A REPORT ON STABLE GRAPHS

D. A. HOLTON

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1. Introduction

It is the aim of this paper to introduce a new concept relating various subgroups of the automorphism group of a graph to corresponding subgraphs. Throughout $\mathscr{G}$ will denote a (Michigan) graph on a vertex set $V(|V|=n)$ and $\Gamma(\mathscr{G})=G$ will be the automorphism group of $\mathscr{G}$ considered as a permutation group on V. E_n, C_n, D_n and S_n are the identity, cyclic, dihedral, and symmetric groups acting on a set of size n, while $S_p(q)$ is the permutation group of pq objects which is isomorphic to S_p but is q-fold in the sense that the objects are permuted q at a time [6]. $H \leqq G$ means that H is a subgroup of G. Other group concepts can be found in Wielandt [7]. The graphs $\mathscr{G}_1 \cup \mathscr{G}_2, \mathscr{G}_1 + \mathscr{G}_2, \mathscr{G}_1 \times \mathscr{G}_2$, and $\mathscr{G}_1[\mathscr{G}_2]$ along with their corresponding groups are as defined in, for example, Harary [4]. Finally we use $\mathscr{K}_n$ for the complete graph on n vertices.

2. Stable Graphs

Let $\mathscr{G}_{v_1 v_2 \cdots v_k}$ be the graph obtained from $\mathscr{G}$ by removing the vertices $v_1, v_2, \cdots, v_k$ and any edges attached to these vertices, from $\mathscr{G}$. As in Wielandt [7] we let $G_{v_1 v_2 \cdots v_k}$ be the set of permutations of G which keep $v_1, v_2, \cdots, v_k$ individually fixed. Then if for $\mathscr{G}$ there exists a sequences $\mathscr{S} = \{v_1, v_2, \cdots, v_n\}$ such that $\Gamma(\mathscr{G}_{v_1 v_2 \cdots v_k}) = G_{v_1 v_2 \cdots v_k}$ for all $k = 1, 2, 3, \cdots, n$ we say that $\mathscr{G}$ is *stable*. Otherwise it is *unstable*. We refer to the sequence $\mathscr{S}$ as the *stabilising sequence* of $\mathscr{G}$.

The concept of stability is not vacuous.

PROPOSITION 1. *$\mathscr{K}_n$ is stable.*

PROOF. For any $v \in$, $V(\mathscr{K}_n)_v = \mathscr{K}_{n-1}$. Now since $\Gamma(\mathscr{K}_n) = S_n$, $[\Gamma(\mathscr{K}_n)]_v = \Gamma[(\mathscr{K}_n)_v] = S_{n-1}$. Then induction shows that any ordering of V is a stability sequence of $\mathscr{K}_n$.

The following four propositions are readily established.

PROPOSITION 2. *If $\mathcal{G}_v$ is stable for some $v \in V$, and $\Gamma(\mathcal{G}_v) = G_v$ then $\mathcal{G}$ is stable.*

PROPOSITION 3. *If $\mathcal{G}$ is stable then there exists a $v \in V$ such that $\mathcal{G}_v$ is stable.*

PROPOSITION 4. *If $\tilde{\mathcal{G}}$ is the complement of $\mathcal{G}$ on V we have $\mathcal{G}$ is stable if and only if $\tilde{\mathcal{G}}$ is stable.*

PROPOSITION 5. *If $\mathcal{G}_v$ is unstable for all $v \in V$ then $\mathcal{G}$ is unstable.*

3. A class of unstable graphs

Here we show that, as was to be expected, the attribute of stability depends to a large extent, but not completely, on the automorphism group of the graph.

LEMMA 1. *If $G \leqq D_n$ then $G_v \leqq D_{n-1}$ for all $v \in V$.*

PROOF. Now $G_v \leqq (D_n)_v$ so it is enough to show that $(D_n)_v \leqq D_{n-1}$.

By the orbit-stabiliser relation, and bearing in mind that D_n is transitive we have

$$\left|(D_n)_v\right| = \frac{|D_n|}{|v^{D_n}|} = \frac{2n}{n} = 2.$$

The elements of order two in D_n are the product of $(n-1)/2$ 2–cycles if n is odd and the product or $n/2$ or $(n-2)/2$ 2–cycles if n is even and hence an element of order two in D_n which fixes one vertex belongs to D_{n-1}.

