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ON PAIRS OF LINEAR EQUATIONS IN FOUR PRIME VARIABLES AND POWERS OF TWO

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Abstract

In this paper, we consider the simultaneous representation of pairs of positive integers. We show that every pair of large positive even integers can be represented in the form of a pair of linear equations in four prime variables and k powers of two. Here, $k = 63$ in general and $k = 31$ under the generalised Riemann hypothesis.

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1. Introduction

In 1742, Goldbach proposed the celebrated conjecture that every even integer greater than 2 can be expressed as the sum of two primes. With the original conjecture still unsolved, many variations of the problem have arisen. One of them is the so-called Goldbach–Linnik problem, initiated by Linnik, who showed [4] the existence of an absolute constant k such that every sufficiently large even integer can be written as a sum of two primes and at most k powers of two. (See Gallagher [1] for a simplified proof.) Explicit values for k were not found until 1998, when Liu *et al.* [5] showed that $k = 54000$ is acceptable. The bound for k was dramatically improved by Heath-Brown and Puchta [3] who proved that every sufficiently large even integer is a sum of two primes and at most 7 or 13 powers of two, according to whether the generalised Riemann hypothesis is assumed or not.

In this paper, we study the Goldbach–Linnik problem in an extended way. Instead of considering representations of a single even integer, we attempt to simultaneously represent pairs of positive even integers as sums of primes and powers of two. Our investigation is motivated by the work of several mathematicians on another important kind of problem, concerning linear equations in primes. For details of progress along these lines, we refer readers to the work of Liu and Tsang [8] and Green and Tao [2]. In particular, the results in [8] turn out to be closely related to numerical estimates on the cardinality of exceptional sets in the Goldbach problem.

Throughout this paper, with or without subscripts, ν always denotes a positive integer, and p always denotes a prime. We shall consider the simultaneous representation of pairs of positive even integers B_1, B_2, with $B_1 > B_2$, in the form

$$\begin{cases} B_1 = p_1 + p_2 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}, \\ B_2 = p_3 + p_4 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k} \end{cases} \tag{1.1}$$

where k is a positive integer. Our result is stated as follows.

Theorem 1.1. *For $k = 63$, the simultaneous equations (1.1) are solvable for every pair of sufficiently large positive even integers B_1, B_2; furthermore, $k = 31$ is admissible, assuming the generalised Riemann hypothesis.*

Before giving the proof of the main theorem in the following sections, let us fix some terminology. Let ω be a small positive constant. Set

$$S(\alpha, N) = \sum_{\omega N < p \le N} e(p\alpha) \tag{1.2}$$

and

$$T(\alpha) = \sum_{1 \le \nu \le L} e(2^{\nu}\alpha), \tag{1.3}$$

where $e(x) := \exp(2\pi i x)$ and $L = \log_2^{B_1}$. For any real α_1, α_2, put

$$\beta_B = B_1\alpha_1 + B_2\alpha_2 \quad \text{and} \quad \beta_5 = \alpha_1 + \alpha_2. \tag{1.4}$$

Let $R(B_1, B_2)$ be the number of solutions of (1.1) in $(p_1, p_2, p_3, p_4, \nu_1, \nu_2, \ldots, \nu_k)$ with

$$\omega B_1 < p_1, p_2 \le B_1, \quad \omega B_2 \le p_3, p_4 \le B_2, \quad 1 \le \nu_j \le L \text{ for } j = 1, 2, \ldots, k.$$

In view of (1.2), (1.3) and (1.4),

$$\begin{aligned} R(B_1, B_2) &= \int_0^1 \int_0^1 S^2(\alpha_1, B_1) S^2(\alpha_2, B_2) T^k(\beta_5) e(-\beta_B)\, d\alpha_1\, d\alpha_2 \\ &= \sum_{1 \le \nu_1 \le L} \cdots \sum_{1 \le \nu_k \le L} \int_0^1 \int_0^1 S^2(\alpha_1, B_1) S^2(\alpha_2, B_2) e((2^{\nu_1} + 2^{\nu_2} \\ &\qquad + \cdots + 2^{\nu_k} - B_1)\alpha_1 + (2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k} - B_2)\alpha_2)\, d\alpha_1\, d\alpha_2 \\ &= \sum_{1 \le \nu_1 \le L} \cdots \sum_{1 \le \nu_k \le L} \int_0^1 S^2(\alpha, B_1) e(-n_1\alpha)\, d\alpha \int_0^1 S^2(\alpha, B_2) e(-n_2\alpha)\, d\alpha \end{aligned}$$

where

$$n_i = B_i - 2^{\nu_1} - 2^{\nu_2} - \cdots - 2^{\nu_k} \quad \text{for } i = 1, 2.$$

