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GOTTLIEB SETS AND DUALITY IN HOMOTOPY THEORY

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Introduction. Evaluation subgroups of the homotopy groups have been objects of extensive study recently by Gottlieb, Haslam, Jerrold Siegel, G. E. Lang (Jr), etc. In [**8**] one of the authors has introduced the notions of 'cyclic' and 'cocyclic' maps and studied generalizations of evaluation subgroups and their duals in the set up of Eckmann-Hilton duality. This paper continues the study of these generalized Gottlieb and dual Gottlieb subsets. All the spaces, except the function spaces, will be arc connected locally compact CW-complexes with base point at a vertex. For any X, Y the set of base point preserving homotopy classes of maps of X to Y is denoted by $[X, Y]$. The subset of $[X, Y]$ represented by "cyclic" maps of X to Y in the sense of [**8**] will be referred to as the Gottlieb part of $[X, Y]$ and will be denoted by $\mathscr{G}(X, Y)$. Similarly the subset of $[X, Y]$ represented by "cocyclic" maps of X in Y will be referred to as the dual Gottlieb part of $[X, Y]$ and will be denoted by $\mathscr{D}\mathscr{G}(X, Y)$.

In Section 1 we prove that an element $\alpha \in [A, X]$ lies in $\mathscr{G}(A, X)$ if and only if there exists a Hurewicz fibration $p : E \to \Sigma A$ over ΣA with fibre X satisfying the condition that $\partial[e_A] = \alpha$ in the Eckmann-Hilton exact sequence of the fibration $p : E \to \Sigma A$ where $e : A \to \Omega\Sigma A$ is the map adjoint to the identity map $\Sigma A \to \Sigma A$ and $\partial : [A, \Omega\Sigma A] \to [A, X]$ the boundary homomorphism in the Eckmann-Hilton exact sequence (Proposition 1.2). This generalizes a result of G. E. Lang (Jr) [**6**].

In Section 2 we deal with the following problem. Let $\alpha \in \mathscr{G}(A, X)$ and $\varphi : X \times A \to X$ a map "affiliated" to α, namely $\varphi|X = 1_X$ and $\varphi|A$ represents α. Let

$$E \xrightarrow{p} X$$

be a fibration. Under what conditions can we say that there exists a map $\tilde{\varphi} : E \times A \to E$ with $\tilde{\varphi}|E = 1_E$ and

$$\begin{array}{ccc} E \times A & \xrightarrow{\tilde{\varphi}} & E \\ \downarrow p & A & \downarrow p \\ X \times A & \xrightarrow{\varphi} & X \end{array}$$

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commutative? In case $A = S^q$ and $p : E \to X$ is induced from the standard contractible path space fibration $PB \xrightarrow{d_1} B$ with B a product of Eilenberg-Maclane spaces G. E. Lang finds *sufficient* condition for the existence of such a $\tilde{\varphi}$ [**6**, Theorem 3.1]. In case $A = S^q$ and B is a single Eilenberg-Maclane space, D. Gottlieb gives a necessary and sufficient condition for the existence of such a $\tilde{\varphi}$ [**4**, Theorem 6.3]. In Section 2 we prove a *necessary and sufficient* condition for the existence of such a $\tilde{\varphi}$ with A arbitrary and $E \xrightarrow{p} X$ induced from

$$PB \xrightarrow{d_1} B$$

where B is any product of Eilenberg-Maclane spaces (Theorem 2.1). From our result we show how the earlier results of Gottlieb and Lang can be deduced. Actually when $A = S^q$ our necessary and sufficient condition appears to be weaker than the sufficient condition given by G. E. Lang. However when B is a single Eilenberg-Maclane space and $A = S^q$ our condition exactly agrees with Gottlieb's necessary and sufficient condition.

In Section 3 we deal with the dual question. Let $\alpha \in \mathscr{D}\mathscr{G}(X, A)$ and $\varphi : X \to X \vee A$ a map "co-affiliated" to α, namely $p_A \circ j_{X,A} \circ \varphi \sim \alpha$ and $p_X \circ j_{X,A} \circ \varphi \sim 1_X$ where $j_{X,A} : X \vee A \to X \times A$ denotes the inclusion, $p_X : X \times A \to X$ and $p_A : X \times A \to A$ denote the respective projections. Let $\mu : X \to Y$ be a cofibration. Under what conditions can we say that there exists a map $\chi : Y \to Y \vee A$ such that $p_Y \circ j_{Y,A} \circ \chi \sim 1_Y$ and making

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \vee A \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle \mu \vee 1_A} \\ Y & \xrightarrow{\chi} & Y \vee A \end{array}$$

commutative? We answer this question when the cofibration $\mu : X \to Y$ is the push-out of the standard cofibration $e_1 : B \to CB$ $(e_1(b) = \langle b, 1\rangle$ where CB is got from $B \times I$ by collapsing $B \times 0 \cup * \times I$ to the base point) by means of a map $\theta : B \to X$ with

$$B = \bigvee_{\lambda \in \Lambda} K'(\pi_\lambda, n_\lambda)$$

a wedge of Moore-spaces, each n_λ being an integer $\geqq 3$ (Theorem 3.1). The proof of Theorem 3.1 is not completely dual to the proof of Theorem 2.1. Some complicated homotopies are involved in the proof and we had to also have a recourse to the stronger form of Puppe exact sequence.

