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NOTES ON THE BINDING NUMBERS FOR (a,b,k)-CRITICAL GRAPHS

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Let G be a graph of order n, and let a, b, k be nonnegative integers with $1 \leqslant a < b$. An $[a, b]$-factor of graph G is defined as a spanning subgraph F of G such that $a \leqslant d_F(x) \leqslant b$ for each $x \in V(F)$. Then a graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. In this paper, it is proved that G is an (a, b, k)-critical graph if the binding number

$$\mathrm{bind}(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$$

and

$$n \geqslant \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1}.$$

Furthermore, it is showed that the result in this paper is best possible in some sense.

1. Introduction

All graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the neighbourhood $N_G(x)$ of x is the set vertices of G adjacent to x, and the degree $d_G(x)$ of x is $|N_G(x)|$. The minimum vertex degree of $V(G)$ is denoted by $\delta(G)$. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{x \in S} N_G(x)$ and we denote by $G[S]$ the subgraph of G induced by S, by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. If $S \subseteq V(G)$, then $P_j(G-S)$ denotes the set of vertices in $G-S$ with degree j and $|P_j(G-S)| = p_j(G-S)$. The binding number $\mathrm{bind}(G)$ of G is the minimum value of $|N_G(X)|/|X|$ taken over all non-empty subsets X of $V(G)$ such that $N_G(X) \neq V(G)$. Let a and b be integers with $0 \leqslant a \leqslant b$. An $[a, b]$-factor of graph G is defined as a spanning subgraph F of G such that $a \leqslant d_F(x) \leqslant b$ for every vertex x of G (Where of course d_F denotes the degree in F). If $a = b = k$, then an $[a, b]$-factor is called a k-factor. A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. If G is an (a, b, k)-critical graph, then we also say that G is (a, b, k)-critical. If $a = b = n$,

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then an (a,b,k)-critical graph is simply called an (n,k)-critical graph. In particular, a $(1,k)$-critical graph is simply called a k-critical graph. The other terminologies and notations not given in this paper can be found in [**1**].

Many authors have investigated (g,f)-factors [**10, 11, 12, 7**]. The following results on k-factors and $[a,b]$-factors and (a,b,k)-critical graphs are known.

In [**4**], Katerinis and Woodall proved the following result for the existence of k-factors.

THEOREM 1. ([**4**]) *Let $k \geqslant 2$ be an integer and let G be a graph of order $p \geqslant 4k-6$ and binding number* $\mathrm{bind}(G)$ *such that kp is even and*

$$\mathrm{bind}(G) > \frac{(2k-1)(p-1)}{k(p-2)+3}.$$

Then G has a k-factor.

In [**6**], Li and Cai gave the following result for the existence of $[a,b]$-factors.

THEOREM 2. ([**6**]) *Let a and b be integers such that $1 \leqslant a < b$. Suppose that G is a graph of order $n \geqslant 2a+b+(a^2-a)/b$. If $\delta(G) \geqslant a$ and*

$$\max\{d_G(x), d_G(y)\} \geqslant \frac{an}{a+b}$$

for any two nonadjacent vertices x and y of G, then G has an $[a,b]$-factor.

In [**9**], Matsuda showed the following result for the existence of $[a,b]$-factors.

THEOREM 3. ([**9**]) *Let $1 \leqslant a < b$ be integers and G a graph of order*

$$n \geqslant \frac{(a-1)(a+1)(a+b)(a+b-1)}{a(b-1)} - \frac{(a+b)(ab+b-1)}{ab(b-1)}.$$

Suppose that $\delta(G) \geqslant a$ and

$$\max\{d_G(x), d_G(y)\} \geqslant \frac{an}{a+b}$$

for any vertices x and y of G with $d(x,y)=2$. Then G has an $[a,b]$-factor.

In [**3**], Kano proved the following result for the existence of $[a,b]$-factors.