In order to apply the Hardy–Littlewood method, we choose $P_i = B_i^{45/154}$ with $i = 1, 2$. For any integers h, q satisfying

$$1 \le h \le q \le P_i \quad \text{and} \quad (h, q) = 1, \tag{1.5}$$

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define

$$m_i(h,q) = \left\{ \alpha \in [0,1] : \left| \alpha - \frac{h}{q} \right| \le \frac{P_i}{qB_i} \right\},$$
$$\mathfrak{M}_i = \bigcup m_i(h,q) \quad \text{and} \quad \mathfrak{m}_i = [0,1] \setminus \mathfrak{M}_i,$$

where the union is over all h, q satisfying (1.5). In addition, we set

$$\mathcal{A}_\lambda = \{\alpha \in [0,1] : |T(\alpha)| \ge \lambda L\}.$$

Now, for $i = 1, 2$,

$$\begin{aligned}
\int_0^1 & S^2(\alpha, B_i) e(-n_i\alpha)\, d\alpha \\
&= \int_{\mathfrak{M}_i} S^2(\alpha, B_i) e(-n_i\alpha)\, d\alpha + \int_{\mathfrak{m}_i \cap \mathcal{A}_\lambda} S^2(\alpha, B_i) e(-n_i\alpha)\, d\alpha \\
&\quad + \int_{\mathfrak{m}_i \setminus \mathcal{A}_\lambda} S^2(\alpha, B_i) e(-n_i\alpha)\, d\alpha \\
&= S_1(B_i, n_i) + S_2(B_i, n_i) + S_3(B_i, n_i),
\end{aligned}$$

say. Hence

$$R(B_1, B_2) = \sum_{s,t=1}^{3} \sum_{1 \le \nu_1 \le L} \cdots \sum_{1 \le \nu_k \le L} S_s(B_1, n_1) S_t(B_2, n_2), \tag{1.6}$$

where

$$n_i = B_i - 2^{\nu_1} - 2^{\nu_2} - \cdots - 2^{\nu_k} \quad \text{for } i = 1, 2.$$

We will establish Theorem 1.1 by estimating the term

$$R_{s,t} = \sum_{1 \le \nu_1 \le L} \cdots \sum_{1 \le \nu_k \le L} S_s(B_1, n_1) S_t(B_2, n_2)$$

for all $1 \le s, t \le 3$. A substantial part of this paper, Section 2, will be devoted to an estimate for $R_{1,1}$, where we borrow some ideas from [3] in our proof. Estimates for the remaining eight terms will be gathered in Section 3.

2. Estimate for $R_{1,1}$

In this section, we provide a lower bound for $R_{1,1}$, as contained in the following proposition.

PROPOSITION 2.1. *$R_{1,1}$ is given by*

$$\begin{aligned}
R_{1,1} &= \sum_{1 \le \nu_1 \le L} \cdots \sum_{1 \le \nu_k \le L} S_1(B_1, n_1) S_1(B_2, n_2) \\
&\ge 1.74293(1 - 4\omega)(1 + o(1)) B_1 B_2 (\log B_1 \log B_2)^{-2} L^k.
\end{aligned}$$

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We begin with the following lemma.

