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DUALITY FOR A NON-DIFFERENTIABLE PROGRAMMING PROBLEM

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A generalised dual to a non-differentiable programming problem is given and duality established under general convexity and invexity conditions. A second order dual is also given and duality theorems proved under generalised second order invexity conditions.

1. INTRODUCTION

In [11], Mond considered the class of non-differentiable mathematical programming problems

$$\text{(P)} \qquad \text{Minimise } f(x) + \left(x^T Bx\right)^{1/2}$$

$$\text{(1)} \qquad \text{subject to } g(x) \geqslant 0$$

where f and g are twice differentiable functions from R^n to R and R^m respectively, and B is an $n \times n$ positive semi-definite (symmetric) matrix. Let x_0 satisfy (1); Mond [11] defined the set

$$\begin{aligned} Z_0 = \Big\{z \mid\ & z^T \nabla g_i(x_0) \geqslant 0 \quad (\forall\, i \in Q_0), \quad \text{and} \\ & z^T \nabla f(x_0) + z^T Bx_0 / \left(x_0^T Bx_0\right)^{1/2} < 0, \text{ if } x_0^T Bx_0 > 0; \\ & z^T \nabla f(x_0) + \left(z^T Bz\right)^{1/2} < 0, \text{ if } x_0^T Bx_0 \doteq 0\Big\} \end{aligned}$$

where $Q_0 = \{i \mid g_i(x_0) = 0\}$, and established the following necessary conditions for x_0 to be an optimal solution to (P).

PROPOSITION 1. *If x_0 is an optimal solution of (P) and the corresponding set Z_0 is empty, then there exist $y \in R^m$, $y \geqslant 0$, and $w \in R^n$ such that*

$$y^T g(x_0) = 0, \quad \nabla y^T g(x_0) = \nabla f(x_0) + Bw, \quad w^T Bw \leqslant 1, \quad \left(x_0^T Bx_0\right)^{1/2} = x_0^T Bw.$$

(Mond and Schechter [12] gave a constraint qualification which assures that Z_0 is empty. Additional constraint qualifications were given by Wolkowitz [18].)

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Using these necessary conditions, a Wolfe type dual problem [**17**] was formulated in [**11**]:

(WD)
$$\begin{aligned} &\text{Maximise } f(u) - y^T g(u) + u^T[\nabla y^T g(u) - \nabla f(u)] \\ &\text{subject to } \nabla f(u) + Bw = \nabla y^T g(u) \\ &\qquad w^T B w \leqslant 1 \\ &\qquad y \geqslant 0. \end{aligned}$$

(WD) is a dual to (P) assuming that f is convex and g is concave.

Chandra, Craven and Mond [**4**] weakened the convexity requirements for duality by giving a Mond-Weir type dual [**14**]

(M-WD)
$$\begin{aligned} &\text{Maximise } f(u) + u^T[\nabla y^T g(u) - \nabla f(u)] \\ &\text{subject to } \nabla f(u) - \nabla y^T g(u) + Bw = 0 \\ &\qquad y^T g(u) \leqslant 0 \\ &\qquad w^T B w \leqslant 1 \\ &\qquad y \geqslant 0 \end{aligned}$$

and established duality theorems assuming that $f(.) + (.)^T Bw$ is pseudo-convex for all $w \in R^n$ and that $y^T g$ is quasi-concave.

Mond and Smart [**13**] later generalised the results obtained by Mond [**11**] and Chandra, Craven and Mond [**4**] to invexity conditions ([**3, 5, 7**]). Bector and Chandra [**2**] recently presented two different second order duals to (P), which extended the results obtained by Mangasarian [**8**], Mond [**10**] and Mond and Weir [**15**] for second order duality and the results obtained by Mond [**11**], Mond and Weir [**14**] and Chandra, Craven and Mond [**4**] for first order duality.

In this paper, we propose a general Mond-Weir type dual [**14**] to (P) and establish the duality theorems under both convexity and invexity conditions. A general second order Mond-Weir dual [**15**] to (P) will also be proposed and duality results established under generalised second order invexity conditions [**1**].

We shall make use of the generalised Schwarz inequality ([**6**] and [**16**])

$$(x^T Bw) \leqslant (x^T Bx)^{1/2} (w^T Bw)^{1/2}. \tag{2}$$

Note that equality holds if, for $\lambda \geqslant 0$, $Bx = \lambda Bw$.

2. Duality

We propose the following general dual (GD) to (P).

$$\text{(GD)} \qquad \text{Maximise } f(u) - \sum_{i \in I_0} y_i g_i(u) + u^T B w$$

$$(3) \qquad \text{subject to } \nabla f(u) - \nabla y^T g(u) + Bw = 0$$

$$(4) \qquad \sum_{i \in I_\alpha} y_i g_i(u) \leqslant 0, \quad \alpha = 1, 2, \ldots, r$$

$$(5) \qquad w^T B w \leqslant 1$$

$$y \geqslant 0$$

where $I_\alpha \subseteq M = \{1, 2, \ldots, m\}$, $\alpha = 0, 1, 2, \ldots, r$ with

$$\bigcup_{\alpha=0}^{r} I_\alpha = M \text{ and } I_\alpha \cap I_\beta = \phi \text{ if } \alpha \neq \beta.$$

