PESIN, Y. B. *Dimension theory in dynamical systems: contemporary views and applications*, (University of Chicago Press, Chicago, 1998), xi+304 pp., (cloth) 0 226 66221 7, £44.75 (US$56.00); (paper) 0 226 66222 5, £15.95 (US$19.95).

As the title suggests, the topic of this book is a modern (and innovative) treatment of dimension theory in dynamical systems. The past dozen years have seen significant advances in the interplay between dimension theory and dynamical systems – for example, L.-S. Young's beautiful result from 1982 relating the dynamical parameters of an ergodic measure to its dimensional properties [**15**]:

Theorem. *If $f : M \to M$ is a $C^{1+\alpha}$ map on a compact 2-dimensional manifold M and μ is an ergodic f-invariant probability measure, then*

$$\lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) h_\mu \quad \textit{for } \mu - \textit{a.a. } x, \tag{$*$}$$

where λ_1, λ_2 are the Liapunov exponents with $\lambda_1 \le \lambda_2$ and h_μ is the entropy of μ.

The righthand side of $(*)$ contains the dynamical parameters associated with μ, whereas the lefthand side describes a measure-theoretical dimension property of μ. The book under review, written by one of the world's leading experts in the field, is the first book to address the interplay between dimension theory and dynamical systems. However, what really makes this book unique is the author's use of a general and unifying approach to dimensions, which he refers to as the general Carathéodory construction. This approach was developed by the author in a series of papers in the late 1980s [**11, 12**]. The classical notion of measure-theoretical dimension theory goes back to Constantin Carathéodory [**2**] in 1914 and, in particular, Felix Hausdorff [**7**] in 1919. This construction forms the basis of fractal dimension theory and has been studied in detail from a purely measure-theoretical point of view in the classical text by Federer [**6**] and in the excellent (and recently republished) gem by Rogers [**14**], and more recently in textbooks by Edgar [**3**], Falconer [**4, 5**] and Mattila [**9**]. Following the approach of Carathéodory and Hausdorff the author defines dimensions as follows. Let X be a set and let $\mathcal{F}$ be a collection of subsets of X. Let $\eta, \psi, \xi : \mathcal{F} \to [0, \infty]$ be three set functions (and assume that η and ψ satisfy some mild technical conditions). For each $\alpha \in \mathbb{R}$ the author defines an outer measure $M_C(\cdot, \alpha)$ by

$$M_C(Z,\alpha) = \liminf_{\varepsilon \searrow 0} \left\{ \sum_{i=1}^{\infty} \xi(U_i)\eta(U_i)^\alpha \mid Z \subseteq \bigcup_{i=1}^{\infty} U_i,\ U_i \in \mathcal{F} \text{ for all } i, \psi(U_i) \le \varepsilon \text{ for all } i \right\}$$

for $Z \subseteq X$. It is easily seen that there exists a unique value, $\dim_C(Z)$, of α such that

$$M_C(Z,\alpha) = \begin{cases} \infty & \text{for } \alpha < \dim_C(Z); \\ 0 & \text{for } \dim_C(Z) < \alpha. \end{cases}$$

The number $\dim_C(Z)$ is called the (η, ψ, ξ) Carathéodory dimension of the set Z. This construction provides a very general and natural framework for dimension theory and includes many dimension indices that have been studied in the literature.

(1) *Hausdorff dimension.* Let $X = \mathbb{R}^n$. If $\mathcal{F}$ is the family of open subsets of $\mathbb{R}^n$ and $\eta(U) = \psi(U) = \operatorname{diam}(U)$ and $\xi(U) = 1$, then $\dim_C(Z)$ equals the Hausdorff dimension of Z.

(2) *Rényi dimensions.* Let $X = \mathbb{R}^n$ and let μ be a probability measure on $\mathbb{R}^n$ and $\gamma > 0$. If $\mathcal{F}$ is the family of open balls of $\mathbb{R}^n$ and $\eta(B(x,r)) = \psi(B(x,r)) = r$ and $\xi(B(x,r)) = \mu(B(x,\gamma r))^q$, then

$\tau(Z, q, r) := \dim_C(Z)$ equals the (q, γ) Rényi dimension used in multifractal analysis. Putting Z equal to the support of μ and taking the supremum over all $\gamma > 1$, one obtains the Rényi dimension function $\tau(q) = \sup_{\gamma>1} \tau(Z, q, \gamma)$.

(3) *Topological entropy*. Let (X, d) be a compact metric space and $f : X \to X$ a continuous map. For $\delta > 0, n \in \mathbb{N}$ and $x \in X$ write $B_n(x, \delta) = \{y | d(f^i(x), f^i(y)) \leq \delta \text{ for } 0 \leq i \leq n\}$. If $\mathcal{F} = \{B_n(x, \delta) | x \in X, n \in \mathbb{N}\}$ and $\eta(B_n(x, \delta)) = e^{-n}$, $\psi(B_n(x, \delta)) = \frac{1}{n}$ and $\xi(B_n(x, \delta)) = 1$, then $\dim_C(Z)$ equals the δ-topological entropy of f on Z. The limit superior as δ tends to zero gives the topological entropy.

The book is divided into two parts. Part 1, consisting of Chapters 1–4, develops the general theory of Carathéodory dimensions and Part 2, consisting of Chapters 5–8, applies the general theory to various constructions in dynamical systems.

