

# C\*-Convexity and the Numerical Range

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*Abstract.* If  $A$  is a prime C\*-algebra,  $a \in A$  and  $\lambda$  is in the numerical range  $W(a)$  of  $a$ , then for each  $\varepsilon > 0$  there exists an element  $h \in A$  such that  $\|h\| = 1$  and  $\|h^*(a - \lambda)h\| < \varepsilon$ . If  $\lambda$  is an extreme point of  $W(a)$ , the same conclusion holds without the assumption that  $A$  is prime. Given any element  $a$  in a von Neumann algebra (or in a general C\*-algebra)  $A$ , all normal elements in the weak\* closure (the norm closure, respectively) of the C\*-convex hull of  $a$  are characterized.

## 1 Introduction and Basic Definitions

By the well known Dixmier approximation theorem (see [7] or [17]) the norm closure  $\overline{\text{co}}_{U(R)}(a)$  of the convex hull of the unitary orbit of any element  $a$  in a von Neumann algebra  $R$  intersects the center  $Z$  of  $R$ ; moreover in many cases this intersection can be described precisely (see [25], [15]). In this paper we shall study the weak\* closure and the norm closure of an analogous set, the C\*-convex hull of  $a$ , denoted by  $\text{co}_R(a)$  and called more precisely the  $R$ -convex hull of  $a$ . By definition  $\text{co}_R(a)$  consists of all elements of the form

$$\sum_{j=1}^n v_j^* a v_j,$$

where  $n \in \mathbb{N}$ ,  $v_j \in R$  and  $\sum_{j=1}^n v_j^* v_j = 1$ . Let us call a subset  $S$  of a unital C\*-algebra  $A$   $A$ -convex (or C\*-convex if  $A$  is clear from the context) if  $\sum_{j=1}^n v_j^* x_j v_j \in S$  whenever  $x_j \in S$  and  $v_j \in A$  for all  $j$  and  $\sum_{j=1}^n v_j^* v_j = 1$ . (Such sets were studied explicitly in [18] and they appear frequently in the theory of operator spaces and algebras.) Then clearly  $\text{co}_R(a)$  is the smallest  $R$ -convex subset of  $R$  containing  $a$ , and  $\text{co}_R(a) \supseteq \text{co}_{U(R)}(a)$ .

If  $p$  is a central projection in  $R$  and  $x \in Rp$  let  $W_{Rp}(x)$  be the (algebraic) numerical range of  $x$ , that is, the set of all  $\rho(x)$ , where  $\rho$  is a state on  $Rp$  [4]. For a normal element  $b \in R$  we shall show that  $b$  is contained in the weak\* closure  $\overline{\text{co}}_R(a)$  of  $\text{co}_R(a)$  if and only if  $W_{Rp}(bp) \subseteq W_{Rp}(ap)$  for each central projection  $p \in R$ . (Since central elements are normal, this will also describe the intersection of  $\overline{\text{co}}_R(a)$  with the center of  $R$ .) As a consequence, given a unital C\*-algebra  $A$  and elements  $a, b \in A$  with  $b$  normal, we shall deduce that  $b$  is in the norm closure  $\overline{\text{co}}_A(a)$  of  $\text{co}_A(a)$  if and only if  $W(b+P) \subseteq W(a+P)$  for each primitive ideal  $P$  of  $A$ , where  $a+P$  denotes the coset of  $a$  in  $A/P$  and  $W(a+P)$  denotes the numerical range; when the algebra is clear from the context we write  $W(\cdot)$  instead of  $W_A(\cdot)$  for the numerical range. (We remark that the norm and the weak\* closure of  $\text{co}_{U(R)}(a)$  for a normal element  $a \in R$  have been studied in [16], and some results in [20] can be interpreted as a description of the C\*-analogy of the closed absolutely convex hull of a general element  $a \in R$ .)

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Let  $K$  be an  $A$ -convex subset of a unital  $C^*$ -algebra  $A$ . By definition (introduced in [18]) a point  $x \in K$  is  $C^*$ -extreme (or, more precisely,  $A$ -extreme) for  $K$  if the condition

$$x = \sum_{j=1}^n a_j^* x_j a_j, \quad \sum_{j=1}^n a_j^* a_j = 1, \quad x_j \in K, \quad a_j \text{ invertible in } A, \quad n \in \mathbb{N}$$

implies that all  $x_j$  are unitarily equivalent to  $x$  in  $A$ . As an immediate consequence of our results we shall deduce that for each weak\* compact  $R$ -convex set  $K$  in a von Neumann algebra  $R$  each extreme point of  $K \cap Z$  (where  $Z$  is the center of  $R$ ) is  $C^*$ -extreme in  $K$ . The existence of  $C^*$ -extreme points for compact  $C^*$ -convex subsets of finite dimensional algebras  $\mathcal{M}_n(\mathbb{C})$  was proved by Farenick [9] and the appropriate variant of the Krein-Milman theorem for  $C^*$ -convex subsets in matrix algebras was proved by Morenz [21]. After the first version of the present paper has been already submitted we have found (on the basis of some results presented here) a proof of a variant of the Krein-Milman theorem for  $C^*$ -convex weak\* compact subsets of injective factors, but for more general  $C^*$ -algebras the problem seems to be still open. If  $K$  is a weak\* compact  $R$ -convex subset of a factor  $R$ , then each extreme point of  $K \cap \mathbb{C}$  will turn out to be  $C^*$ -extreme in  $K$ , but  $\overline{\text{co}}_R(K \cap \mathbb{C})$  consists only of elements that have normal dilations (relative to  $R$ ) with the spectrum in  $K \cap \mathbb{C}$  (Proposition 3.7).

The key to the results here is a simple internal characterization of numerical ranges in  $C^*$ -algebras: if  $A$  is a prime  $C^*$ -algebra (= without ideal divisors of 0),  $a \in A$  and  $\lambda \in W(a)$ , or if  $A$  is an arbitrary  $C^*$ -algebra but  $\lambda$  is an extreme point of  $W(a)$ , then for each  $\varepsilon > 0$  there exists a positive  $h \in A$  such that  $\|h\| = 1$  and  $\|h(a - \lambda)h\| \leq \varepsilon$ . (Conversely, the existence of such an element  $h$  clearly implies that  $\lambda \in W(a)$ .) This result partially generalizes the well known characterization of the numerical range of elements in the Calkin algebra [10], where  $\lambda \in W(a)$  if and only if  $p(a - \lambda)p = 0$  for some non-zero projection  $p$ , but of course in a general prime  $C^*$ -algebra we can not expect that the above  $h$  is a projection or that  $\varepsilon = 0$ .

A classical way of proving that some well known examples of unital  $C^*$ -algebras  $A$  are simple (see [6]) is to consider for  $a \in A$  the intersection  $\overline{\text{co}}_{U(A)}(a) \cap \mathbb{C}$ . By [12] a simple unital  $C^*$ -algebra  $A$  has the property that  $\overline{\text{co}}_{U(A)}(a) \cap \mathbb{C} \neq \emptyset$  for each  $a \in A$  if and only if  $A$  has at most one tracial state. (The author is grateful to E. Kirchberg for bringing the article [12] to his attention.) On the other hand, we shall show here that  $\overline{\text{co}}_A(a) \cap \mathbb{C} \neq \emptyset$  for all simple unital  $C^*$ -algebras  $A$ .

## 2 A Characterization of the Numerical Range of Elements in $C^*$ -Algebras

In this section we shall obtain a characterization of the numerical range (Theorem 2.3 below) needed later, but first we state two general lemmas.