In conclusion we want to point out that it is not clear to us whether the dual of Proposition 1.2 is true. The dual result, if it is true, will be the following:

An element $\alpha \in [X, A]$ lies in $\mathscr{D}\mathscr{G}(X, A)$ if and only if there exists a cofibration $\Omega A \xrightarrow{q} E$ with cofibre X such that $\partial[r_A] = \alpha$ where

$$\partial : [\Sigma(\Omega A), A] \to [X, A]$$

is the boundary homomorphism in the Puppe-exact sequence corresponding to the cofibration $\Omega A \xrightarrow{q} E \to X$ and $r_A : \Sigma\Omega A \to A$ is the canonical retraction which is adjoint to the identity map $\Omega A \to \Omega A$.

Our proof of Proposition 1.2 relies on the existence of Guy Allaud's classifying space for Hurewicz fibrations. There is no satisfactory theory of "classifying spaces" for cofibrations.

1. Relationship between $\mathscr{G}(A, X)$ and Guy Allaud's classifying space.

Throughout this and subsequent sections all spaces, except function spaces, are assumed to be path connected locally compact *CW* complexes with base point at a vertex. With the exception of elements of function spaces and homotopies between them, all maps and homotopies preserve base points.

Results of this section have been established for the particular case of evaluation subgroups by D. Gottlieb and G. E. Lang (Jr) in their papers [**4**] and [**6**].

Given X let B_∞ be Guy Allaud's classifying space for (Hurewicz) fibrations with fibre X and $p_\infty : E_\infty \to B_\infty$ be the corresponding universal fibration [**1**].

PROPOSITION 1.1. *For any space A, in the Eckmann-Hilton exact sequence for the fibration $p_\infty : E_\infty \to B_\infty$ we have*

$$\partial[A, \Omega B_\infty] = \mathscr{G}(A, X)$$

Proof. Let $L_\#$ be the space of all maps $X \to E_\infty$ which are homotopy equivalances from X into any fibre of $p_\infty : E_\infty \to B_\infty$ and $i_\infty : X \hookrightarrow E_\infty$ the base point of $L_\#$.

Define $\varphi : L_\# \to B_\infty$ by $\varphi(f) = p_\infty f(x_0)$.

Let X^* be the space of all homotopy equivalences of X into itself with base point 1_X.

Let $w : X^* \to X$ and $w_1 : L_\# \to L_\infty$ be the evaluation maps. Then we have a commutative diagram:

$$\begin{array}{ccc} X^* & \xrightarrow{w} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_\infty} \\ L_\# & \xrightarrow{w_1} & E_\infty \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle p_\infty} \\ B_\infty & = & B_\infty. \end{array}$$

φ is known to be a Serre fibration [**4**]. Since A is a CW complex there is an exact sequence

$$- - - - \to [A, \Omega B_\infty] \xrightarrow{\partial} [A, X^*] \xrightarrow{i_*} [A, L_\#] \xrightarrow{\varphi_*} [A, B_\infty].$$

Similarly there is an exact sequence

$$- - - - - - [A, \Omega B_\infty] \xrightarrow{\partial} [A, X] \xrightarrow{i_{\infty *}} [A, E_\infty] \xrightarrow{p_{\infty *}} [A, B_\infty]$$

and the following diagram is commutative.

$$\begin{tikzcd}
{[A, \Omega L_\#]} \arrow[r] \arrow[d, "\Omega w_{1*}"] & {[A, \Omega B_\infty]} \arrow[r, "\partial"] \arrow[d, equal] & {[A, X^*]} \arrow[r, "i_*"] \arrow[d, "w_*"] & {[A, L_\#]} \arrow[r, "\varphi_*"] \arrow[d, "w_{1*}"] & {[A, B_\infty]} \arrow[d, equal] \\
{[A, \Omega E_\infty]} \arrow[r] & {[A, \Omega B_\infty]} \arrow[r, "\partial"] & {[A, X]} \arrow[r, "i_{\infty *}"] & {[A, E_\infty]} \arrow[r, "p_{\infty *}"] & {[A, B_\infty]}
\end{tikzcd}$$

Diagram 1

It is known that for any CW complex B, $[B, L_\#] = 0$ [4]. Hence $[A, \Omega L_\#] = 0 = [A, L_\#]$ and ∂ is a bijection $[A, \Omega B_\infty] \approx [A, X^*]$.

Let $L(X, X; 1_X)$ be the path connected component of X^* containing 1_X. Since A is path connected we have

$$\mathscr{G}(A, X) = w_*[A, L(X, X; 1_X)] = w_*[A, X^*].$$

From Diagram 1 we get $\partial[A, \Omega B_\infty] = \mathscr{G}(A, X)$ in the lower horizontal sequence. This completes the proof of Proposition 1.1.

Let $e_A : A \to \Omega\Sigma A$ be the adjoint of $1_{\Sigma A}$, i.e., e_A be defined by $e_A(a)(t) = \langle a, t\rangle$ for all $a \in A$ and $t \in I$.

PROPOSITION 1.2. *Let $\alpha \in [A, X]$. Then α is in $\mathscr{G}(A, X)$ if and only if there is a fibration $p : E \to \Sigma A$ with fibre X such that $\partial[e_A] = \alpha$ in the Eckmann-Hilton exact sequence for the fibration $p : E \to \Sigma A$.*

Proof. That $\partial[e_A] \in \mathscr{G}(A, X)$ follows from [**8**, Theorem 6.4].

We only have to prove that if $\alpha \in \mathscr{G}(A, X)$ there exists a fibration $p : E \to \Sigma A$ with $\partial e_A = \alpha$.