Clearly then $(D_n)_v \leqq D_{n-1}$ since $D_{n-1} = \langle (123 \cdots n-1), (1\ n-1)(2\ n-2) \cdots \rangle$ and we would have $(D_n)_v = \langle (1\ n-1)(2\ n-2) \cdots \rangle$ where the members of V have been reordered if necessary.

We can now prove

THEOREM 1. *If $\mathcal{G}$ is a graph on n vertices V with $\Gamma(\mathcal{G}) = G \leqq D_n$ and if $n \geqq 5$ then $\mathcal{G}$ is unstable.*

PROOF. If $n = 5$ then G could be $D_5, C_5, S_2(2) + E_1$, or E_5. However C_5 and E_5 do not appear as automorphism groups for any graph on five vertices.

(i) D_5. The only graph whose automorphism group is D_5 is shown in Figure 1(a). Now for any vertex v_1 of (a) $\Gamma(\mathcal{G}_{v_1}) = (D_5)_{v_1}$, but for another vertex v_2 we find that $(D_5)_{v_1 v_2} = E_3$ and $\Gamma(\mathcal{G}_{v_1 v_2}) \neq E_3$. Hence the pentagon of Figure 1(a) is unstable.

(ii) $S_2(2) + E_1$. The graphs (b) to (f) inclusive in Figure 1 all have $S_2(2) + E_1$ as their automorphism groups.

(b), (c). The graphs (b) and (c) are complements and so by Proposition 4 it is sufficient to discuss (b).

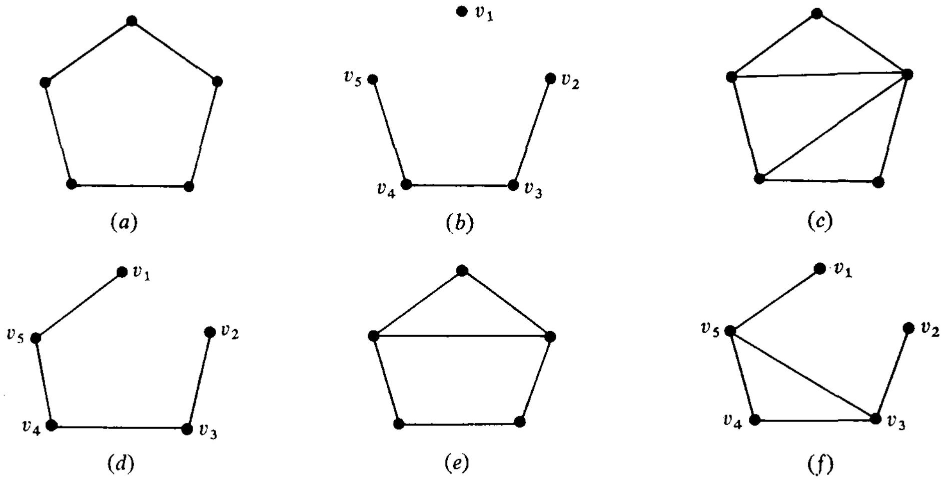

Figure 1.

Clearly $\Gamma(\mathscr{G}_{v_1}) = S_2(2) = (S_2(2)+E_1)_{v_1}$. But $(S_2(2)+E_1)_{v_1v_i} = E_3$ for $i = 2,3,4,5$ and $\Gamma(\mathscr{G}_{v_1v_i}) \neq E_3$. Hence ($b$) (and ($c$)) is unstable.

(d), (e). Again these graphs are complements so we discuss (d) only.

Here $\Gamma(\mathscr{G}) = S_2(2)+E_1 = \{e, (v_1v_2)(v_3v_5)\}$ and hence $(S_2(2)+E_1)_{v_4} = S_2(2)$. But $\Gamma(\mathscr{G}_{v_4}) = S_2[S_2] \neq S_2(2)$. Further for $i \neq 4$ $(S_2(2)+E_1)_{v_i} = E_4$ but $\Gamma(\mathscr{G}_{v_i}) \neq E_4$. Thus ($d$) is not stable.