Theorem 1. *(Weak Duality) Let x be feasible for (P) and (u, y, w) feasible for (GD). If, for all feasible (x, u, y, w), $f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw$ is pseudo-invex and $\sum_{i \in I_\alpha} y_i g_i(.)$, $\alpha = 1, 2, \ldots, r$ is quasi-incave with respect to the same η, then*

$$\textit{infimum (P)} \geqslant \textit{supremum (GD)}.$$

Proof: Since x is feasible for (P) and (u, y, w) is feasible for (GD), we have

$$\sum_{i \in I_\alpha} y_i g_i(x) - \sum_{i \in I_\alpha} y_i g_i(u) \geqslant 0, \quad \alpha = 1, 2, \ldots, r.$$

By the quasi-incavity of $\sum_{i \in I_\alpha} y_i g_i$, $\alpha = 1, 2, \ldots, r$, it follows that

$$\eta(x, u)^T \nabla \sum_{i \in I_\alpha} y_i g_i(u) \geqslant 0, \quad \alpha = 1, 2, \ldots, r.$$

Hence

$$\eta(x, u)^T \nabla \Big(\sum_{i \in M \setminus I_0} y_i g_i(u) \Big) \geqslant 0,$$

then from (3), it follows that

$$\eta(x, u)^T \Big[\nabla f(u) - \nabla \sum_{i \in I_0} y_i g_i(u) + Bw \Big] \geqslant 0.$$

The pseudo-invexity of $f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw$ then yields

$$f(x) - \sum_{i \in I_0} y_i g_i(x) + x^T Bw \geqslant f(u) - \sum_{i \in I_0} y_i g_i(u) + u^T Bw.$$

Thus

$$f(x) + x^T Bw \geqslant f(u) - \sum_{i \in I_0} y_i g_i(u) + u^T Bw, \quad \text{from } y \geqslant 0 \text{ and } g(x) \geqslant 0.$$

Since $w^T Bw \leqslant 1$, by the generalised Schwarz inequality (2), it follows that

$$f(x) + \left(x^T Bx\right)^{1/2} \geqslant f(u) - \sum_{i \in I_0} y_i g_i(u) + u^T Bw.$$

∎

THEOREM 2. *(Strong Duality) If x_0 is an optimal solution of (P) and the corresponding set Z_0 is empty, then there exist $y \in R^m$ and $w \in R^n$ such that (x_0, y, w) is feasible for (GD) and the corresponding values of (P) and (GD) are equal. If, also, $f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw$ is pseudo-invex for all $w \in R^n$ and $\sum_{i \in I_\alpha} y_i g_i(.)$, $\alpha = 1, 2, \ldots, r$ is quasi-incave with respect to the same η, then (x_0, y, w) is optimal for (GD).*

PROOF: Since x_0 is an optimal solution to (P) and the corresponding set Z_0 is empty, then from Proposition 1, there exist $y \in R^m$ and $w \in R^n$ such that

$$y^T g(x_0) = 0,\ \nabla y^T g(x_0) = \nabla f(x_0) + Bw,\ w^T Bw \leqslant 1,\ \left(x_0^T Bx_0\right)^{1/2} = x_0^T Bw,\ y \geqslant 0.$$

So, (x_0, y, w) is feasible for (GD) and the corresponding values of (P) and (GD) are equal. If $f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw$ is pseudo-invex for all $w \in R^n$ and $\sum_{i \in I_\alpha} y_i g_i(.)$, $\alpha = 1, 2, \ldots, r$, is quasi-incave with respect to the same η, then from Theorem 1, (x_0, y, w) must be an optimal solution for (GD). ∎

We now consider some special cases of the dual (GD) and Theorems 1 and 2.

If $I_0 = M$, then (GD) becomes (WD) and from Theorems 1 and 2, (WD) is a dual to (P) if $f(.) - y^T g(.) + (.)^T Bw$ is pseudo-invex with respect to η.

In the case $I_0 = \phi$ and $I_\alpha = M$ (for some $\alpha \in \{1, 2, \ldots, r\}$) then (GD) becomes (M-WD) and from Theorems 1 and 2, (M-WD) is a dual to (P) if $f(.) + (.)^T Bw$ is pseudo-invex and $y^T g$ is quasi-incave with respect to the same η. This extends the results obtained in [4] because pseudo-convex and quasi-concave functions are pseudo-invex and quasi-incave functions respectively.

If $I_0 = \phi, I_1 = \{1\}, \ldots, I_m = \{m\}$ $(r = m)$, then (GD) becomes

(M-WD1)
$$\begin{aligned} &\text{Maximise } f(u) + u^T Bw \\ &\text{subject to } \nabla f(u) - \nabla y^T g(u) + Bw = 0 \\ &\qquad y_i g_i(u) \leqslant 0, \quad i = 1, 2, \ldots, m \\ &\qquad w^T Bw \leqslant 1 \\ &\qquad y \geqslant 0 \end{aligned}$$

and (M-WD1) is a dual to (P) if $f(.) + (.)^T Bw$ is pseudo-invex and each $y_i g_i$, $i = 1, 2, \ldots, m$ is quasi-incave with respect to the same η. Note that if g_i is quasi-incave with respect to η, $y_i \geqslant 0$, then $y_i g_i$ is quasi-incave with respect to the same η; thus (M-WD1) is a dual to (P) if $f(.) + (.)^T Bw$ is pseudo-invex and each g_i, $i = 1, 2, \ldots, m$ is quasi-incave with respect to the same η.

