ such that the Gelfand–Naimark–Segal (GNS) representations $\pi_\phi$ and $\pi_\psi$ cannot be separated by disjoint open subsets of the spectrum $\hat{A}$ then $\frac{1}{2}(\phi+\psi)$ is a weak*-limit of pure states. We extend this to arbitrary $C^*$-algebras (and more general convex combinations) by means of what we hope will be regarded as a transparent proof based on the notion of transition probabilities. As an application, we show that if $J$ is a proper primal ideal of a separable $C^*$-algebra $A$ then there exists a state $\phi$ in $\overline{P(A)}$ (the pure state space) such that $J=\ker\pi_\phi$ (Theorem 3). The significance of this is discussed below after the introduction of further notation and terminology.

The state space $S(A)$ is defined by

$$S(A)=\{\phi\in A^*: \phi\geqq 0,\ \|\phi\|=1\}.$$

The set $P(A)$ of pure states consists of the extreme points of $S(A)$ and the set $F(A)$ of factorial states consists of those $\phi\in S(A)$ such that the von Neumann algebra generated by $\pi_\phi(A)$ is a factor. Unless stated otherwise, $A^*$ should be regarded as being endowed with the weak*-topology. The closures $\overline{P(A)}$ and $\overline{F(A)}$ are known respectively as the pure and factorial state spaces of $A$.

Following [2, Definition 3.1], we say that a (closed two-sided) ideal $J$ of $A$ is primal if whenever $n\geqq 2$ and $J_1, J_2, \ldots, J_n$ are ideals of $A$ such that $J_1J_2\ldots J_n=\{0\}$ then $J_i\subseteq J$ for at least one value of $i$. It is shown in [2, Theorem 3.5] that if $\phi\in S(A)$ then $\phi\in\overline{F(A)}$ if and only if $\ker\pi_\phi$ is primal. Thus there is a mapping $\theta$ from $\overline{F(A)}\cap S(A)$ into the set of proper primal ideals of $A$ given by $\theta(\phi)=\ker\pi_\phi$ The mapping $\theta$ is continuous and open relative to its image [1, Theorem 3.6], and is surjective if $A$ is separable (see [2, p. 62]). Theorem 3 tells us that, if $A$ is separable, the restriction of $\theta$ to $\overline{P(A)}\cap S(A)$ is still surjective.

We note here two related questions to which we have been able to find affirmative answers only in certain special cases.

**Question 1.** Does $\theta(\overline{F(A)}\cap S(A))=\theta(\overline{P(A)}\cap S(A))$ for all $C^*$-algebras $A$?

**Question 2.** Is the restriction of $\theta$ to $\overline{P(A)} \cap S(A)$ an open mapping relative to its image?

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## 2. The generalization of Glimm's result

Let $A$ be a $C^*$-algebra and suppose that pure states $\phi$ and $\psi$ have support projections $p$ and $q$ in $A^{**}$. As in [**4, 12**], we define the transition probability $\langle \phi, \psi \rangle$ between $\phi$ and $\psi$ by

$$\langle \phi, \psi \rangle = \phi(q) = \psi(p).$$

If $\phi$ and $\psi$ are inequivalent then $\langle \phi, \psi \rangle = 0$. On the other hand, if $\phi$ and $\psi$ are equivalent there exists an irreducible representation $\pi$ of $A$ and unit vectors $\xi$ and $\eta$ in the Hilbert space for $\pi$ such that

$$\phi(a) = \langle \pi(a)\xi, \xi \rangle \text{ and } \psi(a) = \langle \pi(a)\eta, \eta \rangle$$

for all $a \in A$. In this case $\langle \phi, \psi \rangle = |\langle \xi, \eta \rangle|^2$.

The following result is noted in [**4**, Remark 2] as a simple consequence of the equality

$$\|\phi - \psi\| = 2(1 - \langle \phi, \psi \rangle)^{1/2}$$

(see [**8**, Corollary 9], [**9**, p. 146] and [**11**, Lemma 2.4]).