**Lemma 2.1** *Let  $A \subseteq B(\mathcal{H})$  be an irreducible  $C^*$ -algebra,  $a = a^* \in A$ ,  $\varepsilon > 0$  and  $\xi \in \mathcal{H}$  a unit vector such that  $|\langle a\xi, \xi \rangle| < \varepsilon$ . Then there exists a positive element  $h \in A$  such that  $\|h\| = 1$ ,  $\|hah\| < \varepsilon$  and  $\|h\xi - \xi\| < \varepsilon$ .*

**Proof** Choose a positive  $\delta < 2^{-1}(1 + \varepsilon)^{-1}(\varepsilon - |\langle a\xi, \xi \rangle|)$ , let  $[\alpha_0, \alpha] = W(a)$  (= the smallest interval containing the spectrum of  $a$ ) and for each subset  $\sigma \subset \mathbb{R}$  let  $e(\sigma)$  be the

spectral projection of  $a$  corresponding to  $\sigma$ . (If  $\sigma = \{\lambda\}$  is a singleton, we shall write  $e(\lambda)$  instead of  $e(\{\lambda\})$ .) Since  $e(\lambda)\xi$  is non-zero for at most countably many values of  $\lambda \in \mathbb{R}$ , we can choose a partition  $\alpha_0 < \alpha_1 < \dots < \alpha_n = \alpha$  of the interval  $[\alpha_0, \alpha]$  such that  $e(\alpha_j)\xi = 0$  for all  $j = 1, \dots, n - 1$  and  $|\alpha_j - \alpha_{j-1}| < \delta$  for all  $j = 1, \dots, n$ . Put  $e_j = e((\alpha_{j-1}, \alpha_j))$  for  $j = 2, \dots, n - 1$  and  $e_1 = e([\alpha_0, \alpha_1))$ ,  $e_n = e((\alpha_{n-1}, \alpha_n])$ . Let  $\lambda_j = \|e_j\xi\|$  ( $j = 1, \dots, n$ ),  $\mathbb{J} = \{j : \lambda_j \neq 0\}$  and  $\xi_j = \lambda_j^{-1}e_j\xi$  for  $j \in \mathbb{J}$ . Then  $\|\xi_j\| = 1$  and  $\sum_{j \in \mathbb{J}} \lambda_j^2 = \|\xi\|^2 = 1$ . Let any  $j_1 \in \mathbb{J}$  be fixed. By the Kadison transitivity theorem for each  $j \in \mathbb{J}$  there exists an  $s_j \in A$  such that  $s_j\xi_{j_1} = \xi_j$  and  $\|s_j\| = 1$ , which implies that  $s_j^*\xi_j = \xi_{j_1}$  (since  $\|s_j^*\xi_j - \xi_{j_1}\|^2 = \|(s_j^*s_j - 1)\xi_{j_1}\|^2 \leq \|s_j\xi_{j_1}\|^2 - 2\langle s_j^*s_j\xi_{j_1}, \xi_{j_1} \rangle + 1 = 0$ ). Since the characteristic function  $\chi$  of an open subset of the spectrum can be approximated pointwise by a sequence of positive continuous functions  $\phi_k \leq \chi$ , it follows easily by the spectral theorem that for each  $j \in \mathbb{J}$  there exists an element  $g_j \in A$  such that  $0 \leq g_j \leq e_j$  and

$$(2.1) \quad \|g_j\xi_j - \xi_j\| < \frac{\delta}{4\sqrt{n}}.$$

Put

$$v_j = g_j s_j g_{j_1} \quad (j \in \mathbb{J}) \quad \text{and} \quad v = \sum_{j \in \mathbb{J}} \lambda_j v_j.$$

Then

$$\|v\|^2 \leq \sum_{j \in \mathbb{J}} \lambda_j^2 \left\| \sum_{j \in \mathbb{J}} v_j v_j^* \right\| \leq \max_{j \in \mathbb{J}} \|g_j s_j g_{j_1}^2 s_j^* g_j\| \leq 1.$$

Further, denoting  $\bar{e}_j = e((\alpha_{j-1}, \alpha_j])$  for  $j = 2, \dots, n$  and  $\bar{e}_1 = e([\alpha_0, \alpha_1])$ , we have

$$\begin{aligned} \|v^*av\| &\leq \left\| a - \sum_{j=1}^n \alpha_j \bar{e}_j \right\| + \left\| v^* \left( \sum_{j=1}^n \alpha_j \bar{e}_j \right) v \right\| \\ &< \delta + \left\| \sum_{j=1}^n \sum_{i,k \in \mathbb{J}} \alpha_j \lambda_i \lambda_k v_i^* \bar{e}_j v_k \right\| \\ &= \delta + \left\| \sum_{j \in \mathbb{J}} \alpha_j \lambda_j^2 g_{j_1} s_j^* g_{j_1}^2 s_j g_{j_1} \right\| \quad (\text{since } 0 \leq g_j \leq e_j) \\ &\leq \delta + \left| \sum_{j \in \mathbb{J}} \alpha_j \lambda_j^2 \right| \quad (\text{since } 0 \leq g_{j_1} s_j^* g_{j_1}^2 s_j g_{j_1} \leq 1) \\ &= \delta + \left| \left\langle \sum_{j=1}^n \alpha_j \bar{e}_j \xi, \xi \right\rangle \right| \\ &\leq \delta + \left\| a - \sum_{j=1}^n \alpha_j \bar{e}_j \right\| + |\langle a\xi, \xi \rangle|, \end{aligned}$$

thus

$$(2.2) \quad \|v^*av\| < 2\delta + |\langle a\xi, \xi \rangle|.$$

Observe also that

$$\begin{aligned} \|v\xi_{j_1} - \xi\| &= \left\| \sum_{j \in J} \lambda_j (v_j \xi_{j_1} - \xi_j) \right\| \\ &= \left\| \sum_{j \in J} \lambda_j [g_j s_j (g_{j_1} \xi_{j_1} - \xi_{j_1}) + (g_j \xi_j - \xi_j)] \right\| \\ &\leq \left( \sum_{j \in J} \|g_j s_j (g_{j_1} \xi_{j_1} - \xi_{j_1}) + (g_j \xi_j - \xi_j)\|^2 \right)^{1/2} \\ &< \frac{\delta}{2} \end{aligned}$$

by (2.1), and

$$\begin{aligned} \|v^* \xi - \xi_{j_1}\| &= \left\| \sum_{i, j \in J} \lambda_i \lambda_j v_i^* \xi_j - \xi_{j_1} \right\| \\ &= \left\| \sum_{j \in J} \lambda_j^2 (g_{j_1} s_j^* g_j \xi_j - \xi_{j_1}) \right\| \quad (\text{recall that } \sum_{j \in J} \lambda_j^2 = 1) \\ &= \left\| \sum_{j \in J} \lambda_j^2 [g_{j_1} s_j^* (g_j \xi_j - \xi_j) + (g_{j_1} \xi_{j_1} - \xi_{j_1})] \right\| \\ &< \frac{\delta}{2\sqrt{n}} \sum_{j \in J} \lambda_j^2 \quad (\text{by (2.1)}) \\ &\leq \frac{\delta}{2}. \end{aligned}$$

Thus, we have  $1 - \frac{\delta}{2} < \|v\| \leq 1$  and  $\|vv^* \xi - \xi\| < \delta$ . Finally, put  $h = \|v\|^{-2} vv^*$ . Then  $\|h\xi - \xi\| < 2\delta < \varepsilon$  and (using (2.2) and the choice of  $\delta$ )

$$\|hah\| = \frac{1}{\|v\|^4} \|vv^* avv^*\| \leq \frac{1}{\|v\|^2} \|v^* av\| < \frac{2\delta + |\langle a\xi, \xi \rangle|}{(1 - \frac{\delta}{2})^2} < \varepsilon. \quad \blacksquare$$

**Lemma 2.2** *Let  $A$  be a unital  $C^*$ -algebra,  $a_1, \dots, a_m$  elements of  $A$  and  $\rho$  a state on  $A$  in the weak\* closure of the pure states. Then for each  $\varepsilon > 0$  there exists an element  $h \in A$  such that  $\|h\| = 1$  and  $\|h^*(a_i - \rho(a_i))h\| < \varepsilon$  for  $i = 1, \dots, m$ .*