Lemma 2.2 (Heath-Brown and Puchta). *For $i = 1, 2$,*

$$\begin{aligned} S_1(B_i, n_i) &= 2C_0(1 - 2\omega)B_i(\log B_i)^{-2} \sum_{d|n_i} k(d) \\ &\quad + O\Big(e^{-\omega\eta_i} B_i(\log B_i)^{-2} \frac{n_i}{\phi(n_i)}\Big) \\ &\quad + O\Big(B_i(\log B_i)^{-2} \sum_{\chi_i, \chi_i' \in \mathfrak{P}(\eta_i)} \frac{n_i}{\phi(n_i)} \Big(\frac{m_i}{(m_i, n_i)}\Big)^{-1/3}\Big) \\ &= S_{1,1}(B_i, n_i) + S_{1,2}(B_i, n_i) + S_{1,3}(B_i, n_i), \end{aligned}$$

where

$$0.6601 \le C_0 = \prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big) \le 0.66017,$$

$k(d)$ is a multiplicative function defined by

$$k(p^e) = \begin{cases} 0 & p = 2 \text{ or } e \ge 2, \\ \dfrac{1}{p-2} & \text{otherwise,} \end{cases}$$

η_i is a suitable value in the range $0 \le \eta_i \le \log\log B_i$, $\mathfrak{P}(\eta_i)$ is the set of characters χ_i of conductor $r_i \le P_i$, for which the function $L(s_i, \chi_i)$ has at least one zero ρ_i in the region $\operatorname{Re}\rho_i > 1 - \eta_i/\log B_i$, $|\operatorname{Im}\rho_i| \le B_i$, and $m_i = [r_i, r_i']$ if we let r_i, r_i' be the conductors of χ_i, χ_i', respectively. Moreover,

$$\#\mathfrak{P}(\eta_i) \ll e^{6\eta_i}.$$

The above result can be deduced by a careful examination of the arguments in [3, Sections 3 and 4], which we omit here.

Then, with the definition of $S_{1,j}$ $(j = 1, 2, 3)$ above,

$$R_{1,1} = \sum_{s,t=1}^{3} \sum_{1\le\nu_1\le L} \cdots \sum_{1\le\nu_k\le L} S_{1,s}(B_1, n_1)S_{1,t}(B_2, n_2). \tag{2.1}$$

Obviously, we can deal with one of the terms in (2.1) as follows:

$$\begin{aligned} &\sum_{1\le\nu_1\le L} \cdots \sum_{1\le\nu_k\le L} S_{1,1}(B_1, n_1)S_{1,1}(B_2, n_2) \\ &\quad = 4C_0^2(1 - 2\omega)^2 B_1B_2(\log B_1 \log B_2)^{-2} \sum_{1\le\nu_1\le L} \cdots \sum_{1\le\nu_k\le L} \sum_{d|n_1} k(d) \sum_{l|n_2} k(l) \\ &\quad \ge 4C_0^2(1 - 4\omega)B_1B_2(\log B_1 \log B_2)^{-2} \sum_{d=1}^{\infty} \sum_{l=1}^{\infty} k(d)k(l)\#\{n_1, n_2 : d \mid n_1, l \mid n_2\} \\ &\quad \ge 4C_0^2(1 - 4\omega)B_1B_2(\log B_1 \log B_2)^{-2}L^k \\ &\quad \ge 1.74293(1 - 4\omega)B_1B_2(\log B_1 \log B_2)^{-2}L^k. \end{aligned} \tag{2.2}$$

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The second inequality holds since $k(1) = 1$, $\#\{n_1, n_2\} = L^k$ and $k(d) \geq 0$ for any positive integer d.

To deal with the other eight terms in (2.1), we give the following lemma.

Lemma 2.3. *We have*

$$\sum_{1\leq \nu_1 \leq L} \cdots \sum_{1\leq \nu_k \leq L} \Big(\sum_{d|n_i} k(d)\Big)^2 \leq (38.2229 + o(1))L^k \ll L^k.$$

for $i = 1, 2$.

Proof. Let $i = 1, 2$. From the definition of $k(d)$,

$$\Big(\sum_{d|n_i} k(d)\Big)^2 = \sum_{d|n_i} a(d),$$

where $a(d)$ is the multiplicative function defined by

$$a(p^e) = \begin{cases} 0 & p = 2 \text{ or } e \geq 2, \\ \dfrac{(p-1)^2}{(p-2)^2} - 1 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{1\leq \nu_1 \leq L} \cdots \sum_{1\leq \nu_k \leq L} \Big(\sum_{d|n_i} k(d)\Big)^2 \leq \sum_{d \leq B_i} a(d)\#\{n_i : d \mid n_i\}.$$