By Proposition 1.1, $\mathscr{G}(A, X) = \partial[A, \Omega B_\infty]$. Let $y \in [A, \Omega B_\infty]$ be such that $\partial y = \alpha$ and $h : A \to \Omega B_\infty$ represent y. Let $\tilde{h} : \Sigma A \to B_\infty$ be the adjoint of h. Let $p : E \to \Sigma A$ be the fibration induced by $\tilde{h}$ from the fibration $p_\infty : E_\infty \to B_\infty$.

From the commutative diagram

$$\begin{tikzcd}
{[A, \Omega\Sigma A]} \arrow[r, "\partial"] \arrow[d, "\Omega\tilde{h}_*"] & {[A, X]} \arrow[r] \arrow[d, equal] & {[A, E]} \arrow[r, "p_*"] \arrow[d] & {[A, \Sigma A} \arrow[d, "\tilde{h}_*"] \\
{[A, \Omega B_\infty]} \arrow[r, "\partial"] & {[A, X]} \arrow[r] & {[A, E_\infty]} \arrow[r, "p_{\infty *}"] & {[A, B_\infty]}
\end{tikzcd}$$

we have $\partial[e_A] = \partial\Omega\tilde{h}_*[e_A] = \partial[h] = \partial y = \alpha$.

For the proof of the following lemma with $A = S^n$, Lang [**6**] refers to [**7**]. But to our knowledge an explicit proof is not found in [**7**]. Since we feel that this needs a proof we include it here.

Let $p : E \to \Sigma A$ be a fibration with fibre X. Let $\partial[e_A] = \alpha \in [A, X]$. From Proposition 1.2 we know that $\alpha \in \mathscr{G}(A, X)$.

Write ΣA as $C_+A \cup C_-A$ with $C_+A \cap C_-A = A$. Then there exist fibre homotopy equivalences

$$g_+ : p^{-1}(C_+A) \to X \times C_+A, \quad f_- : X \times C_-A \to p^{-1}(C_-A)$$

satisfying the additional requirements that

$$g_{+/X} : X \to X \quad \text{and} \quad f_{-/X} : X \to X$$

are homotopic to the identity map.

The map $\mu : X \times A \to X$ defined by $(\mu(x, a), a) = g_+f_-(x, a)$ for $x \in X$ and $a \in A$ is called the clutching function for the fibration $p : E \to \Sigma A$. Clearly $\mu_{/X} : X \to X$ is homotopic to 1_X.

PROPOSITION 1.3. *$\mu_{/A} : A \to X$ represents α.*

Proof. Let $\Omega_p = \{(e, w) \in E \times (\Sigma A)^I / w(0) = p(e)\}$. Define $q : X \to \Omega_p$ and $k : \Omega\Sigma A \to \Omega_p$ by

$$q(x) = (x, w_*) \quad \text{and} \quad k(w) = (x_0, w)$$

where w_* is the constant path at $*$, the base point of ΣA. The q is a homotopy equivalence and ∂ is defined by the commutative diagram

$$\begin{tikzcd}
{[A, \Omega\Sigma A]} \arrow[rr, "\partial"] \arrow[dr, "k_*"'] & & {[A, X]} \arrow[dl, "q_*", "\approx"'] \\
& {[A, \Omega_p]} &
\end{tikzcd}$$

Hence it suffices to prove that $k \circ e_A \sim q \circ (\mu/A)$.

Define $F : * \sim 1_{C_-A}$ and $G : 1_{C_+A} \sim *$ by

$$F(\langle a, t\rangle, s) = \langle a, st\rangle \quad \text{and} \quad G(\langle a, t\rangle, s) = \langle a, t + (1 - t)s\rangle.$$

Since p is a fibration the dotted arrows exist in the following commutative diagrams.

$$\begin{tikzcd}
X \times C_-A \arrow[rr, "p_1"] \arrow[d, hook] & & p^{-1}(C_-A) \arrow[d] \\
X \times C_-A \times I \arrow[r, "p_2 \times 1"'] \arrow[urr, dashed, "F'"] & C_-A \times I \arrow[r, "F"'] & C_-A
\end{tikzcd}
\qquad
\begin{tikzcd}
p^{-1}(C_-A) \arrow[rr, equal] \arrow[d, hook] & & p^{-1}(C_+A) \arrow[d, "p"] \\
p^{-1}(C_+A) \times I \arrow[r, "p \times 1"'] \arrow[urr, dashed, "G'"] & C_+A \times I \arrow[r, "G"'] & C_+A
\end{tikzcd}$$

where p_i stands for projection to ith factor.

Then f_- and g_+ are defined by $f_- = F_1'$ and $g_+(e) = (G_1'(e), p(e))$ and hence $\mu(x_0, a) = G_1'F_1'(x_0, \langle a, \frac{1}{2}\rangle)$.

Define a homotopy $H : A \times I \to E$ from the constant map $A \to E$ to the composite of

$$A \xrightarrow{\mu/A} X \hookrightarrow E$$

by

$$H(a, t) = \begin{cases} F'(x_0, \langle a, \frac{1}{2}\rangle, 2t), & \text{if } t \leqq \frac{1}{2} \\ G'(F_1'(x_0, \langle a, \frac{1}{2}\rangle), 2t - 1), & \text{if } t \geqq \frac{1}{2}. \end{cases}$$

Define a homotopy $L : A \times I \to (\Sigma A)^I$ from e_A to the constant map w_* by

$$L(a, t)(s) = \langle a, t + (1 - t)s\rangle.$$

Then for all $t \in I$, $pH(a, t) = \langle a, t\rangle = L(a, t)\ (0)$. Hence there exists $K : A \times I \to \Omega_p$ such that $K(a, t) = (H(a, t), L(a, t))$. Clearly K is a homotopy from $k \circ e_A$ to $q \circ (\mu/A)$. Hence the result.