The following corollaries obviously hold because pseudo-convex and quasi-concave functions are, respectively, pseudo-invex and quasi-incave functions.

COROLLARY 1. *(Weak Duality) Let x be feasible for (P) and (u, y, w) feasible for (GD). If, for all feasible (x, u, y, w), $f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw$ is pseudo-convex and $\sum_{i \in I_\alpha} y_i g_i(.)$, $\alpha = 1, 2, \ldots, r$ is quasi-concave, then*

$$\textit{infimum (P)} \geqslant \textit{supremum (GD)}.$$

COROLLARY 2. *(Strong Duality) If x_0 is an optimal solution to (P) and the corresponding set Z_0 is empty, then there exist $y \in R^m$ and $w \in R^n$ such that (x_0, y, w) is feasible for (GD) and the corresponding values of (P) and (GD) are equal. If, also, $f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw$ is pseudo-convex for all $w \in R^n$ and $\sum_{i \in I_\alpha} y_i g_i(.)$, $\alpha = 1, 2, \ldots, r$ is quasi-concave, then (x_0, y, w) is an optimal for (GD).*

THEOREM 3. *(Converse Duality) Let (x^*, y^*, w^*) be optimal to (GD) at which the matrix*

$$\nabla^2 f(x^*) - \nabla^2 y^{*T} g(x^*)$$

is positive or negative definite and the vectors

$$\left\{ \sum_{i \in I_\alpha} \nabla y_i^* g_i(x^*), \quad \alpha = 1, 2, \ldots, r \right\}$$

are linearly independent. If, for all feasible (x, u, y, w), $f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw$ is pseudo-convex and $\sum_{i \in I_\alpha} y_i g_i(.)$, $\alpha = 1, 2, \ldots, r$ is quasi-concave, or $f(.) - \sum_{i \in I_0} y_i g_i(.) +$

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$(.)^T Bw$ is pseudo-invex and $\sum_{i\in I_\alpha} y_i g_i(.)$, $\alpha = 1,2,\ldots,r$ is quasi-incave with respect to the same η, then x^* is an optimal to (P).

PROOF: Since (x^*, y^*, w^*) is an optimal solution to (GD), by the generalised Fritz-John theorem [9], there exist $\tau_0 \in R$, $\nu \in R^n$, $\tau_\alpha \in R$, $\alpha = 1,2,\ldots,r$, $\beta \in R$ and $\gamma \in R^m$ such that

$$\tau_0\Big(-\nabla f(x^*) + \sum_{i\in I_0}\nabla y_i^* g_i(x^*) - Bw^*\Big) + \nu^T\Big(\nabla^2 f(x^*) - \nabla^2 y^{*T} g(x^*)\Big) + \sum_{\alpha=1}^{r}\tau_\alpha\Big(\sum_{i\in I_\alpha}\nabla y_i^* g_i(x^*)\Big) = 0 \tag{6}$$

$$\tau_0 g_i(x^*) - \nu^T \nabla g_i(x^*) - \gamma_i = 0, \quad i \in I_0 \tag{7}$$

$$\nu^T\nabla g_i(x^*) - \tau_\alpha g_i(x^*) + \gamma_i = 0, \quad i \in I_\alpha,\ \alpha = 1,2,\ldots,r, \tag{8}$$

$$\tau_0\big(Bx^*\big) - \nu^T B - 2\beta\big(Bw^*\big) = 0 \tag{9}$$

$$\tau_\alpha\Big(\sum_{i\in I_\alpha} y_i^* g_i(x^*)\Big) = 0, \quad \alpha = 1,2,\ldots,r, \tag{10}$$

$$\beta\Big(w^{*T}Bw^* - 1\Big) = 0 \tag{11}$$

$$\gamma^T y^* = 0 \tag{12}$$

$$(\tau_0, \tau_1, \ldots, \tau_\alpha, \beta, \gamma) \geqslant 0 \tag{13}$$

$$(\tau_0, \tau_1, \ldots, \tau_\alpha, \beta, \gamma, \nu) \neq 0. \tag{14}$$

Multiplying (8) by $y_i^* \geqslant 0$, $i \in I_\alpha$, $\alpha = 1,2,\ldots,r$ and using (12) yields

$$\nu^T\nabla y_i^* g_i(x^*) - \tau_\alpha y_i^* g_i(x^*) = 0, \quad i \in I_\alpha, \quad \alpha = 1,2,\ldots,r.$$

Hence

$$\nu^T\sum_{i\in I_\alpha}\nabla y_i^* g_i(x^*) - \tau_\alpha\sum_{i\in I_\alpha} y_i^* g_i(x^*) = 0, \quad \alpha = 1,2,\ldots,r.$$

From (10), it follows that

$$\nu^T\sum_{i\in I_\alpha}\nabla y_i^* g_i(x^*) = 0, \quad \alpha = 1,2,\ldots,r. \tag{15}$$

Using the equation constraint (3), (6) becomes

$$\sum_{\alpha=1}^{r}(\tau_\alpha - \tau_0)\Big(\sum_{i\in I_\alpha}\nabla y_i^* g_i(x^*)\Big) + \nu^T\Big(\nabla^2 f(x^*) - \nabla^2 y^{*T} g(x^*)\Big) = 0. \tag{16}$$