**Proof** Clearly we may assume that all  $a_i$  are self-adjoint (otherwise consider the set of all real and imaginary parts of  $a_i$ 's) and (by translation) that  $\rho(a_i) = 0$  for all  $i$ . Then by the hypothesis there exists a pure state  $\omega$  on  $A$  such that  $|\omega(a_i)| < \delta$  for all  $i$ , where  $\delta = 8^{-m}\varepsilon$ . Let  $\pi: A \rightarrow B(\mathcal{H})$  be the representation constructed from  $\omega$  by the GNS construction and let  $\xi \in \mathcal{H}$  be the corresponding unit cyclic vector. Put  $b_i = \pi(a_i)$  and note that

$$|\langle b_i \xi, \xi \rangle| < \delta \quad (i = 1, \dots, m).$$

Observe that it suffices to construct an element  $w \in \pi(A) \cong A/\ker \pi$  such that  $\|w\| = 1$  and  $\|w^*b_iw\| < \varepsilon$ . Namely, then  $w$  can be lifted to an element  $s \in A$  with  $\|s\| = 1$  and, if  $\{e_k\}_k$  is an approximate unit in  $\ker \pi$  (with  $0 \leq e_k \leq 1$ ), then from the well known identity  $\|\pi(x)\| = \inf_k \|(1 - e_k)x(1 - e_k)\|$  ( $x \in A$ ) we have that  $\|w^*b_iw\| = \inf_k \|(1 - e_k)s^*a_i s(1 - e_k)\|$ , which implies that for an appropriate  $k$  the element  $h := s(1 - e_k)$  satisfies  $\|h^*a_ih\| < \varepsilon$  for all  $i = 1, \dots, m$  and clearly  $\|h\| = 1$  (since  $\|h\| \leq \|s\| = 1$  and  $\|h\| \geq \|\pi(h)\| = \|w\| = 1$ ).

The required element  $w \in \pi(A)$  can be constructed by an induction on  $m$ , using Lemma 2.1. We may assume that  $\varepsilon < 1$  and  $\|a_i\| \leq 1$  for all  $i = 1, \dots, m$ . Suppose inductively that for some  $k \in \{1, \dots, m - 1\}$  we already have found an element  $u \in \pi(A)$  such that

$$(2.3) \quad \|u\| = 1, \quad \|u\xi - \xi\| < 8^{k-1}\delta \quad \text{and} \quad \|u^*b_iu\| < 8^{k-1}\delta \quad (i = 1, \dots, k).$$

Since

$$\begin{aligned} |\langle u^*b_{k+1}u\xi, \xi \rangle| &\leq |\langle b_{k+1}\xi, \xi \rangle| + |\langle b_{k+1}(u\xi - \xi), \xi \rangle| + |\langle b_{k+1}u\xi, u\xi - \xi \rangle| \\ &< \delta + 2\|u\xi - \xi\| \\ &< 3 \cdot 8^{k-1}\delta \end{aligned}$$

by (2.3), by Lemma 2.1 there exists an element  $v \in \pi(A)$  such that

$$\|v\| = 1, \quad \|v\xi - \xi\| < 3 \cdot 8^{k-1}\delta \quad \text{and} \quad \|v^*(u^*b_{k+1}u)v\| < 3 \cdot 8^{k-1}\delta.$$

Then  $\|uv\| \leq 1$ ,  $\|v^*u^*b_iuv\| < 3 \cdot 8^{k-1}\delta$  for  $i = 1, \dots, k + 1$  and

$$\|uv\xi - \xi\| \leq \|u\xi - \xi\| + \|u(v\xi - \xi)\| < 4 \cdot 8^{k-1}\delta.$$

The last estimate implies in particular that  $\|uv\| > 1 - 4 \cdot 8^{k-1}\delta$ . Put  $w = \|uv\|^{-1}uv$ . Then  $\|w\| = 1$ ,  $\|w\xi - \xi\| \leq \|w\xi - uv\xi\| + \|uv\xi - \xi\| < 8^k\delta$  and

$$\|w^*b_iw\| = \frac{1}{\|uv\|^2} \|v^*u^*b_iuv\| < \frac{3 \cdot 8^{k-1}\delta}{(1 - 4 \cdot 8^{k-1}\delta)^2} < 8^k\delta$$

for  $i = 1, \dots, k + 1$ , since  $k < m$  and  $\delta < 8^{-m}$ . ■

**Theorem 2.3** *Let  $A$  be a C\*-algebra,  $a \in A$  and  $\lambda \in W(a)$ . If  $A$  is prime or if  $\lambda$  is an extreme point of  $W(a)$ , then for each  $\varepsilon > 0$  there exists a positive element  $h \in A$  such that  $\|h\| = 1$  and  $\|h(a - \lambda)h\| < \varepsilon$ . Moreover, if  $A$  is a von Neumann algebra, we can choose  $h$  to be a projection.*

**Proof** If  $\lambda$  is an extreme point of  $W(a)$  then there exists a pure state  $\rho$  on  $A$  (or in the unitization of  $A$  if  $A$  does not have a unit) such that  $\rho(a) = \lambda$ , hence the theorem follows in this case at once from Lemma 2.2.

If  $A$  is a prime C\*-algebra, then there exists a separable prime C\*-subalgebra  $B$  of  $A$  such that  $a \in A$  (this was first proved in [8, Proposition 3.1] and a very elementary proof can also

be found in [19, Lemma 3.2]), but separable prime  $C^*$ -algebras have faithful irreducible representations [24, p. 102], so we may assume that  $B$  is an irreducible  $C^*$ -algebra on a Hilbert space  $\mathcal{H}$ . Since  $\lambda \in W(a)$ , there exists for each  $n = 1, 2, \dots$  a unit vector  $\xi_n \in \mathcal{H}$  such that  $|\langle (a - \lambda)\xi_n, \xi_n \rangle| < \frac{1}{n}$  [4], hence  $\lambda = \rho(a)$ , where  $\rho$  is a weak\* limit point of the vector states induced by the vectors  $\xi_n$ . Since  $B$  is irreducible, the vector states on  $B$  are pure, hence by Lemma 2.2 there exists  $h \in B \subseteq A$  such that  $\|h\| = 1$  and  $\|h^*(a - \lambda)h\| < \varepsilon$ . Replacing  $h$  by  $hh^*$ , we may assume that  $h$  is positive. Finally, if  $A$  is a von Neumann algebra, we may replace  $h$  by the spectral projection of  $h$  corresponding to the interval  $(1 - \delta, 1]$  for some sufficiently small  $\delta > 0$ . ■

**Remark 2.4** In general, if  $A$  is, say, an abelian  $C^*$ -algebra,  $a \in A$  and  $\lambda \in W(a)$  is not an extreme point of  $W(a)$ , then  $\inf\{\|h^*(a - \lambda)h\| : h \in A, \|h\| = 1\} > 0$ . To see this, consider for example the diagonal  $2 \times 2$  matrix  $a$  with 1 and  $-1$  along the diagonal and  $\lambda = 0$ .

### 3 Weak\* Compact $C^*$ -Convex Hulls

Our first goal in this section is to characterize all normal elements in  $\overline{\text{co}}_R(a)$ , where  $a$  is an arbitrary element of a von Neumann algebra  $R$ . Throughout the rest of the paper we will denote by  $Z$  the center of  $R$  and by  $\Delta$  the spectrum of  $Z$ . By the Gelfand transform we shall identify  $Z$  with the  $C^*$ -algebra  $C(\Delta)$  of all complex valued continuous functions on  $\Delta$ . For each  $t \in \Delta$  we denote by  $Z_t$  the kernel of  $t$ , by  $R_t$  the closed ideal in  $R$  generated by  $Z_t$  (called the *Glimm ideal* at  $t$ ) and by  $R(t)$  the quotient algebra  $R/R_t$ . Further, the coset of an element  $a \in R$  in  $R(t)$  will be denoted by  $a(t)$ . By [11]  $\|a\| = \sup_{t \in \Delta} \|a(t)\|$  and the function  $t \mapsto \|a(t)\|$  is continuous on  $\Delta$  for each  $a \in R$ .

**Lemma 3.1** Let  $a \in R$ ,  $t \in \Delta$  and  $\lambda \in W(a(t))$ . Then for each  $\varepsilon > 0$  there exists a projection  $p \in Z$  such that  $p(t) = 1$  and  $d(\lambda p, \overline{\text{co}}_{R_p}(ap)) < \varepsilon$ , where  $d$  denotes the distance.

