Proceedings of the Edinburgh Mathematical Society (1976-77), 20, 211-213 ©

ON GROUPS OF FIBONACCI TYPE

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(Received 24th March 1975)

1. An embedding

Let $w = w(a_0, a_1, \ldots, a_{n-1})$ be a word in the free group freely generated by $a_0, a_1, \ldots, a_{n-1}$; let w_i denote the word $w(a_i, a_{i+1}, \ldots, a_{i+n-1})$, where the subscripts j in a_j are reduced modulo n; and let

$$G(n; w) = \langle a_0, a_1, \ldots, a_{n-1}: \quad w_i = 1; \quad i = 0, 1, \ldots, n-1\rangle.$$

Amongst the groups $G(n; w)$ are those said to be of "Fibonacci type" (see (2)). The Fibonacci groups are the groups $F(r, n) = G(n; w)$, where $w = a_0a_1 \ldots a_{r-1}a_r^{-1}$ (see (3)).

Now let

$$E(n; w) = \langle a, b; \quad b^n = 1, w(a_0, a_1, \ldots, a_{n-1}) = 1\rangle,$$

where here a_j denotes the element $b^{-j}ab^j$ $(0 \leq j \leq n-1)$. Then, it is a direct consequence of the Reidemeister-Schreier subgroup theorem (see Theorem 2.9 of (4)) that $G(n; w)$ is embedded as the least normal subgroup of $E(n; w)$ containing the element a; one choses the elements $1, b, \ldots, b^{n-1}$ as Schreier coset representatives.

It is often easier and more instructive to consider $E(n; w)$ rather than $G(n; w)$. Apparently this was the case in R. C. Lyndon's proof that $F(r; n)$ is infinite if $n \geq 11$. As $G(n; w)$ is a normal subgroup of index n in $E(n; w)$ it is infinite if $E(n; w)$ is infinite, finite of order s/n if $E(n; w)$ is finite of order s. Also, $G(n; w)$ inherits subgroup properties. In any case, $E(n; w)$ is a two-generator two-relator group whose relators can often be easily transformed into simple and workable forms.

This approach may be used to give a simple solution* to part of Problem 3 of (3) (another solution is given in (1)). Indeed, the comments above, together with the argument of Theorem 5 of (3), yield immediately that $F(r; n)$ is metacyclic of order $r^n - 1$ when $r \equiv 1$ modulo n.

2. On a conjecture

In (2), the group $H(r, n, s) = G(n; w)$, where

$$w = a_0a_1 \ldots a_{r-1}(a_ra_{r+1} \ldots a_{r+s-1})^{-1},$$

* I understand from the referee, that M. J. Dunwoody has preceded me with this observation, which was made at the International Conference of Mathematicians, Vancouver, 1974.

https://doi.org/10.1017/S0013091500026286 Published online by Cambridge University Press

is defined. According to §1, the group $\boldsymbol{H}(r, n, s)$ is embedded as a normal subgroup of index n in the group $\boldsymbol{E}(n; w) = \langle c, b; b^n = 1, b^r c^r = c^s b^s \rangle$, where $c = ab^{-1}$.

C. M. Campbell and E. F. Robertson conjecture (in (2)) that $\boldsymbol{H}(r, 4, 2)$ is metacyclic if r is odd. This is so; in fact, it will follow below that $\boldsymbol{H}(r, 4, 2)$ is metacyclic of order $(r-2)(2^{\xi} - 2.2^{\eta} + 2)/2$, where $\xi \equiv 4\alpha$, $\eta \equiv 2\alpha$ reduced modulo $4(r-2)$, with α given by $4\alpha \equiv 1$ modulo $r-2$.

Consider $\boldsymbol{E}(4; w) = \langle c, b; b^4 = 1, b^r c^r = c^2 b^2 \rangle$. If r is even then $\boldsymbol{E}(4; w)$ is infinite, as it has the group $\langle b, c; b^2 = c^2 = 1 \rangle$ as an epimorph. Thus we assume in the sequel that r is odd.

