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LATIN SQUARES WITH PRESCRIBED DIAGONALS

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1. Introduction. An *incomplete latin rectangle* on t symbols $\sigma_1, \ldots, \sigma_t$ of size $r \times s$ is an $r \times s$ matrix in which each cell is occupied by exactly one of the symbols $\sigma_1, \ldots, \sigma_t$ in such a way that no symbol occurs more than once in any row or more than once in any column. If $r = t$ or $s = t$ then it is a *latin rectangle*; if $r = s < t$ it is an *incomplete latin square*; if $r = s = t$ it is a *latin square*. The *diagonal* of a latin square consists of the cells (i, i) $(1 \leqq i \leqq t)$ together with the symbols occupying those cells. Let an *allowed sequence* of length t be a sequence of length t in which no symbol occurs exactly $t - 1$ times. Let an *allowed diagonal* of length t be a diagonal occupied by an allowed sequence. If an incomplete latin square R of side r is embedded in the top right hand corner of a latin square T then let the *diagonal of T outside R* be the cells (i, i) $(r + 1 \leqq i \leqq t)$ together with the symbols occupying those cells.

An *edge-colouring* of a graph G with n colours is a surjection $\phi : E(G) \rightarrow C$, where $E(G)$ is the edge set of G and C is a set of n colours, such that $\phi(e_1) \neq \phi(e_2)$ whenever $e_1, e_2 \in E(G)$ and e_1 and e_2 have a common vertex. If G contains $2t$ vertices, then a *partial matching* is a set M of at most t edges such that no two edges of M have a common vertex. If $|M| = t$ then M is a (complete) *matching*.

The following result was proved by G. J. Chang [**5**] using abelian groups.

THEOREM 1. *For any given allowed diagonal D, there exists a latin square with diagonal D.*

A stronger result that includes this theorem was proved by Andersen and Hilton [**3**] by a long, rather delicate, argument.

We present in this paper a short, combinatorial proof of Theorem 1; we also prove Theorem 2 below. It is convenient to prove the two together, but it is not necessary, as either could be proved first and used to prove the other.

THEOREM 2. *Let P be a latin square of side s, $s \geqq 2$ on the symbols $\sigma_1, \ldots, \sigma_s$. Then P can be embedded in a latin square L on symbols $\sigma_1, \ldots, \sigma_{2s+x}$, where $x = 1$, or 2, in which the diagonal δ of L outside P is prescribed, if and only if the following are satisfied:*

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(a) *δ is occupied by an allowed sequence,*
(b) $f(\sigma_i) \leqq x$ $(s + 1 \leqq i \leqq 2s + x)$,
(c) *if* $x = 1$ *then* $\sum_{i=1}^{s} f(\sigma_i) \neq 1$,
where $f(\sigma_i)$ *is the number of times* σ_i *occurs in* δ.

For further work on the wider problem of embedding an incomplete latin square R in a latin square T, where the diagonal of T outside R is prescribed, see [**1**], [**2**], [**4**] and [**7**]. For the case $x \geqq 3$ we hope to show elsewhere by different arguments that Theorem 2 remains true; these different arguments fail for $x = 1$ or 2.

We shall use the following result.

THEOREM 3. ([**6**]). *If* $r < n$ *then an* $r \times n$ *latin rectangle can be embedded in a latin square of side* n.

2. Proof of theorems 1 and 2.

Necessity of Theorem 2. The proof of the necessity of (a) and (b) is left to the reader. For (c), suppose P is embedded in L and that L has prescribed diagonal outside P, but that (c) does not hold. Let L be in the form

$$(1) \quad L = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}.$$

Then each of the symbols $\sigma_{s+1}, \ldots, \sigma_{2s+1}$ occurs once in S, so one symbol, say σ_{2s+1}, occurs in a non-diagonal cell of S and so occurs in a row or column which contains one of $\sigma_{s+1}, \ldots, \sigma_{2s}$ as well. Then, in that row or column of S, one of $\sigma_1, \ldots, \sigma_s$, say σ_1, will not occur. But σ_1 cannot occur in Q or R either, so σ_1 will not occur anywhere in that row or column, a contradiction. Therefore (c) holds.

Sufficiency of Theorem 2 *and proof of Theorem* 1. We shall suppose that the length of D in Theorem 1 will be $2s + 1$ or $2s + 2$ and we shall prove the two results jointly by induction on s. We leave it to the reader to verify that both theorems are true for applicable values of $s \leqq 3$. Suppose that $s \geqq 4$ and that both theorems are true for $1, 2, \ldots, s - 1$.

Suppose first that D is an allowed diagonal of length $2s + x$, $x = 1$ or 2. If $s = 4$ and D consists of one symbol occurring three times and another symbol occurring $2s + x - 3$ times then it is not hard to form a latin square with diagonal, D. We shall now ignore this possibility for D. In any other case, D can be separated into D_1 and δ_1, where D_1 has length s, δ_1 has length $s + x$ and, after relabelling the symbols,

I. D_1 is an allowed diagonal of length s on symbols $\sigma_1, \ldots, \sigma_s$,
II. Each symbol from $\sigma_{s+1}, \ldots, \sigma_{2s+x}$ occurs at most x times in δ_1, and
III. (a) and (c) are satisfied by δ_1.