**Proof** Since  $R(t)$  is a prime  $C^*$ -algebra by [11] (in fact it is even primitive by [14]), by Theorem 2.3 there exists a positive element  $h \in R(t)$  such that  $\|h\| = 1$  and  $\|h(a - \lambda)h\| < \varepsilon$ . Using the same approximate unit argument as in the beginning of the proof of Lemma 2.2 we can lift  $h$  to an element  $s \in R$  such that  $\|s\| = 1$  and  $\|s^*(a - \lambda)s\| < \varepsilon$ . Then by the polar decomposition of  $s$  and applying the spectral theorem to  $|s|$  it follows easily that there exists a projection  $e \in R$  such that  $e(t) \neq 0$  and  $\|(e(a - \lambda)e)(t)\| < \varepsilon$ . By the continuity of the norm and since  $\Delta$  is extremely disconnected there exists a clopen neighborhood (that is, a neighborhood which is closed and open)  $\sigma$  of  $t$  such that  $e(s) \neq 0$  and  $\|(e(a - \lambda)e)(s)\| < \varepsilon$  for each  $s \in \sigma$ . Let  $p \in Z$  be the projection corresponding to  $\sigma$  (that is, the Gelfand transform of  $p$  is the characteristic function of  $\sigma$ ). Replacing  $e$  by  $ep$ , we may assume that the central carrier of  $e$  in  $R$  is  $p$  (since  $e(s) \neq 0$  for all  $s \in \sigma$ ) and  $\|e(a - \lambda)e\| < \varepsilon$ .

Since the central carrier  $p$  of  $e$  can be written as an orthogonal sum of projections equivalent to subprojections of  $e$ , there exists a family  $\{u_k\}$  of partial isometries in  $R$  such that  $u_k u_k^* \leq e$  and  $\sum_k u_k^* u_k = p$ . Then

$$\begin{aligned} \left\| \sum_k u_k^* a p u_k - \lambda p \right\| &= \left\| \sum_k u_k^* e(a - \lambda) e u_k \right\| \\ &= \sup_k \|u_k^* e(a - \lambda) e u_k\| \end{aligned}$$

$$\begin{aligned} &\leq \|e(a - \lambda)e\| \\ &< \varepsilon \end{aligned}$$

since the range projections of  $u_k^*$  are mutually orthogonal, hence

$$d(\lambda p, \overline{\text{co}}_{R_p}(ap)) < \varepsilon. \quad \blacksquare$$

Later in this section we shall need matrix ranges, so the following lemma is formulated for matrix ranges, although for the proof of Theorem 3.3 below the usual numerical range is sufficient. Recall ([3], [5]) that for each  $n = 1, 2, \dots$  the  $n$ -th matrix range  $W_n(a)$  of an element  $a \in A$ , where  $A$  is a unital C\*-algebra, is defined as the set of all matrices in  $\mathcal{M}_n(\mathbb{C})$  of the form  $\varphi(a)$ , where  $\varphi: A \rightarrow \mathcal{M}_n(\mathbb{C})$  is a unital completely positive map (we refer to [23] for the definition and basic properties of completely positive maps). In particular,  $W_1(a)$  coincides with the usual numerical range  $W(a)$ .

**Lemma 3.2** *Let  $A$  be a unital C\*-algebra,  $C$  the center of  $A$ ,  $\Delta$  the spectrum of  $C$  and  $t \in \Delta$ . For each closed set  $\sigma \subseteq \Delta$  let  $C_\sigma = \{c \in C : c|_\sigma = 0\}$  (where we have identified  $C$  with the continuous functions on  $\Delta$ ), let  $A_\sigma$  be the closed ideal in  $A$  generated by  $C_\sigma$  and put  $A(t) = A/A_{\{t\}}$  and  $a(t) = a + A_{\{t\}}$ . Then for each  $a \in A$  and  $n = 1, 2, \dots$  we have*

$$(3.1) \quad W_n(a(t)) = \bigcap_{\sigma \in \mathbb{B}_t} W_n(a + A_\sigma),$$

where the intersection is over some basis  $\mathbb{B}_t$  of closed neighborhoods  $\sigma$  of  $t$ .

**Proof** Since  $A(t)$  is a quotient algebra of  $A/A_\sigma$  for each  $\sigma \in \mathbb{B}_t$ , the left hand side of (3.1) is contained in the right hand side. To prove the reverse inclusion, suppose that  $\lambda \in W_n(a + A_\sigma)$  for all  $\sigma \in \mathbb{B}_t$ . Then

$$(3.2) \quad \|\alpha \otimes 1 + \beta \otimes \lambda\| \leq \|\alpha \otimes 1 + \beta \otimes (a + A_\sigma)\|$$

for all  $\alpha, \beta \in \mathcal{M}_m(\mathbb{C})$  and all  $m \in \mathbb{N}$  (since there exists a unital completely positive, hence completely contractive, map from  $A/A_\sigma$  to  $\mathcal{M}_m(\mathbb{C})$  sending  $a + A_\sigma$  to  $\lambda$ ). Since  $\mathcal{M}_m(\mathbb{C}) \otimes A/A_\sigma \cong \mathcal{M}_m(A/A_\sigma)$  sits isometrically in  $\bigoplus_{s \in \sigma} \mathcal{M}_m(A(s))$ , we have

$$(3.3) \quad \|\alpha \otimes 1 + \beta \otimes (a + A_\sigma)\| = \sup_{s \in \sigma} \|\alpha \otimes 1 + \beta \otimes a(s)\|.$$

Note that the function  $s \mapsto \|\alpha \otimes 1 + \beta \otimes a(s)\|$  is upper semicontinuous on  $\Delta$  by [11]. Thus, given  $\varepsilon > 0$ , for each fixed  $\alpha, \beta \in \mathcal{M}_m(\mathbb{C})$  there exists a neighborhood  $\sigma \in \mathbb{B}_t$  such that the right hand side of (3.3) is less than  $\|\alpha \otimes 1 + \beta \otimes a(t)\| + \varepsilon$  and it follows then from (3.2) and (3.3) that

$$\|\alpha \otimes 1 + \beta \otimes \lambda\| \leq \|\alpha \otimes 1 + \beta \otimes a(t)\| + \varepsilon.$$

Since this holds for all  $\varepsilon > 0$  and  $\alpha, \beta \in \mathcal{M}_m(\mathbb{C})$  ( $m \in \mathbb{N}$ ), the unital map

$$A(t) \supseteq \text{span}\{1, a(t)\} \rightarrow \mathcal{M}_n(\mathbb{C}), \quad \alpha + \beta a(t) \mapsto \alpha + \beta \lambda \quad (\alpha, \beta \in \mathbb{C})$$

is completely contractive, hence by the Arveson extension theorem (see [23, 3.4, 6.5]) it can be extended to a unital completely positive map from  $A(t)$  to  $\mathcal{M}_n(\mathbb{C})$  sending  $a(t)$  to  $\lambda$ , hence  $\lambda \in W_n(a(t))$ . ■

**Theorem 3.3** *Let  $a$  and  $b$  be elements of a von Neumann algebra  $R$ . If  $b$  is normal, then  $b \in \overline{\text{co}}_R(a)$  if and only if  $W(b(t)) \subseteq W(a(t))$  for each  $t$  in the spectrum  $\Delta$  of the center  $Z$  of  $R$ . Moreover, this is equivalent to the condition that  $W_{Rp}(bp) \subseteq W_{Rp}(ap)$  for each projection  $p \in Z$ .*

**Proof** If  $b \in \overline{\text{co}}_R(a)$ , then there exists a net  $\{b_k\}_k \subseteq \text{co}_R(a)$  converging to  $b$  in the strong operator topology. For each clopen subset  $\sigma$  of  $\Delta$  let  $p_\sigma$  be the corresponding projection in  $Z$ . Since  $b_k p_\sigma \in \text{co}_{Rp_\sigma}(ap_\sigma)$ , it follows easily from the definitions of the numerical range and the  $C^*$ -convex hulls that  $W_{Rp_\sigma}(b_k p_\sigma) \subseteq W_{Rp_\sigma}(ap_\sigma)$ . Since the net  $\{b_k p_\sigma\}_k$  converges to  $b p_\sigma$ , it follows (by the weak\* density of normal states in the set of all states) that  $W_{Rp_\sigma}(b p_\sigma) \subseteq W_{Rp_\sigma}(ap_\sigma)$ . Now for each fixed  $t \in \Delta$  Lemma 3.2 implies that

$$W(b(t)) = \bigcap_{\sigma} W_{Rp_\sigma}(b p_\sigma) \subseteq \bigcap_{\sigma} W_{Rp_\sigma}(ap_\sigma) = W(a(t)),$$

where the intersection is over all clopen neighborhoods  $\sigma$  of  $t$ . (Observe that we have not needed the normality of  $b$  for this implication and that a similar argument shows that  $W_n(b(t)) \subseteq W_n(a(t))$  for all  $n = 1, 2, \dots$ .)