For an odd integer d,

$$\#\{\nu : 1 \leq \nu \leq L, 2^\nu \equiv m \pmod{d}\} \leq \frac{L}{\epsilon(d)} + 1,$$

where $\epsilon(d)$ is the order of 2 in the multiplicative group Z_d, and then

$$\#\{n_i : d \mid n_i\} \leq \frac{L^k}{\epsilon(d)} + L^{k-1}.$$

Hence

$$\sum_{1\leq \nu_1 \leq L} \cdots \sum_{1\leq \nu_k \leq L} \Big(\sum_{d|n_i} k(d)\Big)^2 \leq L^k \sum_{d \leq B_i} \frac{a(d)}{\epsilon(d)} + \sum_{d \leq B_i} a(d) L^{k-1}.$$

Following the method of proof of Lemma 4 in Gallagher [1], and setting

$$R = \prod_{x/2 < \epsilon \leq x} (2^\epsilon - 1),$$

we have

$$\begin{aligned} \sum_{x/2 < \epsilon(d) \leq x} \frac{a(d)}{\epsilon(d)} &\leq 2x^{-1} \sum_{d|R} a(d) \\ &\leq 2x^{-1} \prod_{p|R, p>2} \frac{(p-1)^2}{(p-2)^2} \end{aligned}$$

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$$\leq 2x^{-1}\Big(\prod_{p>2}\frac{(p-1)^2}{p(p-2)}\prod_{p|R}\frac{p}{p-1}\Big)^2$$
$$= 2x^{-1}C_0^{-2}\Big(\frac{R}{\phi(R)}\Big)^2.$$

According to inequality (3.9) of Liu *et al.* [6], for $x \geq 9$, we have $R/\phi(R) \leq e^{\gamma} \log x$, where $\gamma \leq 0.577216$ is Euler's constant. If we let x run over powers of 2 and sum the corresponding bound, then it follows that

$$\sum_{d}\frac{a(d)}{\epsilon(d)} \leq \sum_{m=1}^{4}\frac{1}{m}\sum_{\epsilon(d)=m} a(d) + 2C_0^{-2}e^{2\gamma}\sum_{r=3}^{\infty}\frac{(\log 2^r)^2}{2^r}$$
$$\leq \sum_{m=1}^{4}\frac{1}{m}\sum_{d\leq 2^m-1} a(d) + 31.4897$$
$$= 38.2229.$$

In addition,

$$\sum_{d\leq B_i} a(d) \leq \prod_{p|B_i, p>2}\frac{(p-1)^2}{(p-2)^2} \leq C_0^{-2}\prod_{p|B_i}\frac{p^2}{(p-1)^2} \ll (\log\log B_i)^2.$$

Therefore

$$\sum_{1\leq\nu_1\leq L}\cdots\sum_{1\leq\nu_k\leq L}\Big(\sum_{d|n_i} k(d)\Big)^2 \leq (38.2229 + o(1))L^k \ll L^k.$$

This completes the proof. □

Corollary 2.4. *For $i = 1, 2$,*

$$\sum_{1\leq\nu_1\leq L}\cdots\sum_{1\leq\nu_k\leq L}\Big(\frac{n_i}{\phi(n_i)}\Big)^2 \ll L^k.$$

Proof. Note that

$$\Big(\frac{n_i}{\phi(n_i)}\Big)^2 \ll \prod_{p|n_i, p\neq 2}\Big(\frac{p-1}{p-2}\Big)^2 = \sum_{d|n_i} a(d),$$

for $i = 1, 2$. □

Now a straightforward combination of Lemma 2.3, Corollary 2.4 and Cauchy's inequality yields the following corollary.

Corollary 2.5. *We have*

$$\sum_{1\leq\nu_1\leq L}\cdots\sum_{1\leq\nu_k\leq L} S_{1,2}(B_1, n_1)S_{1,2}(B_2, n_2) = O(e^{-\omega(\eta_1+\eta_2)}B_1B_2(\log B_1\log B_2)^{-2}L^k), \tag{2.3}$$

$$\sum_{1\leq\nu_1\leq L}\cdots\sum_{1\leq\nu_k\leq L} S_{1,1}(B_1, n_1)S_{1,2}(B_2, n_2) = O(e^{-\omega\eta_2}B_1B_2(\log B_1\log B_2)^{-2}L^k), \tag{2.4}$$

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and

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_{1,2}(B_1,n_1)S_{1,1}(B_2,n_2)=O(e^{-\omega\eta_1}B_1B_2(\log B_1\log B_2)^{-2}L^k). \quad (2.5)$$

To handle the remaining five terms which are related to $S_{1,3}(B_i,n_i)$, we need the following lemma whose proof is similar to that of inequality (31) in [3].

