A NOTE ON NORMAL MATRICES

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(received October 7, 1960)

Introduction. In 1954 A. J. Hoffman and O. Taussky [1] showed that if A is an n-square complex matrix with eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ and P is a permutation matrix for which $\alpha A + \beta A^*$ has eigenvalues $\alpha\lambda + \beta P\overline{\lambda}$ for some $\alpha\beta \neq 0$ then A is normal. Here $\overline{\lambda}$ is the conjugate vector of λ. As a companion result they also proved that if the eigenvalues of AA^* are $\lambda_i \overline{(P\lambda)}_i$, $i = 1, \ldots, n$ then A is normal. (See footnote.)

In this note we obtain similar characterizations of normal matrices. Our main results are contained in the

THEOREM. Let α be a complex number, $0 \neq |\alpha| \neq 1$, and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of A. If S is a real orthogonal matrix and $\alpha A + A^*$ has eigenvalues $S(\alpha\lambda + \overline{\lambda})$ then A is normal. If U is unitary and AA^* has eigenvalues $\lambda_i(\overline{U\lambda})_i$, $i = 1, \ldots, n$ then it also follows that A is normal.

We prove the first part of the theorem in a sequence of lemmas.

The second part is very easy and we indicate this at the end of the paper.

Editor's footnote. A^* denotes the conjugate complex transpose of A. A is called normal if it commutes with A^*. By a theorem of Schur and Toeplitz, A is normal if and only if there exists a unitary matrix U and a diagonal matrix D such that $U^*AU = D$. Cf. Linear Algebra and Matrices, by H. W. E. Schwerdtfeger, (Groningen, 1950), p. 204.

[1]The work o this author was completed under a Postdoctorate Fellowship of the National Research Council of Canada.

A lower triangular matrix L satisfies $l_{ij} = 0$ for $i \leq j$. An upper triangular matrix is the transpose of a lower triangular matrix. If X is any n-square matrix $\|X\|$ will denote the Frobenius norm of X:

$$(\Sigma_{i,j=1}^{n} |x_{ij}|^2)^{\frac{1}{2}} = (tr(XX^*))^{\frac{1}{2}}$$

LEMMA 1. If L is lower triangular and α, β are complex numbers then

$$tr[(\alpha L + \beta L^*)^2] = 2\alpha\beta \|L\|^2.$$

Proof.

$$tr[(\alpha L + \beta L^*)^2] = \alpha^2 tr(L^2) + 2\alpha\beta tr(LL^*) + \beta^2 tr(L^{*2}).$$

Note that the set of lower triangular matrices is closed under multiplication and hence this last expression becomes $2\alpha\beta \|L\|^2$.

LEMMA 2. If $d = (d_1, \ldots, d_n)$ satisfies $\alpha d + \bar{d} = S(\alpha\lambda + \bar{\lambda})$ for $|\alpha| \neq 1$ and S is real orthogonal then

$$\Sigma_{k=1}^{n} d_k^2 = \Sigma_{k=1}^{n} \lambda_k^2.$$

Proof. Let $\lambda = a + ib$, $d = x + iy$, $\alpha = w + i\delta$ where a, b, x, y, w, and δ are real. Then equating real and imaginary parts separately of $\alpha d + \bar{d} = S(\alpha\lambda + \bar{\lambda})$ we have

$$\begin{aligned} (w+1)x - \delta y &= S[(w+1)a - \delta b] \\ \delta x + (w-1)y &= S[\delta a + (w-1)b]. \end{aligned} \tag{1}$$

Denote the 2n-square matrix

$$\begin{pmatrix} (w+1)I & -\delta I \\ \delta I & (w-1)I \end{pmatrix}$$

by F. Now $\det F = (|\alpha|^2 - 1)^n \neq 0$ and hence from (1) we conclude

$$x \dotplus y = (S \dotplus S)(a \dotplus b) = Sa \dotplus Sb$$

where $\dotplus$ indicates direct sum. Thus

https://doi.org/10.4153/CMB-1961-004-8 Published online by Cambridge University Press

$$\Sigma_{k=1}^{n} d_k^2 = \Sigma_{k=1}^{n} (x_k + iy_k)^2$$

$$= (x, x) + 2i(x, y) - (y, y)$$

$$= (Sa, Sa) + 2i(Sa, Sb) - (Sb, Sb)$$

$$= (a, a) + 2i(a, b) - (b, b)$$

$$= \Sigma_{k=1}^{n} (a_k + ib_k)^2$$

$$= \Sigma_{k=1}^{n} \lambda_k^2.$$

Here (,) indicates the usual unitary inner product of two n-tuples.

To proceed to the proof of the first part of the theorem select a unitary matrix R that brings $\alpha A + A^*$ to triangular form with zero below the main diagonal. Let $B = RAR^*$ and set

$$B = D + L + V$$

where $D = \text{diag}(d_1, \ldots, d_n)$ and L and V are lower and upper triangular matrices. Then

$$\alpha B + B^* = (\alpha D + \overline{D}) + (\alpha L + V^*) + (\alpha V + L^*).$$

Let $d = (d_1, \ldots, d_n)$ and let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be the n-tuple of numbers on the main diagonal of $\alpha B + B^*$, i.e., the eigenvalues of $\alpha A + A^*$. Then

$$\alpha d + \overline{d} = \gamma = S(\alpha\lambda + \overline{\lambda}),$$

$$\alpha L + V^* = 0,$$

and we conclude that

$$B = D + L - \overline{\alpha} L^*.$$

By lemma 2,

$$\Sigma_{k=1}^{n} d_k^2 = \Sigma_{k=1}^{n} \lambda_k^2$$

and hence

$$tr(B^2) = \Sigma_{k=1}^{n} \lambda_k^2 = tr(D^2).$$

But

$$tr(B^2) = tr[(D + L - \bar{\alpha} L^*)^2]$$
$$= tr(D^2) + 2tr[(L - \bar{\alpha} L^*)D] + tr[(L - \bar{\alpha} L^*)^2].$$

Note that $tr[(L - \bar{\alpha} L^*)D] = 0$ and thus

$$tr[(L - \bar{\alpha} L^*)^2] = 0.$$

By lemma 1

$$-2\bar{\alpha} \|L\|^2 = 0$$

and hence

$$L = 0.$$

Thus $B = RAR^*$ is a diagonal matrix and from this it follows that A is normal.

To see the last part of the theorem note that

$$0 \leq tr(AA^*) = (\lambda, U\lambda) = |(\lambda, U\lambda)| \leq \|\lambda\| \ \|U\lambda\|$$
$$= \|\lambda\|^2 = \Sigma_{i=1}^{n} |\lambda_i|^2.$$

Thus if $RAR^* = diag(\lambda_1, \ldots, \lambda_n) + L$, where L is lower triangular and R is unitary then

$$tr(AA^*) = \Sigma_{i=1}^{n} |\lambda_i|^2 + \|L\|^2 \leq \Sigma_{i=1}^{n} |\lambda_i|^2$$

and $L = 0$. This completes the proof.

REFERENCE

1. A. J. Hoffman and O. Taussky, A characterization of normal matrices, J. Research, Nat. Bur. Standards 52 (1954), 17-19.

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https://doi.org/10.4153/CMB-1961-004-8 Published online by Cambridge University Press