(f). In this case $\Gamma(\mathscr{G}) = \{e, (v_1v_2)(v_3v_5)\}$. For $i = 1,2$ $\Gamma(\mathscr{G}_{v_i}) = \{e, (v_4v_5\})$ which is not of the form $(S_2(2)+E_1)_{v_i}$. With $j = 3,5$ $\Gamma(\mathscr{G}_{v_j}) = \{e, (v_iv_4)\}$ $i = 1,2$, and again this is not a subgroup of $S_2(2)+E_1$. However $\Gamma(\mathscr{G}_{v_4}) = (S_2(2)+E_1)_{v_4}$, but then $(S_2(2)+E_1)_{v_4v_l} = E_3$ $(l = 1,2,3,5)$ and $\Gamma(\mathscr{G}_{v_4v_l}) \neq E_3$.

Thus the theorem is true for $n = 5$.

We now assume that the theorem is true for $n-1 \geqq 5$. Let $|V| = n$ and $V' = \{v \in V : G_v = \Gamma(\mathscr{G}_v)\}$. We need not worry about the vertices of $V - V'$ since in trying to find a first member of the stabilising sequence for $\mathscr{G}$ these vertices must be ignored.

But $G_v \leqq D_{n-1}$ by Lemma 1 and so by the induction hypotheses $\mathscr{G}_v$ is unstable. This is true for all $v \in V - V'$ and so $\mathscr{G}$ is unstable.

So we can see that the automorphism group of the graph plays a large role in the determination of stability. However it is not the sole factor involved.

Consider the graphs of Figure 2. For both graphs the automorphism group is $S_2(2)+S_2$. But in fact (a) is stable while (b) is not. For (a) $\Gamma(\mathscr{G}) = \{e, (v_2v_5), (v_1v_4)(v_3v_6), (v_1v_4)(v_2v_5)(v_3v_6)\}$ and $v_1, v_3, v_4, v_6, v_2, v_5$ is a stabilising sequence.

For (b) $\Gamma(\mathscr{G}) = \{e, (v_5v_6), (v_1v_4)(v_2v_3), (v_1v_4)(v_2v_3)(v_5v_6)\}$. Trouble arises in the stabilising sequence when one tries to remove one of the points v_1, v_2, v_3, or

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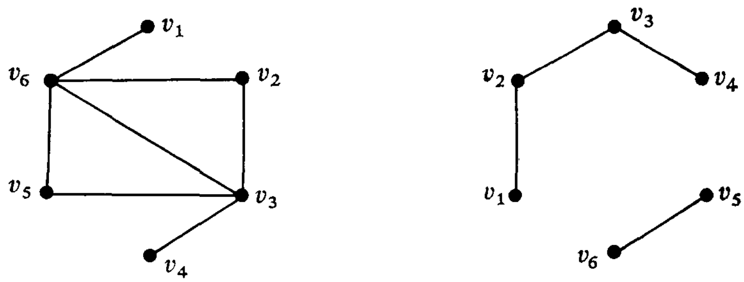

Figure 2.

v_4 from the graph. (As it happens $\mathscr{G}_{v_5 v_6}$ is unstable and so the unstability of $\mathscr{G}$ is guaranteed by Theorem 5.)

At this stage it can only be said that most likely connectedness plays a part in stability. One sees that for (a) $\mathscr{G}$ and $\tilde{\mathscr{G}}$ are connected while for (b) $\tilde{\mathscr{G}}$ is connected while $\mathscr{G}$ obviously is not. We conjecture then that for graphs $\mathscr{G}$ and $\mathscr{K}$, if $\mathscr{G}$, $\tilde{\mathscr{G}}$, $\mathscr{K}$ and $\tilde{\mathscr{K}}$ are all connected and if $\Gamma(\mathscr{G}) = \Gamma(\mathscr{K})$ then $\mathscr{G}$ is stable if $\mathscr{K}$ is. Similar conjectures arise for unstability and also in the cases where $\mathscr{G}$ and $\mathscr{K}$ are connected and $\tilde{\mathscr{G}}$ and $\tilde{\mathscr{K}}$ are not.

4. The union of stable graphs

Here we confine our attention to stable graphs and show that the union of n graphs $(\bigcup_{i=1}^{n} \mathscr{G}_i)$ is stable if and only if each of the graphs $\mathscr{G}_i$ is. We first note the well known Theorems 2 and 3, and prove Theorem 4.

THEOREM 2. $\Gamma(\mathscr{G}_1 \cup \mathscr{G}_2) = \Gamma(\mathscr{G}_1) + \Gamma(\mathscr{G}_2)$ *if and only if no component of* $\mathscr{G}_1$ *is isomorphic to a component of* $\mathscr{G}_2$.