Lemma 2.6. *For a particular pair of characters $\chi_i,\chi_i'\in\mathfrak{P}(\eta_i)$,*

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\Big(\frac{n_i}{\phi(n_i)}\Big)^2\Big(\frac{m_i}{(m_i,n_i)}\Big)^{-2/3}\ll L^k(\log\log B_i)^{-1/5}.$$

Proof. We first consider

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\sum_{g\mid n_i}\Big(\frac{n_i}{\phi(n_i)}\Big)^2.$$

Note, as in the proof of Corollary 2.4, that

$$\Big(\frac{n_i}{\phi(n_i)}\Big)^2\ll\sum_{d\mid n_i}a(d).$$

It follows as in Lemma 2.3 that

$$\begin{aligned}\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\sum_{g\mid n_i}\Big(\frac{n_i}{\phi(n_i)}\Big)^2&\ll\sum_{d\le B_i}a(d)\#\{n_i:[g,d]\mid n_i\}\\&\ll\sum_{d\le B_i}a(d)\Big(L^{k-1}+\frac{L^k}{\epsilon([g,d])}\Big),\end{aligned}$$

where $[g,d]$ denotes the least common multiple of g and d. Observing that, according to the proof of Lemma 2.3,

$$\sum_{d\le B_i}a(d)\ll(\log\log B_i)^2\ll L^{1/2},$$

and

$$\sum_{d\le B_i}\frac{a(d)}{\epsilon([g,d])}\le\frac{1}{\epsilon(g)^{1/2}}\sum_{d\le B_i}\frac{a(d)}{\epsilon(d)^{1/2}}\ll\frac{1}{\epsilon(g)^{1/2}},$$

we can write

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\sum_{g\mid n_i}\Big(\frac{n_i}{\phi(n_i)}\Big)^2\ll L^k(L^{-1/2}+\epsilon(g)^{-1/2})\ll L^k(L^{-1/2}+(\log g)^{-1/2})$$

since $g\le 2^{\epsilon(g)}-1$.

Write $m_i=2^{u_i}f_i$. Put $g_i=(f_i,n_i)$ so that

$$\frac{m_i}{(m_i,n_i)}\ge\frac{f_i}{(f_i,n_i)}=\frac{f_i}{g_i}.$$

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Then

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\Big(\frac{n_i}{\phi(n_i)}\Big)^2\Big(\frac{m_i}{(m_i,n_i)}\Big)^{-2/3}\le\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\Big(\frac{n_i}{\phi(n_i)}\Big)^2\Big(\frac{f_i}{g_i}\Big)^{-2/3}.$$

Let x_i be a parameter to be fixed in due course. Then the terms in which $g_i\le f_i/x_i$ contribute at most

$$x_i^{-2/3}\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\Big(\frac{n_i}{\phi(n_i)}\Big)^2\ll x_i^{-2/3}L^k,$$

by Corollary 2.4. The remaining terms contribute at most

$$\begin{aligned}&\sum_{g_i|f_i,g_i\ge f_i/x_i}\Big(\frac{f_i}{g_i}\Big)^{-2/3}\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\sum_{g_i|n_i}\Big(\frac{n_i}{\phi(n_i)}\Big)^2\\&\quad\ll L^k\sum_{g_i|f_i,g_i\ge f_i/x_i}\Big(\frac{f_i}{g_i}\Big)^{-1/2}(L^{-1/2}+(\log g_i)^{-1/2})\\&\quad\ll L^k\sum_{g_i|f_i,g_i\ge f_i/x_i}(L^{-1/2}+(\log f_i)^{-1/2})\\&\quad\ll x_iL^k(L^{-1/2}+(\log f_i)^{-1/2}).\end{aligned}$$