Chapter 1 develops the theory of general Carathéodory dimensions in abstract spaces, i.e. spaces without any metric structure. In Chapters 2, 3 and 4 the author studies in detail the Carathéodory dimensions associated with the choices of $\mathcal{F}$ and (η, ψ, ξ) in (1), (2) and (3) respectively.

Part 2 of the book treats in detail a number of interesting and well-chosen examples. In Chapter 5 the author considers fractal sets that appear naturally in dynamics, for example limit sets of self-similar iterated function systems, Moran constructions and self-conformal iterated function systems. The major part of Part 2 of the book, namely Chapters 6 and 7, is devoted to a mathematically rigorous treatment of the multifractal structure of Gibbs states and self-conformal measures. Multifractal analysis, which has become one of the most fashionable topics in geometric measure theory and dynamical systems in the 1990s, studies the local structure of measures with "widely varying intensity". Let μ be a (locally finite) Borel measure on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we define the local dimension of μ at X by $d_\mu(x) = \lim_{r \searrow 0}((\log \mu(B(x, r)))/\log r)$. For a measure μ with "widely varying intensity" we might expect that the sets $\{x | d_\mu(x) = \alpha\}$ of points with a given local dimension α display a fractal character for a range of values of α. If this is the case, then the measure is called a multifractal measure and it is natural to study the sizes of the sets $\{x | d_\mu(x) = \alpha\}$ as α varies. Typically these sets have zero μ measure except for a few exceptional values of α and we therefore study the Hausdorff dimension of the sets $\{x | d_\mu(x) = \alpha\}$ as a function of $\alpha : f_\mu(\alpha) = \dim\{x | d_\mu(x) = \alpha\}$. This and similar functions are generically known as the multifractal spectrum of μ. The celebrated "Multifractal Formalism" asserts that for good measures we have $f_\mu(\alpha) = \tau^*(\alpha)$, where τ^* denotes the Legendre transform of the Rényi dimension function in (2). The author gives a detailed discussion of these concepts and verifies the "Multifractal Formalism" for Gibbs measures and equilibrium measures of Axiom A diffeomorphisms. Similar multifractal frameworks based on general Carathéodory constructions have also been developed and studied independently by Peyriére [**1**, **13**] and the reviewer [**10**]. The final chapter discusses the Eckmann-Ruelle conjecture (a higher dimensional version of the above theorem of Young).

The book is well written; the exposition is self-contained, intelligent and well paced; misprints are extremely few. The book is at the graduate level, but not specifically intended as a textbook (for example, the book does not contain any exercises). There are, as mentioned earlier, a number of good texts on dimension theory, for example [**3**, **4**, **5**, **9**, **14**], and dynamical systems and ergodic theory, for example [**8**, **16**]. However, the strength of the present book is due to the author's general and unifying approach to dimension theory and dynamical systems based on general Carathéodory constructions. This book is therefore useful both to people working in dynamical systems and to people working in geometric measure theory (in particular, those interested in multifractal analysis). The reviewer is convinced that the general Carathéodory dimension construction which the book describes and develops will become a standard approach to the study of dimensions in, and also outside, the field of dynamical systems. This book is highly recommended to anybody interested in a modern and unifying approach to dimension theory in dynamical systems and geometric measure theory.

L. OLSEN

REFERENCES

1. G. Brown, G. Michon and J. Peyrière, On the multifractal analysis of measures, *J. Statist. Phys.* **66** (1992), 775–790.

2. C. Carathéodory, Über das lineare Maß von Punktmengen – eine Verallgemeinerung des Längenbegriffs, *Nachr. K. Ges. Wiss. Göttingen*, Math.-phys. Klasse (1914), 414–426.

3. G. Edgar, *Integral, probability, and fractal measures* (Springer-Verlag, 1997).

4. K. J. Falconer, *The geometry of fractal sets* (Cambridge University Press, 1985).

5. K. J. Falconer, *Fractal geometry – mathematical foundations and applications* (John Wiley & Sons, 1990).

6. H. Federer, *Geometric measure theory* (Springer-Verlag, 1969).

7. F. Hausdorff, Dimension und äußeres Maß, *Math. Ann.* **79** (1918), 157–179.

8. A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems* (Cambridge University Press, 1995).

9. P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability* (Cambridge University Press, 1995).

10. L. Olsen, A multifractal formalism, *Adv. Math.* **116** (1995), 82–196.

11. Y. B. Pesin, Dimension type characteristics for invariant sets of dynamical systems, *Russian Math. Surveys* **43** (1988), 111–151.

12. Y. B. Pesin, Generalized spectrum for the dimension: the approach based on Carathéodory's construction, in *Constantin Carathéodory: an international tribute* (World Sci. Publishing, Teaneck, 1991), 1108–1119.

13. J. Peyrière, Multifractal measures, in *NATO ASI Series, Series C: Mathematical and Physical Sciences*, Vol. 372 (Kluwer Academic Press, 1992), 175–186.

14. C. A. Rogers, *Hausdorff measures* (2nd edition) (Cambridge University Press, 1998).

15. L.-S. Young, Dimension, entropy and Liapunov exponents, *Ergodic Theory Dynamical Systems* **2** (1982), 109–124.

16. P. Walters, *An introduction to ergodic theory* (Springer-Verlag, 1982).