**Lemma 1.** *Let $A$ be a $C^*$-algebra, let $T: P(A) \times P(A) \to [0, 1]$ be defined by*

$$T(\phi, \psi) = \langle \phi, \psi \rangle \quad ((\phi, \psi) \in P(A) \times P(A)),$$

*and let $P(A) \times P(A)$ be endowed with the product weak*-topology. Then $T$ is continuous at $(\phi, \psi)$ whenever $\langle \phi, \psi \rangle = 0$.*

The next lemma is the key to the generalization of Glimm's result.

**Lemma 2.** *Let $\phi$ and $\psi$ be inequivalent pure states of a $C^*$-algebra $A$. Suppose that there exists a net $(\pi_\alpha)$ of irreducible representations of $A$ and, for each $\alpha$, unit vectors $\xi_\alpha$ and $\eta_\alpha$ such that*

$$\langle \pi_\alpha(\cdot)\xi_\alpha, \xi_\alpha \rangle \to \phi \text{ and } \langle \pi_\alpha(\cdot)\eta_\alpha, \eta_\alpha \rangle \to \psi.$$

*Then* $\langle \xi_\alpha, \eta_\alpha \rangle \to 0$ *and* $\langle \pi_\alpha(a)\xi_\alpha, \eta_\alpha \rangle \to 0$ *for all* $a \in A$.

**Proof.** Let $\tilde{A}$ be the $C^*$-algebra generated by $A$ and 1 in $A^{**}$ and let $\tilde{\pi}_\alpha$ be the canonical extension of $\pi_\alpha$ to $\tilde{A}$. Let $u$ be a unitary element of $\tilde{A}$. By linearity, it suffices to show that $\langle \tilde{\pi}_\alpha(u)\xi_\alpha, \eta_\alpha \rangle \to 0$.

Writing $v_\alpha = \tilde{\pi}_\alpha(u)$, we have

$$\langle \pi_\alpha(a) v_\alpha \xi_\alpha, v_\alpha \xi_\alpha \rangle \to \phi(u^* a u)$$

for all $a \in A$. Since $\psi$ and $\phi(u^* \cdot u)$ are inequivalent we have $\langle \phi(u^* \cdot u), \psi \rangle = 0$ and hence by Lemma 1

$$|\langle v_\alpha \xi_\alpha, \eta_\alpha \rangle|^2 \to 0.$$

Glimm's proof of the above lemma in the liminal case uses functional calculus to convert a compact operator into one of finite rank. This leads to an element of $A$ which exhibits the difference between $\ker \pi_\phi$ and $\ker \pi_\psi$. In general, however, these kernels might coincide. Nevertheless, it is in fact possible to modify Glimm's argument as follows (although, for conceptual reasons, we prefer the proof given above). Let $\xi$ and $\eta$ be the GNS vectors for $\phi$ and $\psi$ respectively. Given $b \in A$ there exists, by Kadison's transitivity theorem, $a = a^* \in A$ such that $\pi_\psi(a)\pi_\psi(b)\eta = 0$ and $\pi_\phi(a)\xi = \xi$. By functional calculus we may assume $0 \leqq a \leqq 1$. Then we may use Glimm's calculations in [7, pp. 603–604] to show that $\langle \pi_\alpha(b)\eta_\alpha, \xi_\alpha \rangle \to 0$ and $\langle \xi_\alpha, \eta_\alpha \rangle \to 0$.

**Theorem 1.** *Let $A$ be a $C^*$-algebra and let $\phi_1, \phi_2, \ldots, \phi_n$ be pure states of $A$ which are pairwise inequivalent. The following conditions are equivalent.*

(i) *There exist positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ with unit sum such that*

$$\sum_{i=1}^{n} \lambda_i \phi_i \in \overline{P(A)}.$$

(ii) *There exists a net $(\pi_\alpha)$ in $\hat{A}$ such that $\pi_\alpha \to \pi_{\phi_i}$ for each $i$ $(1 \leqq i \leqq n)$.*