Choosing $x_i=(L^{-1/2}+(\log f_i)^{-1/2})^{-3/5}$,

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\Big(\frac{n_i}{\phi(n_i)}\Big)^2\Big(\frac{m_i}{(m_i,n_i)}\Big)^{-2/3}\ll L^k(L^{-1/2}+(\log f_i)^{-1/2})^{2/5}.$$

Then the bound $\log f_i\gg\log\log B_i$ (see [3, p. 552]) yields

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\Big(\frac{n_i}{\phi(n_i)}\Big)^2\Big(\frac{m_i}{(m_i,n_i)}\Big)^{-2/3}\ll L^k(\log\log B_i)^{-1/5}.$$

This completes the proof. □

Using Lemmas 2.3, 2.6, Corollary 2.4, Cauchy's inequality and

$$\#\mathfrak{P}(\eta_i)\le e^{6\eta_i}$$

(see [3, Lemma 5]), we deduce the following corollary.

Corollary 2.7. *We have*

$$\begin{aligned}&\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}S_{1,1}(B_1,n_1)S_{1,3}(B_2,n_2)\\&\quad=O(B_1B_2(\log B_1\log B_2)^{-2}L^ke^{12\eta_2}(\log\log B_2)^{-1/5}),\end{aligned}\tag{2.6}$$

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_{1,3}(B_1,n_1)S_{1,1}(B_2,n_2)$$
$$= O(B_1B_2(\log B_1 \log B_2)^{-2}L^k e^{12\eta_1}(\log\log B_1)^{-1/5}), \tag{2.7}$$
$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_{1,2}(B_1,n_1)S_{1,3}(B_2,n_2)$$
$$= O(B_1B_2(\log B_1 \log B_2)^{-2}L^k e^{-\omega\eta_1+12\eta_2}(\log\log B_2)^{-1/5}), \tag{2.8}$$
$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_{1,3}(B_1,n_1)S_{1,2}(B_2,n_2)$$
$$= O(B_1B_2(\log B_1 \log B_2)^{-2}L^k e^{12\eta_1-\omega\eta_2}(\log\log B_1)^{-1/5}), \tag{2.9}$$

and

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_{1,3}(B_1,n_1)S_{1,3}(B_2,n_2)$$
$$= O(B_1B_2(\log B_1 \log B_2)^{-2}L^k e^{12(\eta_1+\eta_2)}(\log\log B_1 \log\log B_2)^{-1/5}). \tag{2.10}$$

Inserting (2.2)–(2.10) into (2.1), with $\eta_i = (\log\log B_i)^{1/61}$ for $i = 1, 2$, we deduce Proposition 2.1.

3. Completion of the proof

We begin by providing an upper bound for $R_{3,3}$, as contained in the following proposition.

PROPOSITION 3.1. *We have*

$$|R_{3,3}| = \left|\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_3(B_1,n_1)S_3(B_2,n_2)\right|$$
$$\le 67.3739(1+o(1))\lambda^{k-4}B_1B_2(\log B_1 \log B_2)^{-2}L^k.$$

PROOF. Applying Cauchy's inequality to the definition of $T(\alpha)$ yields

$$|T(\alpha_1+\alpha_2)| \le \sqrt{|T(2\alpha_1)T(2\alpha_2)|}.$$

Hence

$$\left|\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_3(B_1,n_1)S_3(B_2,n_2)\right|$$
$$= \left|\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\int_{\mathfrak{m}_1\backslash\mathcal{A}_\lambda} S^2(\alpha,B_1)e(-n_1\alpha)\,d\alpha\int_{\mathfrak{m}_2\backslash\mathcal{A}_\lambda} S^2(\alpha,B_2)e(-n_2\alpha)\,d\alpha\right|$$
$$= \left|\int_{\mathfrak{m}_1\backslash\mathcal{A}_\lambda}\int_{\mathfrak{m}_2\backslash\mathcal{A}_\lambda} S^2(\alpha_1,B_1)S^2(\alpha_2,B_2)T^k(\alpha_1+\alpha_2)e(-B_1\alpha_1-B_2\alpha_2)\,d\alpha_1\,d\alpha_2\right|$$

