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A NEW CHARACTERIZATION OF REFLEXIVITY

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Abstract

In this paper, we give a new characterization of reflexive Banach spaces in terms of the sum of two closed bounded convex sets.

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We know that the sum of a compact set and a closed set is closed; it is also known that the sum of two closed sets need not to be closed. In this paper we shall show that reflexivity of a Banach space can be characterized by the property that the sum of any two closed bounded convex sets in the Banach space remains closed.

Throughout this paper, E will be a Banach space, and we shall use $S(E)$ and $B(E)$ to denote the unit sphere of E and the unit ball of E, respectively. Now we present our main theorem.

MAIN THEOREM. *The Banach space E is reflexive if and only if the sum of any two closed bounded convex sets in E is still closed.*

PROOF. First, assume that E is reflexive. Let A, $B \subset E$ and suppose that both of these sets are closed bounded convex sets. Then A and B are compact in the weak topology of E, and hence $A + B$ is closed in the weak topology of E. It is obvious that $A + B$ is convex, so we deduce that $A + B$ is closed in the norm topology.

To prove the converse, suppose that E is not reflexive. Then, by James's well-known characterization of reflexivity in terms of the supremum of linear functionals [1], there exists $x^* \in S(E^*)$ such that x^* does not attain its norm on $B(E)$. Let $\{x_n\} \subset B(E)$ such that

$$x^*(x_n) > 1 - \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N}.$$

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Set $H = \{x \in E : x^*(x) = 1\}$. It is easy to see that $H \cap B(E) = \emptyset$.

Now fix $x_0 \in H$. For $n \in \mathbb{N}$, let $y_n = x_n + (1 - x^*(x_n))x_0$; then $x^*(y_n) = 1$ and

$$\|x_n - y_n\| = \|(1 - x^*(x_n))x_0\| < \frac{1}{2^n}\|x_0\|.$$

Letting $B = \overline{co}\{-x_n\}$ and $A = \overline{co}\{y_n\}$, we have $A \subset H$ and $-B \subset B(E)$. Since

$$\lim_{n\to\infty} \|x_n - y_n\| = 0,$$

we obtain $\theta \in \overline{A + B}$.

But $A \cap (-B) = \emptyset$, so it must be that $\theta \notin A + B$.

Therefore $A + B$ is not closed, which contradicts our assumption. Thus we conclude that E is reflexive. □

References

[1] R. C. James, 'Reflexivity and the sup of linear functionals', *Israel J. Math.* **13** (1972), 289–300.

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