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For since there are at most s symbols that occur $x + 1$ or more times in D, we can separate D into D_1' and δ_1' such that any symbol occurring $x + 1$ or more times in D occurs at least once in D_1' and, therefore, after relabelling, will be one of $\sigma_1, \ldots, \sigma_s$. Thus II is satisfied. It is easy to check that we can interchange symbols between D_1' and δ_1' and relabel to get I and III satisfied as well. By induction we may complete D_1 with $\sigma_1, \ldots, \sigma_s$ to obtain a latin square P and a diagonal outside P, δ, which satisfies the hypotheses of Theorem 2.

Now suppose that in any case we have a latin square P and a diagonal δ outside P which satisfy the hypotheses of Theorem 2. If $x = 1$ we proceed as follows. Let δ_2 be the allowed diagonal on symbols $\sigma_0, \sigma_1, \ldots, \sigma_s$ of length $s + 1$ formed by replacing each symbol $\sigma_j(s + 1 \leqq j \leqq 2s + 1)$ in δ by a new symbol σ_0. Let S^* be a latin square of side $s + 1$ on symbols $\sigma_0, \ldots, \sigma_s$ with δ_2 as diagonal. By induction, S^* exists. Now replace the symbol σ_0 in S^* by the symbols $\sigma_{s+1}, \ldots, \sigma_{2s+1}$ in such a way that each symbol occurs once and the diagonal is δ. Call this incomplete latin square, S. Let $q_1(r_1)$ be the row (column) vector formed by projecting the elements $\sigma_{s+1}, \ldots, \sigma_{2s+1}$ in S vertically (horizontally). Let Q^* and R^* be latin squares on $\sigma_{s+1}, \ldots, \sigma_{2s+1}$ with q_1 and r_1 as the first row and column, respectively. Let Q and R be formed from Q^* and R^* by omitting q_1 and r_1 respectively. Now form the latin square L of (1). It is the required latin square.

If $x = 2$ let δ_2 be an allowed diagonal on symbols $\sigma_{-1}, \sigma_0, \sigma_1, \ldots, \sigma_s$ of length $s + 2$ formed by replacing each symbol from $\sigma_{s+1}, \ldots, \sigma_{2s+2}$ which occurs twice in δ by σ_{-1} and σ_0 (each once) and each symbol which occurs once by either σ_0 or σ_{-1}. By induction, form a latin square S_2 of side $s + 2$ on symbols $\sigma_{-1}, \sigma_0, \sigma_1, \ldots, \sigma_s$ with diagonal δ_2.

Let the cells occupied by σ_{-1} and σ_0 be $x_1, \ldots, x_{s+2}$ and $y_1, \ldots, y_{s+2}$ respectively, and form a bipartite graph G with the x_i's and y_j's as vertices by joining x_i and y_j if the cells x_i and y_j are not in the same row or column. Then G is regular of degree s. Colour some of the vertices by the rule: a vertex is coloured with a symbol σ_k if the corresponding cell is in δ and is occupied by σ_k. The set of edges joining pairs of vertices with the same colour form a partial matching M of G of at most $\lfloor (s + 2)/2 \rfloor$ edges. M can be extended greedily to a partial matching M' of s edges where we can assume that x_1, x_2, y_1 and y_2 are the four vertices in G that are incident with no edge in M'. If x_1, x_2, y_1 and y_2 have no edges between them then x_1 and x_2 are adjacent to all of $y_3, y_4, \ldots, y_{s+2}$ while y_1 and y_2 are adjacent to all of $x_3, x_4, \ldots, x_{s+2}$. If $|M| \leqq s - 2$ then at least two edges in M', say (x_3, y_3) and (x_4, y_4), do not occur in M and so replacing these two edges in M' by (x_1, y_3), (x_2, y_4), (x_3, y_1) and (x_4, y_2) produces a complete matching of G. Since $s \geqq 4$ and $|M| \leqq \lfloor (s + 2)/2 \rfloor$, $|M| \leqq s - 2$ unless $s = 4$ and $|M| = \lfloor (s + 2)/2 \rfloor$. A similar argument can be used to produce a complete matching if exactly one of x_1 and x_2 is adjacent to

y_1 or y_2. We then assign colours to the remainder of the x_i's and y_j's giving the two vertices at the ends of an edge of the complete matching the same colour. The symbols are then placed in the corresponding cells of S^* to give the required matrix S. If $s = 4$ and $|M| = \lfloor (s + 2)/2 \rfloor$ then the partial matching consists of three edges, and this corresponds to the case when the diagonal δ has length 6 and has 3 of $\sigma_5, \ldots, \sigma_{10}$, each occurring twice. This case is easy to handle.

Then the vertical and horizontal projections of the symbols $\sigma_{s+1}, \ldots, \sigma_{2s+2}$ in S are formed into two rows q_1 and q_2 and into two columns r_1 and r_2 respectively. Then, by Theorem 3, two latin squares Q^* and R^* are formed on the symbols $\sigma_{s+1}, \ldots, \sigma_{2s+2}$ with q_1 and q_2 and r_1 and r_2 as the first two rows and the first two columns respectively. Finally Q and R are formed by deleting q_1 and q_2 and r_1 and r_2 from Q^* and R^* respectively, and then are assembled according to (1) into the required latin square.

The two theorems now follow by induction.

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