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Multiplying (16) by ν and using (15) gives

$$\nu^T\Big(\nabla^2 f(x^*) - \nabla^2 y^{*T} g(x^*)\Big)\nu = 0.$$

By assuming that $\nabla^2 f - \nabla^2 y^T g$ is positive or negative definite at (x^*, y^*, w^*) it follows that

$$\nu = 0. \tag{17}$$

Then (16) gives

$$\sum_{\alpha=1}^{r}(\tau_\alpha - \tau_0)\Big(\sum_{i\in I_\alpha}\nabla y_i^* g_i(x^*)\Big) = 0. \tag{18}$$

Since the vectors $\Big\{\sum_{i\in I_\alpha}\nabla y_i^* g_i(x^*),\quad \alpha = 1, 2, \ldots, r\Big\}$ are linearly independent, (18) then yields

$$\tau_\alpha - \tau_0 = 0, \quad \alpha = 1, 2, \ldots, r. \tag{19}$$

If $\tau_0 = 0$, then $\tau_\alpha = 0$, $\alpha = 1, 2, \ldots, r$ from (19), $\gamma = 0$ from (7) and (8), and $\beta = 0$ from (9) and (11), but $(\tau_0, \tau_1, \tau_2, \ldots, \tau_r, \nu, \gamma, \beta) = 0$ contradicts (14). So $\tau_0 > 0$. This gives $\tau_\alpha > 0$, $\alpha = 1, 2, \ldots, r$. Then (7), (8), (13) and $\tau_\alpha > 0$, $\alpha = 0, 1, 2, \ldots, r$ yield $g(x^*) \geqslant 0$. Therefore, x^* is feasible for (P).

Multiplying (7) by y_i^*, $i \in I_0$ and using (12) gives

$$\tau_0 y_i^* g_i(x^*) = 0,\ i \in I_0.$$

Then from $\tau_0 > 0$, it follows that

$$y_i g_i(x^*) = 0,\ i \in I_0. \tag{20}$$

Also, $\nu = 0$, $\tau_0 > 0$ and (9) give

$$Bx^* = (2\beta/\tau_0)Bw^*. \tag{21}$$

Hence

$$(x^{*T}Bw^*) = \big(x^{*T}Bx^*\big)^{1/2}\big(w^{*T}Bw^*\big)^{1/2}. \tag{22}$$

If $\beta > 0$, then (11) gives $w^{*T}Bw^* = 1$ and so (22) yields

$$(x^{*T}Bw^*) = \big(x^{*T}Bx^*\big)^{1/2}.$$

If $\beta = 0$ then (21) gives $Bx^* = 0$. So we still get

$$\left(x^{*T}Bw^*\right) = \left(x^{*T}Bx^*\right)^{1/2}.$$

Thus in either case, we obtain

$$\left(x^{*T}Bw^*\right) = \left(x^{*T}Bx^*\right)^{1/2}. \tag{23}$$

Therefore from (20) and (23), we have

$$f(x^*) + \left(x^{*T}Bx^*\right)^{1/2} = f(x^*) - \sum_{i\in I_0} y_i^* g_i(x^*) + x^{*T}Bw^*.$$

If, for all feasible (x, u, y, w), $f(.) - \sum_{i\in I_0} y_i g_i(.) + (.)^T Bw$ is pseudo-convex and $\sum_{i\in I_\alpha} y_i g_i(.)$, $\alpha = 1, 2, \ldots, r$ is quasi-concave, or $f(.) - \sum_{i\in I_0} y_i g_i(.) + (.)^T Bw$ is pseudo-invex and $\sum_{i\in I_\alpha} y_i g_i(.)$, $\alpha = 1, 2, \ldots, r$ is quasi-incave with respect to the same η, then fromTheorem 1 or Corollary 1, x^* is an optimal solution to (P). □

3. Second order duality

In this section, we present a general non-differentiable second order Mond-Weir dual [**15**] to (P). We shall make use of the following definitions.

Definition 1: [**1**] f is second order pseudo-invex if for all $p \in R^n$, there exists an $\eta(x, u)$ such that

$$\eta(x,u)^T\left[\nabla f(u) + \nabla^2 f(u)p\right] \geqslant 0 \implies f(x) \geqslant f(u) - \frac{1}{2}p^T\nabla^2 f(u)p.$$

Definition 2: [**1**] f is second order quasi-invex if for all $p \in R^n$, there exists an $\eta(x, u)$ such that

$$f(x) \leqslant f(u) - \frac{1}{2}p^T\nabla^2 f(u)p \implies \eta(x,u)^T\left[\nabla f(u) + \nabla^2 f(u)p\right] \leqslant 0.$$

A function g is said to be second order pseudo-incave or second order quasi-incave if $-g$ is second order pseudo-invex and second order qausi-invex respectively.

The second order Mangasarian type [**8**] and Mond-Weir type [**15**] duals to (P) were regarded in [**2**] as the following problems:

(2MD)

$$\begin{aligned} \text{Maximise} \quad & f(u) - y^Tg(u) + u^TBw - \frac{1}{2}p^T\nabla^2\left[f(u) - y^Tg(u)\right]p \\ \text{subject to} \quad & \nabla f(u) - \nabla y^Tg(u) + Bw + \nabla^2 f(u)p - \nabla^2 y^Tg(u)p = 0 \\ & w^TBw \leqslant 1 \\ & y \geqslant 0 \end{aligned}$$

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where $u, w, p, \in R^n$ and $y \in R^m$.