THEOREM 3. *Frucht* [2]: *If* $\mathscr{G}_1 = \mathscr{G}_2$ *and* $\mathscr{G}_1$ *is connected then* $\Gamma(\mathscr{G}_1 \cup \mathscr{G}_2) = S_2[\Gamma(\mathscr{G}_1)]$.

The extensions of the above two theorems to the union of n graphs are clear.

THEOREM 4[1]. *For all positive integers* n, $S_n[G]_v = S_{n-1}[G] + G_v$.

PROOF. Let S_n act on $\Omega = \{1,2,3,\cdots,n\}$ and G act on

$\Sigma = \{1,2,3,\cdots,m\}$. Form $\Omega \times \Sigma$ and let
$\Sigma_i = \{(i,j) : i \in \Omega$ is fixed and $j \in \Sigma\}$. Clearly Σ_i is just a copy of Σ.

Now $\Sigma_i^g = \Sigma_{i'}$ for all $g \in S_n[G]$ and for some $i' \in \Omega$. Further if $v = (\alpha, \beta)$ and $h \in S_n[G]_v$ then $\Sigma_\alpha^h = \Sigma_\alpha$. But $S_n[G]_v$ acts on this set as G_v (if we identify β with (α, β) here) and as $S_{n-1}[G]$ on $\Sigma' = (\Omega \times \Sigma) - \Sigma_\alpha$.

[1] The proof here was suggested by Prof. W. Jonsson of McGill University.

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Then since $\Omega \times \Sigma = \Sigma' \cup \Sigma_\alpha$ and $\Sigma' \cap \Sigma_\alpha = \phi$ it can be seen that $S_n[G]_v = S_{n-1}[G] + G_v$.

We can now prove

THEOREM 5. *$\bigcup_{i=1}^n \mathscr{G}i$ is stable if and only if each $\mathscr{G}i$ is stable.*

PROOF. (*a*) We first assume that all the $\mathscr{G}_i$ are connected and ordered so that for $j < k$, $|V_j| \leqq |V_k|$. ($\Leftarrow$) If $\mathscr{G}_i$ has stabilising sequence $\{v_{i1}, v_{i2}, \cdots, v_{im_i}\}$ then $v_{11}, v_{12}, \cdots, v_{1m_1}, v_{21}, \cdots, v_{2m_2}, \cdots, v_{n1}, \cdots, v_{nm_n}\}$ is a stabilising sequence of $\bigcup_{i=1}^n \mathscr{G}_i$. For if $\mathscr{K} = \bigcup_{i=1}^n \mathscr{G}_i$ then $\mathscr{K}_{v_{11}} = (\mathscr{G}_1)_{v_{11}} \cup (\bigcup_{i=2}^n \mathscr{G}_i)$ and by the stability of $\mathscr{G}_1$ and using theorems 2, 3 and 4 we have $\Gamma(\mathscr{K}_{v_{11}}) = \Gamma(\mathscr{K})_{v_{11}}$. Induction then produces the required result.

($\Rightarrow$) Let $\{v_1, v_2, \cdots, v_m\}$ be a stabilising sequence of $\mathscr{K} = \bigcup_{i=1}^n \mathscr{G}_i$. Assume $v_1 \in \mathscr{G}_1$ (reorder if necessary). Since $\mathscr{K}$ is stable $\Gamma(\mathscr{K}_{v_1}) = \Gamma(\mathscr{K})_{v_1}$.

$$\text{Now} \qquad \Gamma(\mathscr{K}_{v_1}) = \Gamma\left([\mathscr{G}_1]_{v_1} \cup \bigcup_{i=2}^n \mathscr{G}_i\right)$$
$$= \Gamma([\mathscr{G}_1]_{v_1}) + \Gamma\left(\bigcup_{i=2}^n \mathscr{G}_i\right) \qquad \text{provided}$$

$[\mathscr{G}_1]_{v_1} \neq \mathscr{G}_{i_0}$ for some $i_0 = 2, 3, \cdots, n$, when a wreath product would occur

$$\text{Further} \quad \Gamma(\mathscr{K}) = \begin{cases} S_r[\mathscr{G}_1] + \Gamma(\bigcup \mathscr{G}_i) \\ \Gamma(\mathscr{G}_1) + \Gamma\left(\bigcup_{i=2}^n \mathscr{G}_i\right) \end{cases}$$

depending on whether $\mathscr{K}$ contains r connected components identical with $\mathscr{G}_1$ or not. In the first case the $\bigcup \mathscr{G}_i$ is over all $\mathscr{G}_i \neq \mathscr{G}_1$.