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$$\le \int_{\mathfrak{m}_1\backslash\mathcal{A}_\lambda}\int_{\mathfrak{m}_2\backslash\mathcal{A}_\lambda} |S^2(\alpha_1,B_1)S^2(\alpha_2,B_2)T^4(\alpha_1+\alpha_2)|\cdot|T(\alpha_1+\alpha_2)|^{k-4}\,d\alpha_1\,d\alpha_2$$
$$\le \prod_{i=1}^{2}\int_{\mathfrak{m}_i\backslash\mathcal{A}_\lambda} |S^2(\alpha_i,B_i)T^2(2\alpha_i)|\cdot|T(2\alpha_i)|^{(k-4)/2}\,d\alpha_i.$$

Note that, for $\alpha_i\in\mathfrak{m}_i\setminus\mathcal{A}_\lambda$ and sufficiently large B_i,

$$|T(2\alpha_i)|\le|T(\alpha_i)|+2\le\lambda L+2\le(1+o(1))\lambda L.$$

Then

$$\Big|\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}S_3(B_1,n_1)S_3(B_2,n_2)\Big|$$
$$\le(1+o(1))(\lambda L)^{k-4}\prod_{i=1}^{2}\int_{\mathfrak{m}_i\backslash\mathcal{A}_\lambda}|S^2(\alpha_i,B_i)T^2(2\alpha_i)|\,d\alpha_i. \tag{3.1}$$

A careful examination of the argument of [3, Lemma 9] yields

$$\int_{\mathfrak{m}_i\backslash\mathcal{A}_\lambda}|S^2(\alpha_i,B_i)T^2(2\alpha_i)|\,d\alpha_i$$
$$\le(C_0(C_1-2)C_2+1.1056\log 2+o(1))B_iL^2(\log B_i)^{-2}. \tag{3.2}$$

where $C_1=7.8209$ according to Wu [11, Theorem 1], $C_2\le1.93657$ according to Pintz [9, Lemma 2′] and 1.1056 comes from the Chebyshev inequality for $\pi(B_i)$. Inserting (3.2) into (3.1),

$$\Big|\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}S_3(B_1,n_1)S_3(B_2,n_2)\Big|$$
$$\le 67.3739(1+o(1))(\lambda L)^{k-4}\frac{B_1B_2L^4}{(\log B_1\log B_2)^2}.$$

This completes the proof of Proposition 3.1. □

To complete the proof of the main theorem, it remains to estimate the other seven terms in (1.6), which we will do now.

Firstly, we consider

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}S_1(B_1,n_1)S_3(B_2,n_2),$$

which equals, according to Lemma 2.2,

$$2C_0(1-2\omega)B_1(\log B_1)^{-2}\sum_{1\le\nu_1\le L}\sum_{1\le\nu_k\le L}\sum_{d|n_1}k(d)\int_{\mathfrak{m}_2\backslash\mathcal{A}_\lambda}S(\alpha,B_2)e(-n_2\alpha)\,d\alpha$$
$$+O\Big(e^{-\omega\eta_1}B_1(\log B_1)^{-2}\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\frac{n_1}{\phi(n_1)}\cdot\int_{\mathfrak{m}_2\backslash\mathcal{A}_\lambda}S(\alpha,B_2)e(-n_2\alpha)\,d\alpha\Big)$$

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$$+ O\Big(B_1(\log B_1)^{-2} \sum_{\chi_1,\chi_1'\in\mathfrak{P}(\eta_1)} \sum_{1\le\nu_1\le L} \sum_{1\le\nu_k\le L} \frac{n_1}{\phi(n_1)}\Big(\frac{m_1}{(m_1,n_1)}\Big)^{-1/3}$$
$$\times \int_{\mathfrak{m}_2\setminus\mathcal{A}_\lambda} S(\alpha,B_2)e(-n_2\alpha)\,d\alpha\Big)$$
$$= T_1 + T_2 + T_3,$$

say. By Lemma 2.3, Proposition 3.1 and Cauchy's inequality,

$$T_1 \le 2C_0(1-2\omega)B_1(\log B_1)^{-2}\sqrt{\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\Big(\sum_{d|n_1}k(d)\Big)^2}$$
$$\times\sqrt{\Big|\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\Big(\int_{\mathfrak{m}_2\setminus\mathcal{A}_\lambda} S(\alpha,B_2)e(-n_2\alpha)\,d\alpha\Big)^2\Big|}$$
$$\le 2C_0(1+o(1))\frac{B_1}{(\log B_1)^2}\sqrt{38.2229L^k}\cdot\sqrt{\frac{67.3739\lambda^{k-4}B_2^2L^k}{(\log B_2)^4}}$$
$$\le 67.0029(1+o(1))\lambda^{k/2-2}B_1B_2L^k(\log B_1\log B_2)^{-2}.$$