2. Lifting $\varphi : X \times A \to X$ affiliated to $\alpha \in \mathscr{G}(A, X)$. For any space X let PX be the space of all paths in X starting at the base point and $d_1 : PX \to X$ be the fibration defined by $d_1(\nu) = \nu(1)$ for all $\nu \in PX$. Let

$$B = \prod_{j \in J} K(\pi_j, n_j)$$

where J is any indexing set, $n_j \geqq 1$ and π_j abelian for $j \in J$. Let $p : E \to X$ be a principal fibration induced by a map $\theta : X \to B$ from the fibration $d_1 : PB \to B$. Suppose there is a map $\varphi : X \times A \to X$ such that $\varphi/X = 1_X$. Under what conditions does there exist a map $\tilde{\varphi} : E \times A \to E$ such that $\tilde{\varphi}/E = 1_E$ and the following diagram is commutative?

$$\begin{array}{ccc} E \times A & \xrightarrow{\tilde{\varphi}} & E \\ \downarrow{\scriptstyle p \times 1} & & \downarrow{\scriptstyle p} \\ X \times A & \xrightarrow{\varphi} & X \end{array}$$

This problem was considered by D. H. Gottlieb [**3**] and Lang [**6**] for the case $A = S^n$. We obtain a necessary and sufficient condition for such a $\tilde{\varphi}$ to exist and from that derive the results of [**3**] and [**6**].

For $j \in J$ let $i_j \in H^{n_j}(K(\pi_j, n_j); \pi_j)$ be n_j-characteristic for $K(\pi_j, n_j)$. Let CA be the cone over A, i.e., $A \times I$ with $* \times I \cup A \times 0$ collapsed to a single point. Let $\theta_j = p_j \circ \theta$ where p_j is the projection $B \to K(\pi_j, n_j)$.

THEOREM 2.1. *With the above notation, there exists $\tilde{\varphi} : E \times A \to E$ such that*

$\tilde{\varphi}/E = 1_E$ *and making the diagram*

$$\begin{array}{ccc} E \times A & \xrightarrow{\tilde{\varphi}} & E \\ \Big\downarrow p \times 1_A & & \Big\downarrow p \\ X \times A & \xrightarrow{\varphi} & X \end{array}$$

commutative if and only if $(p \times 1_{CA})^* \ \partial\varphi^* \ \theta_j^*(i_j) = 0$ *in* $H^{n_j+1}(E \times CA, E \times A; \pi_j)$ for all $j \in J$ *where*

$$\partial : H^{n_j}(X \times A; \pi_j) \to H^{n_j+1}(X \times CA, X \times A; \pi_j)$$

is the boundary map in cohomology.

Proof. Assume that the condition $(p \times 1_{CA})^*\partial\varphi_j^*(i_j) = 0$ is satisfied for all $j \in J$. Then $(p \times 1_A)^* \circ \varphi^* \circ \theta_j^*(i_j) \in H^{n_j}(E \times A; \pi_j)$ gets mapped into 0 by $\partial : H^{n_j}(E \times A; \pi_j) \to H^{n_j+1}(E \times CA, E \times A; \pi_j)$. Hence $\theta_j \circ \varphi \circ (p \times 1_A)$ can be extended to a map $E \times CA \to K(\pi_j, n_j)$ [**7**, Theorem 8.1.12]. Thus for $j \in J$ we have a map $F_j : E \times A \times I \to K(\pi_j, n_j)$ such that

$$F_j(e, a, 1) = \theta_j \circ \varphi(p(e), a), \quad F_j(e, a, 0) = \theta_j p(e)$$
$$\text{and } F_j(e, *, t) = \theta_j p(e) \text{ for all } t \in I.$$

Let $\bar{\theta} : E \to PB$ be the canonical map with $d_1 \circ \bar{\theta} = \theta \circ p$ and let $\bar{\theta}_j = \bar{p}_j \circ \bar{\theta}$ where $\bar{p}_j$ is the projection

$$PB = \prod_{j \in J} PK(\pi_j, n_j) \to PK(\pi_j, n_j).$$

Define $H_j : E \times A \times I \to K(\pi_j, n_j)$ by

$$H_j(e, a, t) = \begin{cases} \bar{\theta}_j(e)(2t), & \text{if } t \leqq \frac{1}{2} \\ F_j(e, a, 2t - 1), & \text{if } t \geqq \frac{1}{2}. \end{cases}$$

Then $H_j(e, a, 0) = \bar{\theta}_j(e)(0) = *$ for all $e \in E$ and $a \in A$. Hence H_j gives rise to a map $\tilde{H}_j : E \times A \to PK(\pi_j, n_j)$. Define $\tilde{H} : E \times A \to PB$ by $\bar{p}_j \circ \tilde{H} = \tilde{H}_j$ for all $j \in J$. Since $d_1 \circ \tilde{H} = \theta \circ \varphi \circ (p \times 1_A)$, there exists $\bar{\varphi}' : E \times A \to E$ such that $\bar{\theta} \circ \bar{\varphi}' = \tilde{H}$ and the following diagram is commutative:

$$\begin{array}{ccc} E \times A & \xrightarrow{\bar{\varphi}} & E \\ \Big\downarrow p \times 1_A & & \Big\downarrow p \\ X \times A & \xrightarrow{\varphi} & X. \end{array}$$

A homotopy $G : E \times I \to E$ from 1_E to $\bar{\varphi}'/E$ can be defined by $p \circ G(e, s) = p(e)$ and $\bar{\theta}\, G(e, s)(t) = \tilde{H}(e, *)(t/2 + st/2)$.