$$\text{Then} \qquad \Gamma(\mathscr{K})_{v_1} = \begin{cases} (S_r[\mathscr{G}_1] + \Gamma(\bigcup \mathscr{G}_i))_{v_1} \\ \left(\Gamma(\mathscr{G}_1) + \Gamma\left(\bigcup_{i=2}^n \mathscr{G}_i\right)\right)_{v_1} \end{cases}$$
$$= \begin{cases} \Gamma(\mathscr{G}_1)_{v_1} + S_{r-1}[\mathscr{G}_1] + \Gamma(\bigcup \mathscr{G}_i) \\ \Gamma(\mathscr{G}_1)_{v_1} + \Gamma\left(\bigcup_{i=2}^n \mathscr{G}_i\right) \end{cases}$$
$$= \Gamma(\mathscr{G}_1)_{v_1} + \Gamma\left(\bigcup_{i=2}^n \mathscr{G}_i\right).$$

And since $\mathscr{K}$ is stable we have $\Gamma(\mathscr{G}_1)_{v_1} = \Gamma([\mathscr{G}_1])_{v_1})$ provided $[\mathscr{G}_1]_{v_1} \neq \mathscr{G}_{i_0}$.

If $[\mathscr{G}_1]_{v_1} = \mathscr{G}_{i_0}$ then

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$$\Gamma(\mathscr{K}_{v_1}) = S_u[\Gamma([\mathscr{G}_1]_{v_1})] + \Gamma(\cup_{i\in A}\mathscr{G}_i)$$

and $$\Gamma(\mathscr{K})_{v_1} = (S_{u-1}[\Gamma(\mathscr{G}_{i_0})] + \Gamma(\cup_{i\in B}\mathscr{G}_i))_{v_1}$$

$$= S_{u-1}[\Gamma(\mathscr{G}_{i_0})] + \Gamma(\mathscr{G}_1)_{v_1} + \Gamma(\cup_{i\in A}\mathscr{G}_i).$$

Here the union over A excludes only the u copies of $\mathscr{G}_{i_0}$ and that over B excludes the $u-1$ copies of $\mathscr{G}_{i_0}$.

But $S_{u-1}[\Gamma([\mathscr{G}_1]_{v_1}] = S_{u-1}[\Gamma(\mathscr{G}_{i_0})]$. However $S_u[\Gamma([\mathscr{G}_1]_{v_1})] \neq S_{u-1}[\Gamma([\mathscr{G}_1]_{v_1})] + \Gamma(\mathscr{G}_1)_{v_1}$ and so $\Gamma(\mathscr{K}_{v_1}) \neq \Gamma(\mathscr{K})_{v_1}$ which contradicts v_1's position in the stabilising sequence of $\mathscr{K}$. Hence $[\mathscr{G}_1]_{v_1} \neq \mathscr{G}_{i_0}$.

The proof is again completed by induction. We note that a stabilising sequence for $\mathscr{G}_{i'}$ can be obtained from that of $\mathscr{K}$ by taking the vertices of $\mathscr{G}_{i'}$ in the order that they appear in the stabilising sequence of $\mathscr{K}$.

(*b*) If $\mathscr{G}_i$ not connected $\mathscr{G}_i = \cup_{j=1}^{s} \mathscr{F}_{ij}$ where the $\mathscr{F}_{ij}$ are connected.

By part (*a*) $\mathscr{G}_i$ is stable if and only if each $\mathscr{F}_{ij}$ is stable. But $\mathscr{K} = \cup_{i=1}^{n}\mathscr{G}_i = \cup_{i=1}^{n} \cup_{j=1}^{s} \mathscr{F}_{ij}$. So if $\mathscr{K}$ is stable then each $\mathscr{F}_{ij}$ is by part (*a*) and so each $\mathscr{G}_i$ is. On the other hand when each $\mathscr{G}_i$ is stable so is each $\mathscr{F}_{ij}$ and by (*a*) so is $\mathscr{K}$.