Suppose now conversely that  $b$  is normal and  $W(b(t)) \subseteq W(a(t))$  for all  $t \in \Delta$ . Let  $\varepsilon > 0$ . By the spectral theorem there exist scalars  $\beta_j$  in the spectrum  $\sigma(b)$  of  $b$  and projections  $e_j \in R$  (commuting with  $b$ ) with the sum 1 such that

$$(3.4) \quad \left\| b - \sum_{j=1}^n \beta_j e_j \right\| < \frac{\varepsilon}{3}.$$

Then  $\|b(t) - \sum_{j=1}^n \beta_j e_j(t)\| < \frac{\varepsilon}{3}$ , hence  $\|b(t)e_j(t) - \beta_j e_j(t)\| < \frac{\varepsilon}{3}$  and therefore the distance of  $\beta_j$  to the spectrum of  $b(t)e_j(t)$  in  $e_j(t)R(t)e_j(t)$  is less than  $\frac{\varepsilon}{3}$  if  $e_j(t) \neq 0$ . Thus  $d(\beta_j, W(b(t))) < \frac{\varepsilon}{3}$  (where  $d$  denotes the distance) if  $e_j(t) \neq 0$ . Suppose now, for a moment, that  $e_j(t) \neq 0$  for all  $t \in \Delta$  and  $j = 1, \dots, n$ . Then  $d(\beta_j, W(a(t))) < \frac{\varepsilon}{3}$  for all  $j$  and  $t$ , so we can choose  $\gamma_{j,t} \in W(a(t))$  so that  $|\beta_j - \gamma_{j,t}| < \frac{\varepsilon}{3}$ . By Lemma 3.1 for each  $t$  there exists a projection  $p_t \in Z$  such that  $p_t(t) = 1$  and  $d(\gamma_{j,t} p_t, \overline{\text{co}}_{Rp_t}(ap_t)) < \frac{\varepsilon}{3}$ , hence

$$(3.5) \quad d(\beta_j p_t, \overline{\text{co}}_{Rp_t}(ap_t)) < \frac{2\varepsilon}{3} \quad (j = 1, \dots, n).$$

Since the projections  $e_j$  are mutually orthogonal with the sum 1, (3.5) easily implies that

$$d\left(\sum_{j=1}^n \beta_j e_j p_t, \overline{\text{co}}_{Rp_t}(ap_t)\right) < \frac{2\varepsilon}{3},$$



hence, using (3.4),

$$d(bp_t, \overline{\text{co}}_{R p_t}(ap_t)) < \varepsilon.$$

Thus for each  $t \in \Delta$  we can choose an  $x_t \in \overline{\text{co}}_{R p_t}(ap_t)$  such that

$$\|(b - x_t)p_t\| < \varepsilon.$$

Note that  $p_t$  corresponds to a clopen subset  $\sigma_t$  of  $\Delta$  and  $t \in \sigma_t$  (since  $p_t(t) = 1$ ). By compactness we can cover  $\Delta$  by finitely many such clopen subsets  $\sigma_i$ . It follows that there exist projections  $p_1, \dots, p_m$  in  $Z$  with the sum 1 and elements  $x_i \in \overline{\text{co}}_{R p_i}(ap_i)$  such that

$$\|(b - x_i)p_i\| < \varepsilon \quad (i = 1, \dots, m).$$

Let  $x = \sum_{i=1}^m x_i$ . Then  $x \in \overline{\text{co}}_R(a)$  and  $\|b - x\| < \varepsilon$ . Thus,  $d(b, \overline{\text{co}}_R(a)) < \varepsilon$  for each  $\varepsilon > 0$ , which implies that  $b \in \overline{\text{co}}_R(a)$ . This proves the theorem in the case  $e_j(t) \neq 0$  for all  $j = 1, \dots, n$  and  $t \in \Delta$ .

In general (when  $e_j(t) = 0$  for some  $j$  and  $t$ ) the proof can be reduced to the case just considered by a suitable partition of  $\Delta$ . Namely, we can partition  $\Delta$  into a finite union of disjoint clopen subsets  $\Delta_k$  ( $k = 1, \dots, l$ ) each containing an open dense subset  $\Omega_k$  such that for each  $k$  there exists a subset  $\mathbb{F}_k$  of  $\mathbb{F} = \{1, \dots, n\}$  with the property that for  $t \in \Omega_k$  we have  $e_j(t) \neq 0$  if  $j \in \mathbb{F}_k$  and  $e_j(t) = 0$  if  $j \in \mathbb{F} \setminus \mathbb{F}_k$ . (To find such a partition, let  $\Omega_1 = \{t \in \Delta : e_j(t) \neq 0 \text{ for all } j \in \mathbb{F}\}$ . Since  $\Omega_1$  is open and  $\Delta$  is extremely disconnected,  $\Delta_1 := \overline{\Omega_1}$  is clopen. If  $\Delta_1 \neq \Delta$ , let  $\Omega_2 = \{t \in \Delta \setminus \Delta_1 : e_1(t) \neq 0, \dots, e_{n-1}(t) \neq 0\}$  and put  $\Delta_2 = \overline{\Omega_2}$ . Continuing in this way, we find the required partition in less than  $2^n$  steps.) Let  $p_k \in Z$  be the projection corresponding to  $\Delta_k$ . Then  $\|(b - \sum_{j \in \mathbb{F}_k} \beta_j e_j)p_k\| < \frac{\varepsilon}{3}$  by (3.4), and by the same argument as in the previous paragraph we have that

$$(3.6) \quad d(\beta_j, W(a(t))) \leq \frac{\varepsilon}{3}$$

for all  $j \in \mathbb{F}_k$  and  $t \in \Omega_k$ . Since  $\Omega_k$  is dense in  $\Delta_k$ , (3.6) must hold for all  $t \in \Delta_k$  (and  $j \in \mathbb{F}_k$ ). (Indeed, choosing a net  $\{t_\nu\} \subseteq \Omega_k$  converging to  $t \in \Delta_k$ , (3.6) implies that  $W(\beta_j - a(t_\nu))$  intersects the closed disc with the radius  $\varepsilon/3$  around 0, hence for each  $\nu$  there exists a state  $\rho_\nu$  on  $R$  annihilating the Glimm ideal  $R_{t_\nu}$  such that  $|\rho_\nu(\beta_j - a)| \leq \varepsilon/3$ . Now we can use an argument from [11, p. 233]. Namely, with  $\rho$  a weak\* limit point of the net  $\{\rho_\nu\}$ , we have that  $|\rho(\beta_j - a)| \leq \varepsilon/3$ . Moreover, since  $\rho_\nu|_Z = t_\nu$  (for  $\rho_\nu$  annihilates  $R_{t_\nu}$ ) and  $t_\nu$  converge to  $t$ , it follows that  $\rho|_Z = t$ , hence  $\rho(R_t) = 0$ . Thus  $\rho$  can be regarded as a state on  $R(t)$  such that  $|\rho(\beta_j - a(t))| \leq \varepsilon/3$ , which implies that  $d(\beta_j, W(a(t))) \leq \varepsilon/3$ .) We can now use the reasoning from the previous paragraph (with  $\Delta$  replaced by  $\Delta_k$ ) to show that  $bp_k \in \overline{\text{co}}_{R p_k}(ap_k)$  for each  $k = 1, \dots, l$ . Since  $\sum p_k = 1$ , this implies that  $b \in \overline{\text{co}}_R(a)$ . ■

**Corollary 3.4** For each  $a \in R$  the intersection  $\overline{\text{co}}_R(a) \cap Z$  consists of all  $c \in Z$  such that  $c(t) \in W(a(t))$  for all  $t \in \Delta$ , and this is the same as the set of all  $c \in Z$  such that there exists a conditional expectation  $\phi$  from  $R$  onto  $Z$  satisfying  $\phi(a) = c$ .