(2M-WD)

$$\begin{aligned} \text{Maximise} \quad & f(u) + u^T Bw - \frac{1}{2} p^T \nabla^2 f(u) p \\ \text{subject to} \quad & \nabla f(u) - \nabla y^T g(u) + Bw + \nabla^2 f(u) p - \nabla^2 y^T g(u) p = 0 \\ & y^T g(u) - \frac{1}{2} p^T \nabla^2 y^T g(u) p \leqslant 0 \\ & w^T Bw \leqslant 1 \\ & y \geqslant 0. \end{aligned}$$

Using the second order convexity conditions (called bonvexity in [**2**]), Bector and Chandra established duality theorems between (P) and (2MD) and (2M-WD), respectively.

Following Mond-Weir [**15**] we now propose a general second order dual (2GD) to (P)

(2GD)

$$\text{Maximise} \quad f(u) - \sum_{i \in I_0} y_i g_i(u) + u^T Bw - \frac{1}{2} p^T \Big[\nabla^2 f(u) - \nabla^2 \sum_{i \in I_0} y_i g_i(u) \Big] p$$

(24)
$$\text{subject to} \quad \nabla f(u) - y^T g(u) + Bw + \nabla^2 f(u) p - \nabla^2 y^T g(u) p = 0$$

(25)
$$\sum_{i \in I_\alpha} y_i g_i(u) - \frac{1}{2} p^T \nabla^2 \sum_{i \in I_\alpha} y_i g_i(u) p \leqslant 0, \quad \alpha = 1, 2, \dots, r,$$

(26)
$$w^T Bw \leqslant 1$$

(27)
$$y \geqslant 0$$

where $I_\alpha \subseteq M = \{1, 2, \dots, m\}$, $\alpha = 0, 1, 2, \dots, r$ with

$$\bigcup_{\alpha=0}^{r} I_\alpha = M \quad \text{and} \quad I_\alpha \cap I_\beta = \phi \quad \text{if} \quad \alpha \neq \beta.$$

Theorem 4. *(Weak Duality) Let x be feasible for (P) and (u, y, w, p) feasible for (2GD). If, for all feasible (x, u, y, w, p), $f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw$ is second order pseudo-invex, and $\sum_{i \in I_\alpha} y_i g_i(.)$, $\alpha = 1, 2, \dots, r$ is second order quasi-incave with respect to the same η, then*

$$\textit{infimum (P)} \geqslant \textit{supremum (2GD)}.$$

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PROOF: Since x is feasible for (P) and (u, y, w, p) is feasible for (2GD), we have

$$\sum_{i\in I_\alpha} y_i g_i(x) - \sum_{i\in I_\alpha} y_i g_i(u) - \frac{1}{2}p^T\nabla^2 \sum_{i\in I_\alpha} y_i g_i(u)p \geqslant 0, \quad \alpha = 1, 2, \ldots, r.$$

By the second order quasi-incavity of $\sum_{i\in I_\alpha} y_i g_i$, $\alpha = 1, 2, \ldots, r$, it follows that

$$\eta(x, u)^T\Big(\nabla \sum_{i\in I_\alpha} y_i g_i(u) + \nabla^2 \sum_{i\in I_\alpha} y_i g_i(u)p\Big) \geqslant 0, \quad \alpha = 1, 2, \ldots, r.$$

Hence

$$\eta(x, u)^T\Big(\nabla \sum_{i\in M\setminus I_0} y_i g_i(u) + \nabla^2 \sum_{i\in M\setminus I_0} y_i g_i(u)p\Big) \geqslant 0. \tag{28}$$

Then from (24), (28) yields

$$\eta(x, u)^T\Big(\nabla f(u) + \nabla^2 f(u)p - \sum_{i\in I_0} y_i g_i(u) - \nabla^2 \sum_{i\in I_0} y_i g_i(u)p + Bw\Big) \geqslant 0.$$

Since $f(.) - \sum_{i\in I_0} y_i g_i(.) + (.)^T Bw$ is second order pseudo-invex, it follows that

$$f(x) - \sum_{i\in I_0} y_i g_i(x) + x^T Bw$$
$$\geqslant f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw - \frac{1}{2}p^T\nabla^2\Big[f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw\Big]p.$$

Thus, from $y \geqslant 0$, $g(x) \geqslant 0$, we have

$$f(x) + x^T Bw \geqslant f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw - \frac{1}{2}p^T\Big[\nabla^2 f(u) - \nabla^2 \sum_{i\in I_0} y_i g_i(u)\Big]p. \tag{29}$$

Since $w^T Bw \leqslant 1$, by the generalised Schwarz inequality (2), (29) gives that

$$f(x) + \left(x^T Bx\right)^{1/2} \geqslant f(u) - \sum_{i\in I_0} y_i g_i(u) + u^T Bw - \frac{1}{2}p^T\Big[\nabla^2 f(u) - \nabla^2 \sum_{i\in I_0} y_i g_i(u)\Big]p.$$