Let $L : X \times A \times I \to X$ be the constant homotopy from φ to φ. Define $\psi : E \times A \times 0 \cup E \times * \times I \to E$ by

$$\psi(e, a, 0) = \bar{\varphi}'(e, a) \quad \text{and} \quad \psi(e, *, t) = G(e, 1 - t).$$

Then the following diagram is commutative:

$$\begin{array}{ccc} E \times A \times 0 \cap E \times * \times I & \xrightarrow{\psi} & E \\ \downarrow & & \downarrow p \\ E \times A \times I & \xrightarrow[p \times 1_{A\times I}]{} X \times A \times I \xrightarrow[L]{} & X. \end{array}$$

So ψ can be extended to a map $\Phi : E \times A \times I \rightarrow E$ such that $p \circ \Phi = L \circ (p \times 1_{A\times I})$. Then $\bar{\varphi} = \Phi_1$ is the required map.

The converse is clear.

Let us now consider the special case $A = S^q$, $q \geqq 1$. Since $H^*(S^q; \mathbf{Z})$ is torsion free we have from the Kunneth relations

$$H^*(X \times S^q; \pi) \simeq H^*(X; \pi) \otimes H^*(S^q; \mathbf{Z})$$

for any coefficient group π. Thus given a generator s of $H^q(S^q; \mathbf{Z}) = \mathbf{Z}$ any element of $H^n(X \times S^q; \pi)$ can be written uniquely as $x \times 1 + y \times s$ where $x \in H^n(X; \pi)$, $y \in H^{n-q}(X; \pi)$ and $\times$ denotes the cohomology cross product. Since $\varphi/X = 1_X$, for any $u \in H^n(X; \pi)$ we have in $H^n(X \times S^q; \pi)$ $\varphi^*(u) = u \times 1 + v \times s$ for some $v \in H^{n-q}(X; \pi)$. The element v is denoted by $\bar{\lambda}(u)$. Then $\bar{\lambda} : H^*(X; \pi) \rightarrow H^*(X; \pi)$ is a group homorphism of degree $- q$.

COROLLARY 2.2. *Under the situation as in Theorem* 2.1. *with* $A = S^q$ *and* $n_j \geqq 2$, *if* $\bar{\lambda}(\theta_j^*(i_j)) = 0$ *in* $H^{n_j-q}(X; \pi)$ for all $j \in J$ *then there exists* $\bar{\varphi} : E \times S^q \rightarrow E$ *such that* $\bar{\varphi}/X = 1_X$ *and the following diagram is commutative.*

$$\begin{array}{ccc} E \times S^q & \xrightarrow{\bar{\varphi}} & E \\ {\scriptstyle p \times 1_{S^q}}\downarrow & & \downarrow p \\ X \times S^q & \xrightarrow[\varphi]{} & X \end{array}$$

Proof. We have $\varphi^*\theta_j^*(i_j) = \theta_j^*(i_j) \times 1 + \bar{\lambda}(\theta_j^*(i_j)) \times s$. Hence from the properties of cohomology cross product we have in $H^{n_j+1}(X \times E^{q+1}, X \times S^q; \pi_j)$,

$$\partial\varphi^*\theta_j^*(i_j) = (-1)^{n_j}\theta_j^*(i_j) \times \partial 1 + (-1)^{n_j-q}\bar{\lambda}_j(\theta_j^*(i_j)) \times \partial s.$$

Since $H^1(E^{q+1}, S^q; \mathbf{Z}) = 0$, $\partial 1 = 0$. Therefore,

$$\partial\varphi^*\theta_j^*(i_j) = (-1)^{n_j-q}\bar{\lambda}(\theta_j^*(i_j)) \times \partial s.$$

Since $\bar{\lambda}(\theta_j^*(i_j)) = 0$ for all $j \in J$, we have

$$(p \times 1_{E^{q+1}})^*\partial\varphi^*\theta_j^*(i_j) = 0 \quad \text{in } H^{n_j+1}(E \times E^{q+1}, E \times S^q; \pi_j)$$

for all $j \in J$

Hence the result follows from Theorem 2.1.

Let $p : E \rightarrow X$ be a principal fibration induced by a map $\theta : X \rightarrow K(\pi, n)$

where $n \geqq 2$. Let $i \in H^n(K(\pi, n); \pi)$ be n-characteristic for $K(\pi, n)$. Let $\varphi : X \times S^q \to X$ be a map such that $\varphi/_X = 1_X$. Let $\bar{\lambda} : H^*(X; \pi) \to H^*(X; \pi)$ be the homorphism of degree $-q$ defined by φ as above.

COROLLARY 2.3. *With notations as above, there exists a map $\bar{\varphi} : E \times S^q \to E$ such that $\bar{\varphi}/E = 1_E$ and making the diagram*

$$\begin{array}{ccc} E \times S^q & \xrightarrow{\bar{\varphi}} & E \\ {\scriptstyle p \times 1_{S^q}} \downarrow & & \downarrow {\scriptstyle p} \\ X \times S^q & \xrightarrow{\varphi} & X \end{array}$$

commutative if and only if $\bar{\lambda}(\theta^*(i)) = 0$ in $H^{n-q}(X; \pi)$.