COROLLARY. $\mathscr{G}_1 + \mathscr{G}_2 + \cdots + \mathscr{G}_n$ is stable if and only if each $\mathscr{G}_i$ is stable.

PROOF. Now the complement of $\mathscr{G}_1 + \mathscr{G}_2 + \cdots + \mathscr{G}_n = \cup_{i=1}^{n} \tilde{\mathscr{G}}_i$. So by the theorem the complement of $\mathscr{G}_1 + \mathscr{G}_2 + \cdots \mathscr{G}_n$ is stable if and only if each $\tilde{\mathscr{G}}_i$ is.

The corollary then follows by application of Proposition 4.

One would hope that the result of Theorem 6 and its corollary could be extended to the Cartesian product of two (or more) graphs. However consider the following counter example.

The Cartesian product [3], $\mathscr{G}_1 \times \mathscr{G}_2$, of $\mathscr{G}_1$ and $\mathscr{G}_2$ is as shown in Figure 3.

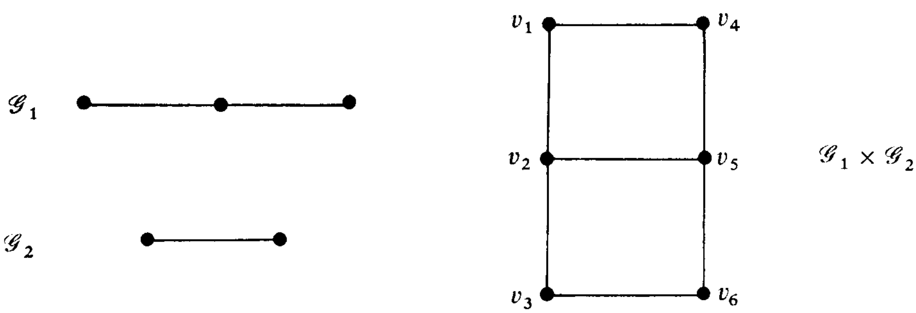

Figure 3.

$\mathscr{G}_1$ and $\mathscr{G}_2$ are quickly seen to be stable. But $\Gamma(\mathscr{G}_1 \times \mathscr{G}_2) < D_6$ and so by Theorem 1, $\mathscr{G}_1 \times \mathscr{G}_2$ is unstable.

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5. Special examples

We consider below the stability of three well known graphs.

(*a*) *Petersen's Graph is Unstable.*

The automorphism group of the Petersen graph, P, Figure 4, is well known [1]. It has degree 10, order 120, is isomorphic to S_5 and $\Gamma(P) = \langle (1429)(3765)(8\ 10),\ (13)(46)(58) \rangle$.

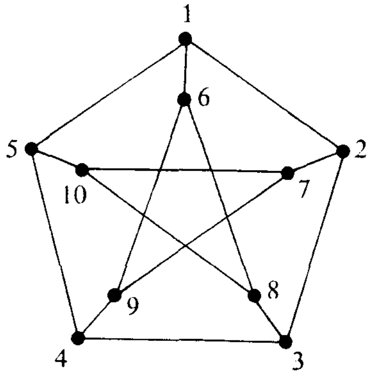

Figure 4.

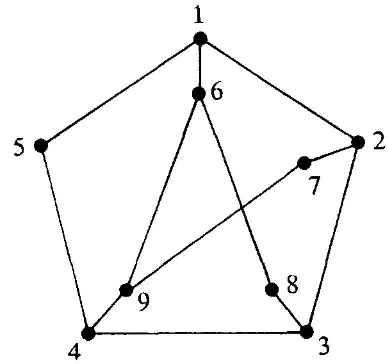

Figure 5.

Now $P_{v_1} = P_{v_2}$ for all $v_1, v_2 \in V$ the vertex set of P, and so to test for stability we remove an arbitrary point, 10 say. Figure 5 shows P_{10}.

After some work we have $\Gamma(P_{10}) = \Gamma(P)_{10} = \langle (169432)\ (587), (14)(23)(69)(78) \rangle$, and $|\Gamma(P_{10})| = 12$.

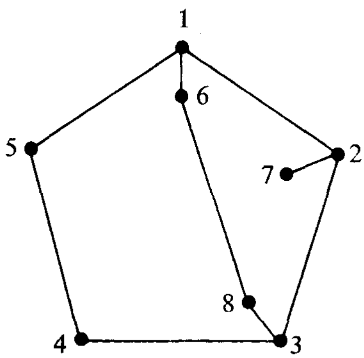

Figure 6.