(iii) *Whenever $\mu_1, \ldots, \mu_n$ are non-negative real numbers with unit sum,*

$$\sum_{i=1}^{n} \mu_i \phi_i \in \overline{P(A)}.$$

**Proof.** (i)$\Rightarrow$(ii). Let $\phi = \sum_{i=1}^{n} \lambda_i \phi_i$ *and let* $(\phi_\alpha)$ be a net in $P(A)$ *such that* $\phi_\alpha \to \phi$. Fix $i \in \{1, 2, \ldots, n\}$ and let $J$ be an ideal of $A$ such that $\pi_{\phi_i}(J) \neq \{0\}$. Then $\phi_i(J) \neq \{0\}$[6, 2.4.9] and hence $\phi(J \neq \{0\}$. Thus there exists $\alpha_0$ such that $\phi_\alpha(J) \neq \{0\}$ for all $\alpha \geqq \alpha_0$. Hence $\pi_{\phi_\alpha}(J) \neq \{0\}$ for all $\alpha \geqq \alpha_0$. This shows that $\pi_{\phi_\alpha} \to \pi_{\phi_i}$ (see [6, Section 3]).

(ii)$\Rightarrow$(iii). Having Lemma 2 at our disposal, we can proceed with a generalization of

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Glimm's method as follows. Let $\mathscr{N}$ be a base of weak*-open neighbourhoods of zero in $A^*$. Let $N \in \mathscr{N}$. The canonical image of $(\phi_i + N) \cap P(A)$ in $\hat{A}$ is an open neighbourhood $V_i$ of $\pi_{\phi_i}$ $(1 \leqq i \leqq n)$ [6, 3.4.11]. By (ii), there exists an irreducible representation $\pi_N$ of $A$ such that $\pi_N \in V_i$ for all $i$ $(1 \leqq i \leqq n)$. Hence there exist unit vectors $\xi_1^{(N)}, \xi_2^{(N)}, \ldots, \xi_n^{(N)}$ in the Hilbert space for $\pi_N$ such that

$$\phi_i^{(N)} \in \phi_i + N \ (1 \leqq i \leqq n),$$

where

$$\phi_i^{(N)}(a) = \langle \pi_N(a)\xi_i^{(N)}, \xi_i^{(N)} \rangle \ (a \in A).$$

For each $i$, the net $(\phi_i^{(N)})$, indexed by $\mathscr{N}$ in the obvious way, is convergent to $\phi_i$. It follows from Lemma 2 that if $i \neq j$ and $a \in A$ then

$$\langle \xi_i^{(N)}, \xi_j^{(N)} \rangle \to 0 \text{ and } \langle \pi_N(a)\xi_i^{(N)}, \xi_j^{(N)} \rangle \to 0. \tag{1}$$

Let $\xi^{(N)} = \sum_{i=1}^n \mu_i^{1/2} \xi_i^{(N)}$. By (1), $\|\xi^{(N)}\| \to 1$. Eventually $\|\xi^{(N)}\| \neq 0$ and then we may form $\eta^{(N)} = \|\xi^{(N)}\|^{-1} \xi^{(N)}$. We define $\psi_N \in P(A)$ by

$$\psi_N(a) = \langle \pi_N(a)\eta^{(N)}, \eta^{(N)} \rangle \ (a \in A).$$

Using (1) again, it is routine to check that

$$\psi_N \to \sum_{i=1}^n \mu_i \phi_i.$$

(iii)⇒(i). This is immediate.

The continuity of the canonical map from $P(A)$ to $\hat{A}$ ensures that the above net $(\pi_N)$ converges to $\pi_{\phi_i}$ for each $i$. However, $(\pi_N)$ might not be a subnet of $(\pi_\alpha)$. If in other circumstances a subnet is required, one may index with pairs $(N, \alpha)$, choosing $\pi_{N,\alpha}$ to be some $\pi_\beta$ where $\beta \geqq \alpha$ and $\pi_\beta \in V_i$ for all $i$.

The fact that (i) implies (ii) in Theorem 1 is also valid without the assumption that the $\phi_i$'s are pairwise inequivalent.