**Proof** If  $c \in Z$  then  $c(t) \in \mathbb{C}$ , hence  $W(c(t)) = \{c(t)\}$  for each  $t \in \Delta$  and the first conclusion of the corollary follows immediately from Theorem 3.3.

If  $\phi: R \rightarrow Z$  is a conditional expectation such that  $\phi(a) = c$ , then, since  $\phi$  is  $Z$ -linear, by a standard argument  $\phi$  induces a state  $\phi_t$  on  $R(t)$  by  $\phi_t(x(t)) = \phi(x)(t)$  ( $x \in R$ ) such that  $\phi_t(a(t)) = c(t)$ , hence  $c(t) \in W(a(t))$ . Conversely, if  $c(t) \in W(a(t))$  for all  $t \in \Delta$ , then the map  $\phi_0: Z + Za \rightarrow Z$ ,  $\phi_0(z_1 + z_2a) = z_1 + z_2c$  is easily seen to be a unital  $Z$ -linear complete contraction, hence by [23, p. 118]  $\phi_0$  can be extended to a conditional expectation  $\phi: R \rightarrow Z$ . ■

Since conditional expectations from  $R$  to  $Z$  separate points of  $R$  (which can be seen directly or by using [13, Theorem 3]), by Corollary 3.4 the set  $\overline{\text{co}}_R(a) \cap Z$  is nonempty (which follows also from the Dixmier approximation theorem), and  $\overline{\text{co}}_R(a) \cap Z$  is a singleton only if  $a \in Z$ .

If  $R$  is injective, then we can characterize all (not just normal) elements in  $\overline{\text{co}}_R(a)$ .

**Proposition 3.5** *Let  $a, b \in R$ . A necessary condition for  $b$  to be in  $\overline{\text{co}}_R(a)$  is that  $W_n(b(t)) \subseteq W_n(a(t))$  for all  $t \in \Delta$  and all  $n = 1, 2, \dots$ ; moreover, if  $R$  is injective, then this condition is also sufficient.*

**Proof** The necessity has already been observed in the beginning of the proof of Theorem 3.3. So, assume that  $R$  is injective and  $W_n(b(t)) \subseteq W_n(a(t))$  for all  $t \in \Delta$  and all  $n = 1, 2, \dots$ . Then by [3, Th. 2.4.2] for each  $t \in \Delta$  the map  $\phi_t: \mathbb{C} + Ca(t) \rightarrow \mathbb{C} + Cb(t)$ ,  $\phi_t(\alpha + \beta a(t)) = \alpha + \beta b(t)$  is a unital complete contraction, hence so is  $\psi_t := \phi_t \pi_{t,a}$ , where  $\pi_{t,a}$  is the restriction of the quotient homomorphism  $\pi_t: R \rightarrow R(t)$  to  $Z + Za$ . This implies that the correspondence

$$\psi: Z + Za \rightarrow Z + Zb, \quad \psi(z_1 + z_2a) = z_1 + z_2b$$

is a well defined  $Z$ -linear unital complete contraction, hence by the injectivity of  $R$ ,  $\psi$  extends to a unital  $Z$ -linear completely contractive (hence completely positive) map  $\theta: R \rightarrow R$ . Since  $R$  is injective, by [1, Corollary 3.7]  $\theta$  is the point weak\* limit of a net of maps  $\theta_k: R \rightarrow R$  of the form  $\theta_k(x) = \sum_{j=1}^{n_k} v_{k,j}^* x v_{k,j}$ , where  $v_{k,j} \in R$  and  $n_k \in \mathbb{N}$ ; moreover, by [2, Lemma 2.2] we may assume that  $\theta_k(1) \leq 1$  for each  $k$ . Put

$$v_{k,0} = \left(1 - \sum_{j=1}^{n_k} v_{k,j}^* v_{k,j}\right)^{1/2}.$$

Since for each vector  $\xi$  in the underlying Hilbert space we have that

$$\|v_{k,0}\xi\|^2 = \|\xi\|^2 - \left\langle \sum_{j=1}^{n_k} v_{k,j}^* v_{k,j} \xi, \xi \right\rangle \xrightarrow{k} \|\xi\|^2 - \langle \theta(1)\xi, \xi \rangle = 0,$$

the net  $\{v_{k,0}\}_k$  converges strongly to 0 and the net of elements

$$b_k := \sum_{j=0}^{n_k} v_{k,j}^* a v_{k,j} = v_{k,0}^* a v_{k,0} + \theta_k(a)$$

converges to  $\theta(a) = b$  in the weak\* topology. Since  $\sum_{j=0}^{n_k} v_{k,j}^* v_{k,j} = 1$  for each  $k$ , this implies that  $b \in \overline{\text{co}}_R(a)$ . ■

If  $A$  is an abelian unital C\*-algebra and  $K$  an  $A$ -convex subset of  $A$  then the  $A$ -extreme points in  $K$  are just the usual extreme points. To see this, let  $x$  be an extreme point of  $K$  and suppose that  $x = ay + (1 - a)z$  where  $y, z \in K, z \neq x$  and  $a \in A$  has the spectrum contained in the open interval  $(0, 1)$ . Identify  $A$  with the algebra of all continuous functions on the spectrum  $\Delta$  of  $A$  and let  $U$  be an open set in  $\Delta$  such that  $y(t), z(t) \neq x(t)$  for all  $t \in U$  and such that  $a(t) \leq 1/2$  for all  $t \in U$  or  $1 - a(t) \leq 1/2$  for all  $t \in U$ . We may assume that  $a(t) \leq 1/2$  for all  $t \in U$  (the other possibility is treated in the same way). Let  $c \in A$  be a non-zero function supported in  $U$  with values in  $[0, 1]$ . Put  $x_1 = (1 - c)x + 2cay + c(1 - 2a)z$  and  $x_2 = (1 - c)x + cz$  and observe that  $x_1, x_2 \in K$  (since  $K$  is  $A$ -convex) and  $\frac{1}{2}(x_1 + x_2) = x$ . Since  $x$  is extreme in  $K$  it follows that  $x_2 = x$ , hence  $cx = cz$ . But this is impossible since  $x(t) \neq z(t)$  for  $t$  in the support of  $c$ .

For matrix algebras  $M_n(\mathbb{C})$  the following corollary was proved by Farenick [9].