In case $r \equiv 3$ modulo 4, place $d = bc^{r-2}$, and in case $r \equiv 1$ modulo 4, place $d = c^{r-2}b^{-1}$. Then,

$$\boldsymbol{E}(4; w) = \langle d, c; dd_{r-2} = d_r, (dc^{-(r-2)})^4 = 1 \rangle,$$

where d_i is used to denote $d^{c^i} = c^{-i}dc^i$ for integers i.

Our first observation is that $(dc^{-(r-2)})^4 = 1$ can be written in the form $c^{4(r-2)} = dd_{r-2}(dd_{r-2})^{c^{2(r-2)}}$; and therefore, since $d_r = dd_{r-2}$, in the form

$$c^{4(r-2)} = dd_{2(r-2)} \tag{i}$$

The relation $dd_{r-2} = d_r$ is the same as $d = d_{r-2}^{c^2} d_{r-2}^{-1}$; hence, using (i), we have $d_{r-2} = d_{2(r-2)}^{c^2} d_{2(r-2)}^{-1} = d_2^{-1} c^{4(r-2)} c^{-4(r-2)} d$, and so

$$d_{r-2} = d_2^{-1} d. \tag{ii}$$

From (i), we have $c^{4(r-2)} = d_r d_{r-2}^{-1} d_{2(r-2)} = d_2 d^{-1} d_{r-2}$. In particular, $d_2 d^{-1} d_{r-2} = d_4 d_2^{-1} d_r = d_4 d_2^{-1} dd_{r-2}$, so that $d_4 = d_2 d^{-2} d_2$. Thus, on the one hand, by (ii) we have $d_r = d_4^{-1} d_2 = d_2^{-1} d^2$, and on the other, $d_r = dd_{r-2} = dd_2^{-1} d$. We conclude that $d_2 d = dd_2$.

Relation (ii) implies $d_{2(r-2)} = d_r^{-1} d_{r-2} = d_2^{-1} d_4 d_2^{-1} d = d^{-1}$, since $d_4 = d_2 d^{-2} d_2$; whence, by (i) we have $c^{4(r-2)} = 1$. Moreover, since $d_r = d_2^{-1} d^2$, we obtain $d_{r+2} = d_4^{-1} d_2^2 = d^2$.

Conversely, the relations $d_{r+2} = d^2$, $d_4 = d_2^2 d^{-2}$ and $c^{4(r-2)} = 1$ imply the original two. Firstly, since $(4(r-2), (r+2)) = 1$ we obtain, using $c^{4(r-2)} = 1$ and $d_{r+2} = d^2$, that $d_1 = d^{\mu}$ for some integer μ, and hence that $d_i d_j = d_j d_i$ for each integer i and j. Also, $d_{r+2} = d^2$ implies $d_r = d_{-2}^2$, $d_{r-2} = d_{-4}^2$; and as $d = d_{-2}^2 d_{-4}^{-2}$, we conclude that $d_r = dd_{r-2}$. Further, the relation $d_4 = d^2 d_2^{-2}$ implies that $d_6 = d_2^2 d^{-4}$ and $d_8 = d^{-4}$; so, as $d_{r-2} = d_{-4}^2$, we have that $d_{2(r-2)} = d_{-8}^4 = d^{-1}$, and this verifies relation (i). It follows that

$$\boldsymbol{E}(4; w) = \langle d, c; d_{r+2} = d^2, d_4 = d_2^2 d^{-2}, c^{4(r-2)} = 1 \rangle.$$

Now let α be given by $\alpha(r+2) \equiv 1$ modulo $4(r-2)$ (equivalently by

$4\alpha \equiv 1$ modulo $r-2$). Then, if $e = c^{r+2}$, we have $c = e^{\alpha}$; consequently $d^e = d^2$, and so $d_4 = d_2^2 d^{-2}$ becomes $d^{2\xi - 2.2^{\eta}+2} = 1$, where ξ and η are given by 4α and 2α, respectively, when these are reduced modulo $4(r-2)$. Since $d^e = d^2$, we conclude that $d^k = 1$. Thus

$$\boldsymbol{E}(4; w) = \langle d, e; d^e = d^2, d^k = 1, e^{4(r-2)} = 1 \rangle.$$

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