□

THEOREM 5. *(Strong Duality) If x_0 is an optimal solution to (P) and the corresponding set Z_0 is empty, then there exist $y \in R^m$ and $w \in R^n$ such that $(x_0, y, w, p = 0)$ is feasible for (2GD), and the corresponding values of (P) and (2GD) are*

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equal. If, for all feasible (x,u,y,p,w), $f(.)-\sum_{i\in I_0} y_ig_i(.)+(.)^TBw$ *is second order pseudo-invex, and* $\sum_{i\in I_\alpha} y_ig_i(.)$, $\alpha=1,2,\ldots,r$ *is second order quasi-incave, then* $(x_0,y,w,p=0)$ *is an optimal solution for (2GD).*

PROOF: Since x_0 is an optimal solution to (P) and the corresponding set Z_0 is empty, then from Proposition 1, there exist $y\in R^m$ and $w\in R^n$ such that

$$y^Tg(x_0)=0,\quad \nabla y^Tg(x_0)=\nabla f(x_0)+Bw,\quad w^TBw\leqslant 1,$$
$$\left(x_0^TBx_0\right)^{1/2}=x_0^TBw,\quad y\geqslant 0.$$

So, $(x_0,y,w,p=0)$ is feasible for (2GD) and the corresponding values of (P) and (2GD) are equal. If $f(.)-\sum_{i\in I_0} y_ig_i(.)+(.)^TBw$ is second order pseudo-invex for all $w\in R^n$ and $\sum_{i\in I_\alpha} y_ig_i(.)$, $\alpha=1,2,\ldots,r$, is second order quasi-incave with respect to the same η, then from Theorem 4, $(x_0,y,w,p=0)$ must be an optimal solution for (2GD). ▯

We now consider some special cases of (2GD) and Theorems 4 and 5.

If $I_0=M$, then (2GD) becomes (2MD), and from Theorems 4 and 5, (2MD) is a second order dual to (P) if $f(.)-y^Tg(.)+(.)^TBw$ is second order pseudo-invex, which extends the results obtained in [**2**] because second order pseudo-convex and second order quasi-concave are second order pseudo-invex and second order quasi-incave respectively [**1**].

If $I_0=\phi$ and $I_\alpha=M$ (for some $\alpha\in\{1,2,\ldots,r\}$), then (2GD) becomes (2M-WD), and from Theorems 4 and 5, (2M-WD) is a second order dual to (P) if $f(.)+(.)^TBw$ is second order pseudo-invex and y^Tg is second order quasi-incave, which extends the results obtained in [**2**].

We now assume that f and g are three times differentiable.

THEOREM 6. *(Converse Duality) Let* (x^*,y^*,w^*,p^*) *be an optimal solution to (2GD) at which the matrix*

$$\nabla\left[\nabla^2f(x^*)-\nabla^2y^{*T}g(x^*)\right]p^*$$

is positive or negative definite and the vectors

$$\left\{\left[\nabla^2f(x^*)-\nabla^2\sum_{i\in I_0}y_i^*g_i(x^*)\right]_j,\quad \left[\nabla^2\sum_{i\in I_\alpha}y_i^*g_i(x^*)\right]_j,\quad \alpha=1,2,\ldots,r,\quad j=1,2,\ldots,n\right\}$$

are linearly independent, where $\left[\nabla^2f-\nabla^2\sum_{i\in I_0}y_ig_i\right]_j$ *is the* j*-th row of* $\nabla^2f-\nabla^2\sum_{i\in I_0}y_ig_i$ *and* $\left[\nabla^2\sum_{i\in I_\alpha}y_ig_i\right]_j$ *is the* j*-th row of* $\nabla^2\sum_{i\in I_\alpha}(y_ig_i)$. *If, for all feasible*

(x, u, y, w, p), $f(.) - \sum_{i \in I_0} y_i g_i(.) + (.)^T Bw$ is second order pseudo-invex and $\sum_{i \in I_\alpha} y_i g_i(.)$, $\alpha = 1, 2, \dots, r$ is second order quasi-incave with respect to the same η, then x^* is an optimal solution to (P).

PROOF: Since (x^*, y^*, w^*, p^*) is an optimal solution to (2GD), by the generalised Fritz-John theorem [**9**], there exist $\tau_0 \in R$, $\nu \in R^n$, $\tau_\alpha \in R$, $\alpha = 1, 2, \dots, r$, $\beta \in R$ and $\gamma \in R^m$ such that

$$\begin{aligned} &\tau_0\Big\{-\nabla f(x^*) + \sum_{i \in I_0} \nabla y_i^* g_i(x^*) - Bw^* + \frac{1}{2}p^{*T}\nabla\Big[\nabla^2 f(x^*) - \nabla^2 \sum_{i \in I_0} y_i^* g_i(x^*)p^*\Big]\Big\} \\ &\quad + \nu^T\Big\{\nabla^2 f(x^*) - \nabla^2 y^{*T} g(x^*) + \nabla\Big[\nabla^2 f(x^*)p^* - \nabla^2 y^{*T} g(x^*)p^*\Big]\Big\} \\ &\quad + \sum_{\alpha=1}^{r} \tau_\alpha\Big\{\nabla \sum_{i \in I_\alpha} y_i^* g_i(x^*) - \frac{1}{2}p^{*T}\nabla\Big[\nabla^2 \sum_{i \in I_\alpha} y_i^* g_i(x^*)p^*\Big]\Big\} = 0 \end{aligned} \tag{30}$$