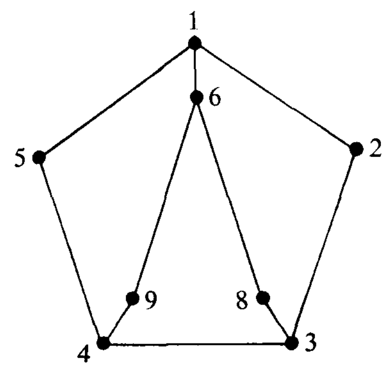

Figure 7.

In P_{10} there are six vertices with degree 3 and three with degree 2. Removing any one of the former set of vertices from P_{10} gives a graph isomorphic to $P_{10,9}$ (Figure 6). On the other hand $P_{10,7}$ is isomorphic to any graph formed by removing a vertex of degree 2 from P_{10} (Figure 7).

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Now 9 is in an orbit of $\Gamma(P_{10})$ of length 6 and 7 is in an orbit of $\Gamma(P_{10})$ of length 3 and so by the orbit stabiliser relation we have

$$|\Gamma(P_{10})_9| = \frac{|\Gamma(P_{10})|}{|_9\Gamma(P_{10})|} = \frac{12}{6} = 2,$$

and

$$|\Gamma(P_{10})_7| = \frac{|\Gamma(P_{10})|}{|_7\Gamma(P_{10})|} = \frac{12}{3} = 4.$$

However it can be quickly seen that

(56) (48), (13) (45) (68), (13) (58) (46) $\in \Gamma(P_{10,9})$ and (25) (34) (89), (14)(36)(29), (13) (46) (58), (16) (34) (29) (58), (16) (28) (59), (1364) (2895), (1463) (2598) $\in$ $\Gamma(P_{10,7})$. Hence $\Gamma(P_{10,9}) \neq \Gamma(P_{10})_9$ and $\Gamma(P_{10,7}) \neq \Gamma(P_{10})_7$, and so P cannot be stable.

(b) *Desargues' graph is unstable.*

As a corollary to (a) then, we see that Desargues' graph is also unstable, since this graph is the complement of P [5].

(c) *The Pappus Graph is Stable*

The complement of the Pappus graph (see for example [5]) is shown in Figure 8.

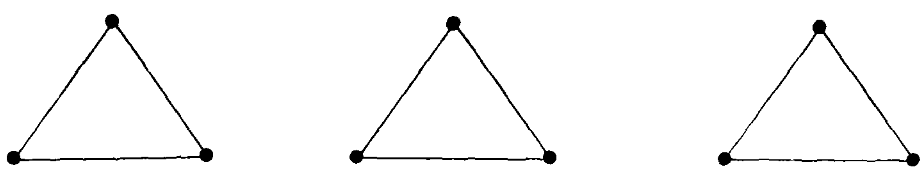

Figure 8.

This is clearly the sum of three graphs K_3. But K_3 is stable (Proposition 1) and hence $K_3 \cup K_3 \cup K_3$ is stable (Theorem 5). Hence by Proposition 4 the Pappus graph is stable.

6. Stability of graphs for $2 \leqq |V| \leqq 6$

Since either a graph or its complement must be connected we consider only connected graphs. A list of these for $2 \leqq |V| \leqq 6$ will be found in Appendix 3 of [6]. For a given $|V|$ we number the graphs of that list in the order they occur. Below we note only the unstable graphs.

(i) $|V| = 2, 3$, all stable

(ii) $|V| = 4$, only graph 1 unstable

(iii) $|V| = 5$, graphs 2, 4, 8, 12, and 16 are unstable

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(iv) $|V| = 6$, graphs 1, 4, 9, 12, 14, 18, 19, 24, 30, 31, 32, 36, 37, 44, 45, 46, 53, 54, 55, 59, 69, 70, 71, 74, 77, 79, 85, 88, 91, 93, 100, 102, and 106 are unstable. (N. B. Graph 20 for $|V| = 6$ should be as in Figure 9).

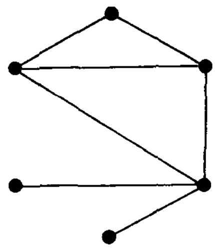

Figure 9.

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Department of Mathematics
University of Melbourne
Parkville, Vic 3052
Australia

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