**Corollary.** *Let $\phi$ and $\psi$ be inequivalent pure states of a $C^*$-algebra $A$. The following conditions are equivalent.*

(i) $\frac{1}{2}(\phi + \psi) \in \overline{P(A)}$.

(ii) *$\pi_\phi$ and $\pi_\psi$ cannot be separated by disjoint open subsets of $\hat{A}$.*

We note that the above Corollary closes gaps in the proofs of [3, Theorem 2.8, (4)⇒(5)] and [5, Theorem 3.5, (iii)⇒(i)] which arose from overlooking the role of liminality in the construction in [7].

The next result is a version of Theorem 1 for infinite convex combinations.

**Theorem 2.** *Let $A$ be a $C^*$-algebra and let $(\phi_i)_{i\geqq 1}$ be a sequence of pairwise inequivalent pure states of $A$. The following conditions are equivalent.*

(i) *There exists a sequence $(\lambda_i)_{i\geqq 1}$ of positive real numbers such that $\sum_{i=1}^{\infty}\lambda_i=1$ and the norm-convergent sum $\sum_{i=1}^{\infty}\lambda_i\phi_i$ lies in $\overline{P(A)}$.*

(ii) *There exists a net $(\pi_\alpha)$ in $\hat{A}$ such that $\pi_\alpha\to\pi_{\phi_i}$ for all $i$.*

(iii) *Whenever $(\mu_i)_{i\geqq 1}$ is a sequence of non-negative real numbers such that $\sum_{i=1}^{\infty}\mu_i=1$, $\sum_{i=1}^{\infty}\mu_i\phi_i\in\overline{P(A)}$.*

**Proof.** (i)$\Rightarrow$(iii). This is proved as in Theorem 1. (ii)$\Rightarrow$(iii). By truncating and scaling, we see that $\sum_{i=1}^{\infty}\mu_i\phi_i$ is the norm-limit of a sequence of finite convex combinations of the $\phi_i$'s, each of which lies in $\overline{P(A)}$ by Theorem 1. Since $\overline{P(A)}$ is norm-closed, it contains $\sum_{i=1}^{\infty}\mu_i\phi_i$. (iii)$\Rightarrow$(i). This is immediate.

## 3. An application

**Theorem 3.** *Let $J$ be a proper primal ideal of a separable $C^*$-algebra $A$. Then there exists $\phi\in\overline{P(A)}\cap S(A)$ such that $J=\ker\pi_\phi$.*

**Proof.** Since $A/J$ is separable, its primitive ideal space is separable [**10**, 4.3.4]. Thus there is a countable set $S$ of distinct primitive ideals of $A$ with intersection equal to $J$. Since $J$ is primal, it follows from [**2**, Proposition 3.2] that there exists a net $(P_\alpha)$ of primitive ideals of $A$ such that $P_\alpha\to P$ for all $P\in S$.

Suppose that $S$ is infinite, say $S=\{P_1,P_2,\ldots,\}$. Let $\phi_i$ be a pure state of $A$ such that $P_i=\ker\pi_{\phi_i}$ for each $i$. Since $P_i\neq P_j$, $\phi_i$ and $\phi_j$ are inequivalent for $i\neq j$. Let $\pi_\alpha$ be an irreducible representation of $A$ such that $P_\alpha=\ker\pi_\alpha$ for each $\alpha$. Then $\pi_\alpha\to\pi_{\phi_i}$ for each $i$ and so

$$\phi=\sum_{i=1}^{\infty}2^{-i}\phi_i\in\overline{P(A)}$$

by Theorem 2. Moreover,

$$\ker\pi_\phi=\bigcap_{i=1}^{\infty}P_i=J.$$

On the other hand, suppose that $S$ is finite, say $S=\{Q_1,Q_2,\ldots,Q_n\}$ for some $n\geqq 1$. Taking $\psi=(1/n)\sum_{i=1}^{n}\psi_i$, where $\psi_i\in P(A)$ and $\ker\pi_{\psi_i}=Q_i$, we have that $\ker\pi_\psi=J$ and $\psi\in\overline{P(A)}$ by Theorem 1.

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