$$\tau_0\Big\{g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 g_i(x^*)p^*\Big\} - \nu^T\Big\{\nabla g(x^*) + \nabla^2 g(x^*)p^*\Big\} - \gamma_i = 0, \quad i \in I_0, \tag{31}$$

$$\begin{aligned} &-\nu^T\Big\{\nabla g(x^*) + \nabla^2 g(x^*)p^*\Big\} + \tau_\alpha\Big\{g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 g_i(x^*)p^*\Big\} - \gamma_i = 0, \\ &\qquad i \in I_\alpha, \quad \alpha = 1, 2, \dots, r, \end{aligned} \tag{32}$$

$$\tau_0 Bx^* - \nu^T B - 2\beta(Bw^*) = 0 \tag{33}$$

$$(\tau_0 p^* + \nu)^T\Big\{\nabla^2 f(x^*) - \nabla^2 \sum_{i \in I_0} y_i^* g_i(x^*)\Big\} - \sum_{\alpha=1}^{r}(\tau_\alpha p^* + \nu)^T\Big\{\nabla^2 \sum_{i \in I_\alpha} y_i^* g_i(x^*)\Big\} = 0 \tag{34}$$

$$\tau_\alpha\Big\{\sum_{i \in I_\alpha} y_i^* g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 \sum_{i \in I_\alpha} y_i^* g_i(x^*)p^*\Big\} = 0, \quad \alpha = 1, 2, \dots, r, \tag{35}$$

$$\beta(w^{*T}Bw^* - 1) = 0 \tag{36}$$

$$\gamma^T y^* = 0 \tag{37}$$

$$(\tau_0, \tau_1, \dots, \tau_r, \beta, \gamma) \geqslant 0 \tag{38}$$

$$(\tau_0, \tau_1, \dots, \tau_r, \beta, \gamma, \nu) \neq 0. \tag{39}$$

Since

$$\Big\{\Big[\nabla^2 f(u) - \nabla^2 \sum_{i \in I_0} y_i g_i(u)\Big]_j, \quad \Big[\nabla^2 \sum_{i \in I_\alpha} y_i g_i(u)\Big]_j, \quad \alpha = 1, 2, \dots, r, \quad j = 1, 2, \dots, n\Big\}$$

are linearly independent at (x^*, y^*, w^*, p^*), (34) then gives

$$\tau_\alpha p^* + \nu = 0, \quad \alpha = 0, 1, 2, \dots, r. \tag{40}$$

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Multiplying (32) by y_i^*, $i \in I_\alpha$, $\alpha = 1, 2, \ldots, r$ and using (37) yields

$$\nu^T\Big\{\nabla y_i^* g_i(x^*) + \nabla^2 y_i^* g_i(x^*) p^*\Big\} - \tau_\alpha\Big\{y_i^* g_i(x^*) - \frac{1}{2} p^{*T} \nabla^2 y_i^* g_i(x^*) p^*\Big\} = 0$$
$$i \in I_\alpha, \quad \alpha = 1, 2, \ldots, r,$$

thus

$$\nu^T\Big\{\sum_{i\in I_\alpha} \nabla y_i^* g_i(x^*) + \sum_{i\in I_\alpha} \nabla^2 y_i^* g_i(x^*) p^*\Big\}$$
$$- \tau_\alpha\Big\{\sum_{i\in I_\alpha} y_i^* g_i(x^*) - \frac{1}{2} p^{*T} \nabla^2 \sum_{i\in I_\alpha} y_i^* g_i(x^*) p^*\Big\} = 0, \quad \alpha = 1, 2, \ldots, r.$$

From (35), it follows that

$$\nu^T\Big\{\sum_{i\in I_\alpha} \nabla y_i^* g_i(x^*) + \sum_{i\in I_\alpha} \nabla^2 y_i^* g_i(x^*) p^*\Big\} = 0, \quad \alpha = 1, 2, \ldots, r. \tag{41}$$

Using (24), (30) gives

$$(\tau_\alpha p^* + \nu)^T\Big\{\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} y_i^* g_i(x^*) + \nabla\Big[\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} y_i^* g_i(x^*)\Big] p^*\Big\}$$
$$- \sum_{\alpha=1}^{r} (\tau_\alpha p^* + \nu)^T\Big\{\nabla^2\Big[\sum_{i\in I_\alpha} y_i^* g_i(x^*)\Big] + \nabla\Big[\nabla^2 \sum_{i\in I_\alpha} y_i^* g_i(x^*)\Big] p^*\Big\}$$
$$- \tau_0\Big\{\nabla \sum_{i\in M\setminus I_0} y_i^* g_i(x^*) + \nabla^2 \sum_{i\in M\setminus I_0} y_i^* g_i(x^*) p^*\Big\}$$
$$- \frac{1}{2}\tau_0 p^{*T}\Big\{\nabla\Big[\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} y_i^* g_i(x^*)\Big] p^*\Big\}$$
$$+ \sum_{\alpha=1}^{r} \tau_\alpha\Big\{\nabla \sum_{i\in I_\alpha} y_i^* g_i(x^*) + \nabla^2\Big[\sum_{i\in I_\alpha} y_i^* g_i(x^*)\Big] p^*\Big\}$$
$$+ \sum_{\alpha=1}^{r} \frac{1}{2}\tau_\alpha p^{*T}\Big\{\nabla\Big[\nabla^2 \sum_{i\in I_\alpha} y_i^* g_i(x^*)\Big] p^*\Big\} = 0.$$