**Corollary 3.6** *Let  $R$  be a von Neumann algebra,  $Z$  the center of  $R$  and  $K$  a weak\* compact  $R$ -convex subset  $R$ . Then each extreme point of  $K_Z := K \cap Z$  is  $R$ -extreme in  $K$ .*

**Proof** For  $a \in R$  put  $W_Z(a) = \overline{\text{co}}_R(a) \cap Z$ . Note that

$$\bigcup_{a \in K} W_Z(a) = K_Z.$$

Let  $c$  be any extreme point of  $K_Z$  (which exists by the classical Krein-Milman theorem). Then  $c$  is  $Z$ -extreme in  $K_Z$  by the remark preceding the lemma. Suppose that

$$(3.7) \quad c = \sum_{j=1}^n a_j^* x_j a_j \quad (x_j \in K),$$

where the elements  $a_j \in R$  are invertible and  $\sum_{j=1}^n a_j^* a_j = 1$ . Let  $\phi: R \rightarrow Z$  be any  $Z$ -state (= a conditional expectation from  $R$  onto  $Z$ ). Then the elements  $c_j := \phi(a_j^* a_j) \in Z$  are invertible with the sum 1. Define  $Z$ -states  $\phi_j$  by  $\phi_j(x) = c_j^{-1} \phi(a_j^* x a_j)$  ( $x \in R, j = 1, \dots, n$ ). Then from (3.7) we have

$$c = \sum_{j=1}^n c_j \phi_j(x_j).$$

Since  $\phi_j(x_j) \in \overline{\text{co}}_R(x_j) \cap Z \subseteq K_Z$  by Corollary 3.4 and  $c$  is  $Z$ -extreme in  $K_Z$ , this implies that  $\phi_j(x_j) = c$  for all  $j$ . By the definition of  $\phi_j$  this means that  $\phi(a_j^*(x_j - c)a_j) = c_j \phi_j(x_j) - c \phi(a_j^* a_j) = c_j c - c c_j = 0$ . Since the  $Z$ -states separate points of  $R$  and the elements  $a_j$  are invertible, we conclude that  $x_j = c$  for all  $j = 1, \dots, n$ . ■

In general the extreme points of  $K_Z$  are not sufficient to generate  $K$  as a weak\* compact  $R$ -convex set. To see what is the weak\* closure of the  $R$ -convex hull of the extreme points

of  $K_Z$ , thus  $\overline{\text{co}}_R(K_Z)$ , suppose for simplicity that  $R$  is a factor, hence  $Z = \mathbb{C}$ . Observe that if  $a \in \text{co}_R(K_{\mathbb{C}})$ , say  $a = \sum_{j=1}^n \lambda_j w_j^* w_j$ , where  $\lambda_j \in K_{\mathbb{C}}$ ,  $w_j \in R$  and  $\sum_{j=1}^n w_j^* w_j = 1$ , then for all  $\alpha, \beta \in \mathcal{M}_n(\mathbb{C})$  and all  $n \in \mathbb{N}$  we have

$$\|\alpha \otimes 1 + \beta \otimes a\| = \left\| \sum_{j=1}^n (1 \otimes w_j)^* (\alpha \otimes 1 + \beta \otimes \lambda_j) (1 \otimes w_j) \right\| \leq \max_{1 \leq j \leq n} \|\alpha + \lambda_j \beta\|.$$

Thus, denoting by  $z$  the identity function on  $K_{\mathbb{C}} \subseteq \mathbb{C}$ ,

$$(3.8) \quad \|\alpha \otimes 1 + \beta \otimes a\| \leq \|\alpha \otimes 1 + \beta \otimes z\|.$$

This inequality persists also for all  $a \in \overline{\text{co}}_R(K_{\mathbb{C}})$ . Put  $A = C(K_{\mathbb{C}})$ . If  $R$  is injective, then (3.8) implies that there exists a unital completely positive map  $\phi: A \rightarrow R$  such that  $\phi(z) = a$ . Then by [22, 5.2, 3.2, 3.7] there exist a self-dual Hilbert (right)  $R$ -module  $E$ , a representation  $\pi: A \rightarrow L(E)$  (the algebra of all adjointable operators on  $E$ ) and an element  $x \in E$  such that  $\phi(w) = \langle x, \pi(w)x \rangle$  ( $w \in A$ ). By [22, Theorem 3.12]  $E$  can be expressed as the ultraweak direct sum  $\bigoplus_{i \in \mathbb{I}} e_i R$ , where  $\{e_i : i \in \mathbb{I}\}$  is a family of projections in  $R$ . Then  $L(E)$  can be regarded as a  $W^*$ -subalgebra of  $\mathcal{M}_{\mathbb{I}}(R) = R \overline{\otimes} B(\ell_2(\mathbb{I}))$  consisting of all matrices  $[a_{ij}] \in \mathcal{M}_{\mathbb{I}}(R)$  such that  $a_{ij} = e_i a_{ij} e_j$  ( $i, j \in \mathbb{I}$ ) and  $x = \sum_{i \in \mathbb{I}} x_i$  can be regarded as a column with the entries  $x_i \in R$  satisfying  $\sum_{i \in \mathbb{I}} x_i^* x_i = \langle x, x \rangle = \phi(1) = 1$ . Put  $b = \pi(z) \in L(E) \subseteq \mathcal{M}_{\mathbb{I}}(R)$ . Then  $b$  is normal,  $W(b) \subseteq W(z) = K_{\mathbb{C}}$ , and the identity

$$a = \phi(z) = x^* \pi(z)x = x^* b x$$

shows that  $a$  is a compression of  $b$ . Conversely, if  $a$  has a normal dilation  $b$  in  $\mathcal{M}_{\mathbb{I}}(R)$  for some index set  $\mathbb{I}$  such that  $W(b) \subseteq K_{\mathbb{C}}$ , then by an application of the spectral theorem we have that  $b \in \overline{\text{co}}_{\mathcal{M}_{\mathbb{I}}(R)}(K_{\mathbb{C}})$ , which easily implies that  $a \in \overline{\text{co}}_R(K_{\mathbb{C}})$ . This proves the following proposition.

**Proposition 3.7** *Let  $K$  be a weak\* compact  $R$ -convex subset of an injective factor  $R$  and  $K_{\mathbb{C}} = K \cap \mathbb{C}$ . Then  $\overline{\text{co}}_R(K_{\mathbb{C}})$  consists of all elements  $a \in R$  that have normal dilations  $b \in \mathcal{M}_{\mathbb{I}}(R)$  for some index set  $\mathbb{I}$  (more precisely,  $a = \sum_{i,j \in \mathbb{I}} x_i^* b_{ij} x_j$ , where  $x_i \in R$  and  $\sum_{i \in \mathbb{I}} x_i^* x_i = 1$ ) such that  $W(b) \subseteq K_{\mathbb{C}}$ .*

### 4 Norm Closed $C^*$ -Convex Hulls

Using the technique of the universal representation [17, Section 10.1], we can deduce from Theorem 3.3 a characterization of normal elements in the norm closure  $\overline{\text{co}}_A(a)$  of the  $C^*$ -convex hull of any element  $a$  in a  $C^*$ -algebra.

**Theorem 4.1** *Let  $A$  be a unital  $C^*$  algebra and  $a, b \in A$  with  $b$  normal. Then  $b \in \overline{\text{co}}_A(a)$  if and only if  $W(b + P) \subseteq W(a + P)$  for each primitive ideal  $P$  of  $A$ .*

**Proof** If  $b \in \overline{\text{co}}_A(a)$ , then  $b + P \in \overline{\text{co}}_{A/P}(a + P)$  for each primitive ideal  $P$ , which implies that  $W(b + P) \subseteq W(a + P)$ .

To prove the converse, let  $R$  be the universal enveloping von Neumann algebra of  $A$  (we shall regard  $A$  as a subalgebra of  $R$ ) and  $Z$  the center of  $R$ . Let  $p \in Z$  be a projection. For each  $\alpha$  in the spectrum of  $bp$  in  $Rp$  there is a pure state  $\rho$  on  $Rp$  such that  $\rho(bp) = \alpha$ . If  $\pi$  is the irreducible representation of  $Rp$  constructed from  $\rho$ , then  $\alpha \in W(\pi(bp))$  and  $\pi(p) = 1$ . We may regard  $\pi$  as a representation of  $R$  by  $\pi(x) := \pi(xp)$  ( $x \in R$ ) and by assumption we have that  $W(\pi(b)) \subseteq W(\pi(a))$ . (Namely, as a closed ideal in  $A$ , the kernel  $J$  of  $\pi|_A$  is an intersection of a family of primitive ideals of  $A$ , say  $J = \bigcap_k P_k$ , and then the isometric embedding  $A/J \rightarrow \bigoplus_k A/P_k$  implies that  $W(x + J)$  is the closure of the convex hull of  $\bigcup_k W(x + P_k)$  for each  $x \in A$ .) It follows that  $\alpha \in W(\pi(a)) = W(\pi(ap)) \subseteq W_{Rp}(ap)$  for all  $\alpha$  in the spectrum of  $bp$  in  $Rp$ . Since for a normal element the numerical range is equal to the convex hull of its spectrum, we conclude that  $W_{Rp}(bp) \subseteq W_{Rp}(ap)$  for each projection  $p \in Z$ . By Theorem 3.3 this implies that  $b \in \overline{\text{co}}_R(a)$ , but we have to prove that  $b \in \overline{\text{co}}_A(a)$ .