Similarly, combining Corollary 2.4, Proposition 3.1 and Cauchy's inequality yields

$$T_2 = o(B_1B_2L^k(\log B_1\log B_2)^{-2}),$$

and combining Lemma 2.6, Proposition 3.1 and Cauchy's inequality yields

$$T_3 = o(B_1B_2L^k(\log B_1\log B_2)^{-2}).$$

In conclusion,

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_1(B_1,n_1)S_3(B_2,n_2)$$
$$\le 67.0029(1+o(1))\lambda^{k/2-2}B_1B_2L^k(\log B_1\log B_2)^{-2}.$$

In the same fashion, we can deduce that

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_3(B_1,n_1)S_1(B_2,n_2)$$
$$\le 67.0029(1+o(1))\lambda^{k/2-2}B_1B_2L^k(\log B_1\log B_2)^{-2}.$$

Next, for $t = 1, 2, 3$, $s = 1, 3$,

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_2(B_1,n_1)S_t(B_2,n_2)$$
$$\le \sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L}\int_{\mathfrak{m}_1\cap\mathcal{A}_\lambda}|S(\alpha,B_1)|^2\,d\alpha\cdot\int_0^1|S(\alpha,B_2)|^2\,d\alpha$$

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$$\ll \sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} \text{meas}\,\mathcal{A}_\lambda B_1^{2\theta}\frac{B_2}{\log B_2}$$
$$\ll B_1^{1-\omega}B_2L^k, \tag{3.3}$$

provided

$$\text{meas}\mathcal{A}_\lambda = B_1^{-E(\lambda)} \ll B_1^{1-\omega-2\theta}$$

with

$$\theta = \begin{cases} 263/308 & \text{in general,} \\ 3/4 & \text{under the generalised Riemann hypothesis,} \end{cases}$$

that is,

$$\lambda = \begin{cases} 0.716344 & \text{in general,} \\ 0.862327 & \text{under the generalised Riemann hypothesis.} \end{cases} \tag{3.4}$$

The second inequality of (3.3) follows from

$$\int_0^1 |S(\alpha, B_i)|^2\,d\alpha = \sum_{\omega B_i<p\le B_i} 1 \ll \frac{B_i}{\log B_i},$$

which is an immediate consequence of the Parseval identity and the prime number theorem, and the last inequality, especially the value of λ, follows from Pintz [10, Corollary 1] and Liu [7, Lemma 3.3]. Similarly, for $j = 1, 3$,

$$\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_j(B_1, n_1)S_2(B_2, n_2) \ll B_1B_2^{1-\omega}L^k \tag{3.5}$$

with (3.4).

Now we reach our conclusion:

$$\begin{aligned} R(B_1, B_2) &= \sum_{j,k=1}^{3}\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_j(B_1, n_1)S_k(B_2, n_2) \\ &\ge \sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_1(B_1, n_1)S_1(B_2, n_2) \\ &\quad - \Big|\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_3(B_1, n_1)S_3(B_2, n_2)\Big| \\ &\quad - \Big|\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_1(B_1, n_1)S_3(B_2, n_2)\Big| \\ &\quad - \Big|\sum_{1\le\nu_1\le L}\cdots\sum_{1\le\nu_k\le L} S_3(B_1, n_1)S_1(B_2, n_2)\Big| + O(B_1B_2^{1-\omega}L^k + B_1^{1-\omega}B_2L^k). \end{aligned}$$

Therefore,

$$R(B_1, B_2) > 0$$

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if

$$1.74293(1-4\omega) > 2\times 67.0029\lambda^{k/2-2} + 67.3739\lambda^{k-4},$$

and ω is a sufficiently small constant. Hence

$$k = \begin{cases} 63 & \text{in general,} \\ 31 & \text{under the generalised Riemann hypothesis} \end{cases}$$

is admissible.

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