Proof. As in the proof of Corollary 2.2, we have in $H^{n+1}(X \times E^{q+1}, X \times S^q; \pi)$,

$$\partial\varphi^*\theta^*(i) = (-1)^{n-q}\bar{\lambda}(\theta^*(i)) \times \partial s$$

and therefore

$$(p \times 1_{E^{q+1}})^*\partial\varphi^*\theta^*(i) = (-1)^{n-q}p^*\bar{\lambda}(\theta^*(i)) \times \partial s.$$

Since ∂s is a generator of $H^{q+1}(E^{q+1}, S^q; \mathbf{Z}) = \mathbf{Z}$ it follows that

$$(p \times 1_{E^{q+1}})^*\partial\varphi^*\theta^*(i) = 0$$

in $H^{n+1}(E \times E^{q+1}, E \times S^q; \pi)$ if and only if $p^*\bar{\lambda}(\theta^*(i)) = 0$ in $H^{n-q}(E; \pi)$. Hence by Theorem 2.1, a map $\bar{\varphi} : E \times S^q \to E$ with required properties exists if and only if $p^*\bar{\lambda}(\theta^*(i)) = 0$ in $H^{n-q}(E; \pi)$.

The fibre of $p : E \to X$ is a $K(\pi, n-1)$ and hence $n-2$ connected with $n - 2 \geqq 0$. By Lemma 2.4 below, $p^* : H^r(X; \pi) \to H^r(E; \pi)$ is a monomorphism for all $r \leqq n - 1$. Hence the result.

LEMMA 2.4. *Let $p : E \to B$ be a fibration with n connected fibre $F(n \geqq 0)$.* Then *for any coefficient group π*

$$p^* : H^q(B; \pi) \to H^q(E; \pi)$$

is an isomorphism for $q \leqq n$ and a monomorphism for $q = n + 1$.

Proof. Consider the homotopy exact sequence of the fibration $p : E \to B$

$$-- \to \pi_{q+1}(B) \xrightarrow{\partial} \pi_q(F) \longrightarrow \pi_q(E) \xrightarrow{p} \pi_q(B) \xrightarrow{\partial} \pi_{q-1}(F) \to --$$

Since $\pi_q(F) = 0$ for all $q \leqq n$, $p_\# : \pi_q(E) \to \pi_q(B)$ is an isomorphism for $q \leqq n$ and an epimorphism for $q = n + 1$. Hence by Whitehead's theorem [**7**, Theorem 7.5.9],

$$p_* : H_q(E) \to H_q(B)$$

is any isomorphism for $q \leqq n$ and an epimorphism for $q = n + 1$.

By the universal coefficient theorem we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}\,(H_q(E), \pi) & \to & H^{q+1}(E; \pi) & \to & \mathrm{Hom}\,(H_{q+1}(E), \pi) & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & \mathrm{Ext}\,(H_q(B), \pi) & \to & H^{q+1}(B; \pi) & \to & \mathrm{Hom}\,(H_{q+1}(B), \pi) & \to 0 \end{array}$$

where all the vertical maps are induced by p.

$\mathrm{Ext}\,(H_q(B), \pi) \to \mathrm{Ext}\,(H_q(E), \pi)$ is an isomorphism for $q \leqq n$ and $\mathrm{Hom}\,(H_q(B), \pi) \to \mathrm{Hom}\,(H_q(E), \pi)$ is an isomorphism for $q \leqq n$ and a monomorphism for $q = n + 1$. Now the result is evident.

Corollary 2.2 is Theorem 3.1 of [**6**] and Corollary 2.3 is Theorem 6.3 of [**4**].

3. Extending a map $\varphi : X \to X \vee A$ "co-affiliated" to $\alpha \in \mathscr{D}\mathscr{G}(X, A)$. In this section we consider the question dual to the one in Section 2.

Let

$$B = \bigvee_{\lambda \in \Lambda} K'(\pi_\lambda, n_\lambda)$$

be a wedge of Moore spaces with each $n_\lambda \geqq 3$. Let $e_1 : B \to CB$ be the canonical inclusion given by $e_1(b) = \langle b, 1 \rangle$. Let $\mu : X \to Y$ be the cofibration induced from the cofibration $e_1 : B \to CB$ by a map $\theta : B \to X$. Let $j_\lambda : K'(\pi_\lambda, n_\lambda) \to B$, $j_{X,A} : X \vee A \to X \times A$ be inclusions and $p_X : X \times A \to X$, $p_A : X \times A \to A$ be projections. Let $\theta_\lambda = \theta \circ j_\lambda$. Suppose $\varphi : X \to X \vee A$ is a map such that $p_X \circ j_{X,A} \circ \varphi \sim 1_X$. We are interested in finding conditions under which there exists a $\psi : Y \to Y \vee A$ such that $p_Y \circ j_{Y,A} \circ \psi \sim 1_Y$ and the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \vee A \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle \mu \vee 1_A} \\ Y & \xrightarrow{\varphi} & Y \vee A \end{array}$$

For this purpose we recall the homotopy exact sequence of a map for homotopy groups with coefficients in an abelian group ([**2**] or [**5**]).

Let π be an abelian group. For any space S and an integer $k \geqq 2$ $\pi_k(\pi; S)$, the kth homotopy group of S with coefficients in π is defined to be $[K'(\pi, k), S]$.

For any two spaces R and S and any map $h : R \to S$, $\pi_k(\pi; h)$, the kth homotopy group of h with coefficients in $\pi (k \geqq 3)$ is defined to be the set of homotopy classes of map pairs (u, v) where

$$\begin{array}{ccc} K'(\pi, k-1) & \xrightarrow{u} & R \\ \downarrow & & \downarrow{\scriptstyle h} \\ CK'(\pi, k-1) & \xrightarrow{v} & S \end{array}$$

is commutative. Then for $k \geqq 3$, $\pi_k(\pi; S) = \pi_k(\pi; \mathscr{O}_S)$ where $\mathscr{O}_S$ is the unique map $* \to S$.

The commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & R \arrow[d, "h"] \\ S \arrow[r, equal] & S \end{tikzcd}$$

defines a homomorphism $J : \pi_k(\pi; S) \to \pi_k(\pi; h)$ and there is an exact sequence

$$\longrightarrow \pi_{k+1}(\pi : h) \longrightarrow \pi_k(\pi : R) \xrightarrow{h_*} \pi_k(\pi : S) \xrightarrow{J} \pi_k(\pi : h) \longrightarrow \ldots\ldots$$

Now we are ready to state the following theorem.

THEOREM 3.1. *With notation as described in the beginning of this section, there exists a* $\psi : Y \to Y \vee A$ *such that* $p_y \circ j_{Y,A} \circ \psi \sim 1_Y$ *and making the diagram*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\mu"'] & X \vee A \arrow[d, "\mu \vee 1_A"] \\ Y \arrow[r, "\psi"'] & Y \vee A \end{tikzcd}$$

commutative if and only if $J((\mu \vee 1_A)_*\varphi_*\theta_{\lambda*}(i_\lambda)) = 0$ *in* $\pi_{n_\lambda}(\pi_\lambda; 1_\lambda \vee d_1)$ *for all* λ, *where* $i_\lambda \in \pi_{n_\lambda}(\pi_\lambda; K'(\pi_\lambda, n_\lambda)) = [K'(\pi_\lambda, n_\lambda), K'(\pi_\lambda, n_\lambda)]$ *is the homotopy class of the identity map,* $d_1 : PA \to A$ *is defined by* $d_1(\sigma) = \sigma(1)$ *and* J *is the homomorphism* $\pi_{n_\lambda}(\pi_\lambda; Y \vee A) \to \pi_{n_\lambda}(\pi_\lambda; 1_Y \vee d_1)$ *in the homotopy exact sequence of the map* $1_Y \vee d_1 : Y \vee PA \to Y \vee A$.

Proof. Assume $J((\mu \vee 1_A)_*\varphi_*\theta_{\lambda*}(i_\lambda)) = 0$ for all λ. Since the sequence

$$\pi_{n_\lambda}(\pi_n; Y \vee PA) \xrightarrow{(1_Y \vee d_1)_*} \pi_{n_\lambda}(\pi_n; Y \vee A) \xrightarrow{\quad J \quad} \pi_{n_\lambda}(\pi_n; 1_Y \vee d_1)$$

is exact and $J[(\mu \vee 1_A) \circ \varphi \circ \theta_\lambda] = J((\mu \vee 1_A)_*\varphi_*\theta_{\lambda*}(i_\lambda)) = 0$ there is $\nu_\lambda : K'(\pi_\lambda, n_\lambda) \to Y \vee PA$ such that the diagram

$$\begin{tikzcd} & Y \vee PA \arrow[d, "1_Y \vee d_1"] \\ K'(\pi_\lambda, n_\lambda) \arrow[ur, "\nu_\lambda"] \arrow[r, "(\mu \vee 1_A) \circ \varphi \circ \theta_\lambda"'] & Y \vee A \end{tikzcd}$$

is homotopy commutative.

Let $L_\lambda : K'(\pi_\lambda, n_\lambda) \times I \to Y \vee A$ be a homotopy with

$$L_\lambda(u, 0) = (1_Y \vee d_1) \circ \nu_\lambda(u) \quad \text{and} \quad L_\lambda(u, 1) = (\mu \vee 1_A) \circ \varphi \circ \theta_\lambda(u)$$
$$\text{for all } u \in K'(\pi_\lambda, n_\lambda).$$

Let $l : Y \vee PA \to (Y \vee A)^I$ be given by

$$l(y) = \text{the constant path at } y \text{ for all } y \in Y$$

and

$$l(\sigma) = \sigma \quad \text{for all } \sigma \in PA.$$

The composite $K'(\pi_\lambda, n_\lambda) \to Y \vee PA \xrightarrow{l} (Y \vee A)^I$ gives a map

$$N_\lambda : K'(\pi_\lambda, n_\lambda) \times I \to Y \vee A$$

such that

$$N_\lambda(u, 0) = p_Y \circ j_{Y,PA} \circ \nu_\lambda(u) \quad \text{and} \quad N_\lambda(u, 1) = (1_Y \vee d_1) \circ \nu_\lambda(u) \quad \text{for all } u \in K'(\pi_\lambda, n_\lambda).$$

Let $K : X \times I \to X$ be a homotopy with

$$K(x, 0) = x \quad \text{and} \quad K(x, 1) = p_X \circ j_{X,A} \circ \varphi(x) \text{ for all } x \in X.$$

Let $\bar{j}_\lambda$ be the inclusion $CK'(\pi_\lambda, n_\lambda) \to \vee\, CK'(\pi_\lambda, n_\lambda) = CB$. Let $\bar{\theta} : CB \to Y$ be the canonical map with $\bar{\theta} \circ e_1 = \mu \circ \theta$ and put $\bar{\theta}_\lambda = \bar{\theta}_\lambda = \bar{\theta} \circ \bar{j}_\lambda$.