From (40), it follows that

$$\sum_{\alpha=1}^{r} (\tau_\alpha - \tau_0)\Big\{\nabla \sum_{i\in I_\alpha} y_i^* g_i(x^*) + \nabla^2 \sum_{i\in I_\alpha} y_i^* g_i(x^*) p^*\Big\}$$
$$+ \frac{1}{2}\nu^T\Big\{\nabla\Big[\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} y_i^* g_i(x^*)\Big] p^* - \nabla\Big[\nabla^2 \sum_{i\in M\setminus I_0} y_i^* g_i(x^*)\Big] p^*\Big\} = 0.$$

That is

$$(42)\quad \sum_{\alpha=1}^{r}(\tau_\alpha-\tau_0)\Big\{\nabla\sum_{i\in I_\alpha}y_i^*g_i(x^*)+\nabla^2\sum_{i\in I_\alpha}y_i^*g_i(x^*)p^*\Big\} + \frac{1}{2}\nu^T\Big\{\nabla\Big[\nabla^2 f(x^*)-\nabla^2 y^{*T}g(x^*)\Big]p^*\Big\}=0.$$

Multiplying (42) by ν and using (41) yields

$$\nu^T\Big\{\nabla\Big[\nabla^2 f(x^*)-\nabla^2 y^{*T}g(x^*)\Big]p^*\Big\}\nu=0.$$

By assuming that $\nabla\left[\nabla^2 f(x^*)-\nabla^2 y^{*T}g(x^*)\right]p^*$ is positive or negative definite, it follows that

$$(43)\qquad \nu=0,$$

so (40) becomes

$$\tau_\alpha p^*=-\nu=0,\quad \alpha=0,1,2,\dots,r.$$

If $\tau_\alpha=0$, $\alpha=0,1,2,\dots,r$, we get $\gamma=0$ from (31) and (32), and $\beta=0$ from (33) and (36); but $(\tau_0,\tau_1,\dots,\tau_r,\beta,\gamma,\nu)=0$ contradicts (39). Thus $\tau_\alpha>0$, $\alpha=0,1,2,\dots,r$; this gives $p^*=0$. Hence from (31) and (32), it follows that

$$(44)\qquad \tau_0 g_i(x^*)-\gamma_i=0,\quad i\in I_0$$

$$(45)\qquad \tau_\alpha g_i(x^*)-\gamma_i=0,\quad i\in I_\alpha,\quad \alpha=1,2,\dots,r.$$

Therefore $g(x^*)\geqslant 0$ since $\gamma\geqslant 0$ and $\tau_\alpha>0$, $\alpha=0,1,2,\dots,r$. Thus x^* is feasible for (P).

Multiplying (44) by y_i, $i\in I_0$ and using (37) gives

$$\tau_0 y_i^* g_i(x^*)=0,\quad i\in I_0.$$

By $\tau_0>0$, it follows that

$$(46)\qquad y_i^*g_i(x^*)=0,\quad i\in I_0.$$

Also, $\nu=0$, $\tau_0>0$ and (33) give

$$(47)\qquad Bx^*=(2\beta/\tau_0)Bw^*.$$

Hence

$$(48)\qquad (x^{*T}Bw^*)=(x^{*T}Bx^*)^{1/2}(w^{*T}Bw^*)^{1/2}.$$

If $\beta > 0$, then (36) gives $w^{*T}Bw^* = 1$, and so (48) yields

$$\left(x^{*T}Bw^*\right) = \left(x^{*T}Bx^*\right)^{1/2}.$$

If $\beta = 0$, then (47) gives $Bx^* = 0$. So we still get

$$\left(x^{*T}Bw^*\right) = \left(x^{*T}Bx^*\right)^{1/2}.$$

Thus, in either case, we have

$$\left(x^{*T}Bw^*\right) = \left(x^{*T}Bx^*\right)^{1/2}. \tag{49}$$

Therefore from (46), (49) and $p^* = 0$, we have

$$f(x^*) + \left(x^{*T}Bx^*\right)^{1/2} = f(x^*) - \sum_{i\in I_0} y_i^* g_i(x^*) + x^{*T}Bw^* - \frac{1}{2}p^{*T}\nabla^2\Big[f(x^*) - y^{*T}g(x^*)\Big]p^*.$$

If, for all feasible (x,u,y,w,p), $f(.) - \sum_{i\in I_0} y_i g_i(.) + (.)^T Bw$ is second order pseudo-invex and $\sum_{i\in I_\alpha} y_i g_i(.)$, $\alpha = 1,2,\ldots,r$ is second order quasi-incave with respect to the same η, then from Theorem 4, x^* is an optimal solution to (P). □

Note that if $p = 0$, then (2GD) becomes (GD), (2MD) becomes (MD) and (2M-WD) becomes (M-WD). This means that second order duality implies first order duality, and so Theorems 4, 5 and 6 imply Theorems 1, 2 and 3 respectively.

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