Since  $\text{co}_A(a)$  is convex,  $\overline{\text{co}}_A(a) = \overline{\text{co}}_A(a) \cap A$  (where one bar denotes the weak\* closure in  $R$ , see [17, p. 713]), hence it suffices now to prove that  $\text{co}_R(a) \subseteq \overline{\text{co}}_A(a)$ . Given any  $x = \sum_{j=1}^n v_j^* a v_j \in \text{co}_R(a)$ , by an application of the Kaplansky density theorem to

$$\begin{bmatrix} v_1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ v_n & 0 & \dots & 0 \end{bmatrix} \in \mathcal{M}_n(R) \supseteq \mathcal{M}_n(A)$$

we see (after an obvious reduction) that there exists a net  $\{a_k\}_k$  of elements  $a_k = (a_{k1}, \dots, a_{kn}) \in A^n$  converging strongly to  $v := (v_1, \dots, v_n)$  and satisfying  $\sum_{j=1}^n a_{kj}^* a_{kj} \leq 1$ . For each  $k$  put

$$a_{k0} = \left(1 - \sum_{j=1}^n a_{kj}^* a_{kj}\right)^{1/2} \quad \text{and} \quad x_k = \sum_{j=0}^n a_{kj}^* a a_{kj}.$$

Then  $x_k \in \text{co}_A(a)$  and the same argument as in the proof of Proposition 3.5 shows that the net  $\{x_k\}$  converges to  $x$  in the weak operator topology, hence  $x \in \overline{\text{co}}_A(a)$ . ■

The following corollary follows immediately from Theorem 4.1.

**Corollary 4.2** *If  $A$  is a simple unital C\*-algebra, then  $\overline{\text{co}}_A(a) \supseteq W(a) \cdot 1$  for each  $a \in A$ .*

**Remark 4.3** For a general unital C\*-algebra  $A$  and an element  $a \in A$  the set  $\overline{\text{co}}_A(a)$  does not necessarily intersect the center of  $A$  even though if  $A$  is primitive. To see this, consider, for example, the C\*-subalgebra of  $B(\mathcal{H})$  ( $\mathcal{H}$  a separable Hilbert space) generated by the ideal  $K(\mathcal{H})$  of all compact operators and two infinite rank projections  $p_1$  and  $p_2$  with  $p_1 + p_2 = 1$ . Now the center of  $A$  is  $\mathbb{C}$ , but  $\overline{\text{co}}_A(p_1 - p_2)$  does not contain any scalar since  $\overline{\text{co}}_{A/K(\mathcal{H})}(p_1 - p_2) = \{p_1 - p_2\}$  (because  $A/K(\mathcal{H})$  is abelian).

Using the known facts concerning the ideal structure of von Neumann algebras, we can

deduce some consequences from Theorem 4.1 and Theorem 3.3. For simplicity we shall consider here only the case of  $\sigma$ -finite factors, where the lattice of (closed two sided) ideals is very simple (see [17, Section 6.8]), so the following corollary is straightforward.

**Corollary 4.4** *Let  $R$  be a  $\sigma$ -finite factor,  $a, b \in R$  and  $b$  normal.*

- (i) *If  $R$  is finite or purely infinite, then  $b \in \overline{\text{co}}_R(a)$  if and only if  $W(b) \subseteq W(a)$ , hence the two sets  $\overline{\text{co}}_R(a)$  and  $\overline{\text{co}}_R(b)$  have the same intersection with the set of all normal elements in  $R$ .*
- (ii) *If  $R$  is semifinite (but infinite), then  $b \in \overline{\text{co}}_R(a)$  if and only if  $W(b) \subseteq W(a)$  and  $W_{\text{ess}}(b) \subseteq W_{\text{ess}}(a)$ , where  $W_{\text{ess}}(x)$  denotes the numerical range of the coset of  $x \in R$  in the quotient of  $R$  by the unique closed two-sided ideal of  $R$  (namely, the closed ideal generated by all finite projections in  $R$ ).*

## References

- [1] C. Anantharaman-Delaroche, *Some remarks on the cone of completely positive maps between von Neumann algebras*. J. London Math. Soc. **55**(1997), 193–208.
- [2] C. Anantharaman-Delaroche and J. F. Havet, *Approximate factorizations of completely positive maps*. J. Funct. Anal. **90**(1990), 411–428.
- [3] W. Arveson, *Subalgebras of  $C^*$ -algebras II*. Acta Math. **128**(1972), 271–308.
- [4] F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and elements of normed algebras*. London Math. Soc. Lecture Note Series **2**, Cambridge Univ. Press, Cambridge, 1971.
- [5] ———, *Numerical ranges II*. London Math. Soc. Lecture Note Series **10**, Cambridge Univ. Press, Cambridge, 1973.
- [6] K. R. Davidson,  *$C^*$ -algebras by example*. Fields Institute Monographs, Amer. Math. Soc., Providence, R.I., 1996.
- [7] J. Dixmier, *Les Algèbres d'Opérateurs dans l'Espace Hilbertien*. Gauthier-Villars, Paris, 1969.
- [8] G. A. Elliott and L. Zsido, *Almost uniformly continuous automorphism groups of operator algebras*. J. Operator Theory **8**(1982), 227–277.
- [9] D. R. Farenick, *Krein-Milman type problems for compact matricially convex sets*. Linear Algebra Appl. **162–164**(1992), 325–334.
- [10] P. A. Fillmore, J. G. Stampfli and J. P. Williams, *On the essential range, the essential spectrum and a problem of Halmos*. Acta Sci. Math. (Szeged) **33**(1972), 179–192.
- [11] J. Glimm, *A Stone-Weierstrass theorem for  $C^*$ -algebras*. Ann. of Math. **72**(1960), 216–244.
- [12] U. Haagerup and L. Zsido, *Sur la propriété de Dixmier pour les  $C^*$ -algebras*. C. R. Acad. Sci. Paris, Ser. I Math. **298**(1984), 173–176.
- [13] H. Halpern, *Module homomorphisms of a von Neumann algebra into its center*. Trans. Amer. Math. Soc. **140**(1969), 183–193.
- [14] ———, *Irreducible module homomorphisms of a von Neumann algebra into its center*. Trans. Amer. Math. Soc. **140**(1969), 195–221.
- [15] ———, *Essential central spectrum and range for elements of a von Neumann algebra*. Pacific J. Math. **43**(1972), 349–380.
- [16] F. Hiai and Y. Nakamura, *Closed convex hulls of unitary orbits in von Neumann algebras*. Trans. Amer. Math. Soc. **323**(1991), 1–38.
- [17] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*. Vols. 1, 2, Academic Press, London, 1983, 1986.
- [18] R. I. Loebl and V. I. Paulsen, *Some remarks on  $C^*$ -convexity*. Linear Algebra and Appl. **35**(1981), 63–78.
- [19] B. Magajna, *The Haagerup norm on the tensor product of operator modules*. J. Funct. Anal. **129**(1995), 325–348.
- [20] ———, *A transitivity problem for completely bounded mappings*. Houston J. Math. **23**(1997), 109–120.
- [21] P. B. Morenz, *The structure of  $C^*$ -convex sets*. Canad. J. Math. **46**(1994), 1007–1026.
- [22] W. L. Paschke, *Inner product modules over  $B^*$ -algebras*. Trans. Amer. Math. Soc. **182**(1973), 443–468.
- [23] V. I. Paulsen, *Completely bounded maps and dilations*. Pitman Res. Notes Math. Ser. **146**, London, 1986.

- [24] G. K. Pedersen,  *$C^*$ -algebras and their automorphism groups*. London Math. Soc. Monographs **14**, Academic Press, London, 1979.
- [25] J. R. Ringrose, *On the Dixmier approximation theorem*. Proc. London Math. Soc. **49**(1984), 37–57.

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