Define $H_\lambda : CK'_\lambda(\pi_\lambda, n_\lambda) \to Y \vee A$ by

$$H_\lambda\langle u, t\rangle = \begin{cases} (\bar{\theta}_\lambda\langle u, 5t\rangle, *), & 0 \leqq t \leqq \frac{1}{5} \\ (\mu K(\theta_\lambda(u), 5t - 1), *), & \frac{1}{5} \leqq t \leqq \frac{2}{5} \\ (p_Y \circ j_{Y,A} \circ L_\lambda(u, 3 - 5t), *), & \frac{2}{5} \leqq t \leqq \frac{3}{5} \\ N_\lambda(u, 5t - 3), & \frac{3}{5} \leqq t \leqq \frac{4}{5} \\ L_\lambda(u, 5t - 4), & \frac{4}{5} \leqq t \leqq 1. \end{cases}$$

Let $H : CB \to Y \vee A$ be such that $H|CK'(\pi_\lambda, n_\lambda) = H_\lambda$. Then $H \circ e_1 = (\mu \vee 1_A) \circ \varphi \circ \theta$. Since

$$\begin{array}{ccc} B & \xrightarrow{\theta} & X \\ {\scriptstyle e_1}\downarrow & & \downarrow{\scriptstyle \mu} \\ CB & \xrightarrow[\bar{\theta}]{} & Y \end{array}$$

is a push out diagram there exists a $\Phi : Y \to Y \vee A$ such that

$$\Phi \circ \bar{\theta} = H \quad \text{and} \quad \Phi \circ \mu = (\mu \vee 1_A) \circ \varphi.$$

Thus we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & X \vee A & \xrightarrow{p_X \circ j_{X,A}} & X \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu \vee 1_A} & & \downarrow{\scriptstyle \mu} \\ Y & \xrightarrow[\Phi]{} & Y \vee A & \xrightarrow[p_Y \circ j_{Y,A}]{} & Y. \end{array}$$

Consider the commutative diagram

$$\begin{array}{ccc}
[X,\, Y \vee A] & \xrightarrow{(p_Y \circ j_{Y,A})_*} & [X,\, Y] \\
\uparrow \mu^* & & \uparrow \mu^* \\
[Y,\, Y \vee A] & \xrightarrow{(p_Y \circ j_{Y,A})} & [Y,\, Y] \\
\uparrow \eta^* & & \uparrow \eta^* \\
[\Sigma B,\, Y \vee A] & \xrightarrow{(p_Y \circ j_{Y,A})} & [\Sigma B,\, Y]
\end{array}$$

where $\eta : Y \to \Sigma B$ is the collapsing map (collapsing $\mu(X)$). The columns are parts of Puppe exact sequences.

Since $p_Y \circ j_{Y,A} \circ \Phi \circ \mu = \mu \circ p_X \circ j_{X,A} \circ \varphi$ and $p_X \circ j_{X,A} \circ \varphi \sim 1_X$ we have $\mu^*[p_Y \circ j_{Y,A} \circ \Phi] = [\mu]$ in $[X,\, Y]$. Also $\mu^*[1_Y] = [\mu]$. Therefore there exists $\alpha \in [\Sigma B,\, Y]$ such that $\alpha \cdot [p_Y \circ j_{Y,A} \circ \Phi] = [1_Y]$ where $\cdot$ denotes the action of $[\Sigma B,\, Y]$ on $[Y,\, Y]$. (See [**7**, 7.2.18]. Note that Y is the mapping cone of $\theta : B \to X$. Clearly $(p_Y \circ j_{Y,A})_* : [\Sigma B,\, Y \vee A] \to [\Sigma B,\, Y]$ is onto. Choose $\beta \in [\Sigma B,\, Y \vee A]$ such that $(p_Y \circ j_{Y,A})_*(\beta) = \alpha$. Let $\chi : Y \to Y \vee A$ represent the element $\beta \cdot [\Phi]$ of $[Y,\, Y \vee A]$. Then we have

$$\begin{aligned}
(p_Y j_{Y,A})_*[\chi] &= (p_Y \circ j_{Y,A})_*(\beta) \cdot (p_Y \circ j_{Y,A})_*[\Phi] \\
&= \alpha \cdot (p_Y \circ j_{Y,A})_*[\Phi] \\
&= [1_Y]
\end{aligned}$$

and $\mu^*[\chi] = \mu^*(\beta \cdot [\Phi]) = \mu^*[\Phi] = [(\mu \vee 1_A) \circ \varphi]$ in $[X,\, Y \vee A]$. Thus $p_Y \circ j_{Y,A} \circ \chi \sim 1_Y$ and $\chi \circ \mu \sim (\mu \vee 1_A) \circ \varphi$. Since $\mu : X \to Y$ is a cofibration there exists a $\psi : Y \to Y \vee A$ such that $\psi \sim \chi$ and the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\varphi} & X \vee A \\
\mu \downarrow & & \downarrow \mu \vee 1_A \\
Y & \xrightarrow[\psi]{} & Y \vee A
\end{array}$$

is commutative. Clearly $p_Y \circ j_{Y,A} \circ \psi \sim 1_Y$.

The converse is clear.

Remark 3.2. In Section 3 we could only deal with a wedge

$$\bigvee_{\lambda \in \Lambda} K'(\pi_\lambda,\, n_\lambda)$$

of Moore spaces with each $n_\lambda \geqq 3$ because the homotopy groups $\pi_k(\pi;\, h)$ of a map h with coefficients in π can be defined only when $k \geqq 3$. In fact the definition of $\pi_k(\pi;\, h)$ involves a $K'(\pi,\, k - 1)$ space and we are sure of its existence only when $k - 1 \geqq 2$ [**9**].

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