THEOREM 4. ([**3**]) *Let a and b be integers such that $2 \leqslant a < b$, and let G be a graph of order n with $n \geqslant 6a+b$. Put $\lambda = (a-1)/b$. Suppose for any subset $X \subset V(G)$, we have*

$$N_G(X) = V(G) \qquad \text{if} \quad |X| \geqslant \left\lfloor \frac{n}{1+\lambda} \right\rfloor; \quad \text{or}$$
$$|N_G(X)| \geqslant (1+\lambda)|X| \qquad \text{if} \quad |X| < \left\lfloor \frac{n}{1+\lambda} \right\rfloor.$$

Then G has an $[a,b]$-factor.

In [5], Li obtained the following result for the existence of (a,b,n)-critical graphs.

Theorem 5. ([5]) *Let a, b, m and n be integers such that $1 \leqslant a < b$, and let G be a graph of order m with*

$$m \geqslant \frac{(a+b)(k(a+b)-2)}{b} + n.$$

If $\delta(G) \geqslant (k-1)a + n$, and

$$\left|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)\right| \geqslant \frac{am+bn}{a+b}$$

for any independent subset $\{x_1, x_2, \cdots, x_k\}$ of $V(G)$, where $k \geqslant 2$, then G is an (a,b,k)-critical graph.

Theorem 6. ([5]) *Let a, b, m and n be integers such that $1 \leqslant a < b$, and let G be a graph of order m with*

$$m \geqslant \frac{(a+b)(a+b+k-3+(a-2)(k-2)+1}{b} + n.$$

If $\delta(G) \geqslant (k-1)a + n$, and

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_k)\} \geqslant \frac{am+bn}{a+b}$$

for any independent subset $\{x_1, x_2, \cdots, x_k\}$ of $V(G)$, where $k \geqslant 2$, then G is an (a,b,k)-critical graph.

In [13], Zhou obtained the following result for the existence of (a,b,k)-critical graphs.

Theorem 7. ([13]) *Let a, b, k be integers with $1 \leqslant a < b$, $k \geqslant 0$, and let G be a graph of order $n \geqslant a+k+1$. If*

$$\delta(G) > n + a + b - 2\sqrt{bn - bk + 1},$$

then G is an (a,b,k)-critical graph.

In this paper, we discuss a binding number condition for a graph to be (a,b,k)-critical. The main results will be given in the following section.

2. The Proof of Main Theorems

Now we give our main theorems.

Theorem 8. *Let G be a graph of order n, and let a, b and k be nonnegative integers such that $1 \leqslant a < b$. If the binding number*

$$\operatorname{bind}(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$$

and

$$n \geqslant \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1},$$

then G is an (a,b,k)-critical graph.

In Theorem 8, if $k = 0$, then we get the following Corollary.

COROLLARY 1. *Let G be a graph of order n, and let a and b be two integers such that $1 \leqslant a < b$. If the binding number*

$$\operatorname{bind}(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)+2}$$

and

$$n \geqslant \frac{(a+b-1)(a+b-2)}{b},$$

then G has an $[a,b]$-factor.

According to Corollary 1, the following theorem obviously holds.

THEOREM 9. ([2]) *Let G be a graph of order n, $1 \leqslant a < b$. If the binding number*

$$\operatorname{bind}(G) > \frac{(a+b-1)(n-1)}{bn-2b+3}$$

and

$$n \geqslant \frac{(a+b-1)(a+b-2)}{b},$$

then G has an $[a,b]$-factor.

Let a, b and k be nonnegative integers such that $1 \leqslant a < b$. The proof of Theorem 8 relies heavily on the following lemma.

LEMMA 2.1. ([8]) *Let G be a graph of order $n \geqslant a+k+1$. Then G is (a,b,k)-critical if and only if for any $\subseteq V(G)$ and $|S| \geqslant k$*

$$\sum_{j=0}^{a-1}(a-j)p_j(G-S) \leqslant b|S| - bk, \quad \text{or}$$
$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \geqslant bk,$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \leqslant a-1\}$.

PROOF OF THEOREM 8: Suppose a graph G satisfies the condition of the theorem, but it is not an (a,b,k)-critical graph. Then, by Lemma2.1, there exists a subset S of $V(G)$ with $|S| \geqslant k$ such that

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \leqslant bk-1, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \leqslant a-1\}$. We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (1).

If $T = \varnothing$, then by (1), $bk - 1 \geqslant \delta_G(S,T) = b|S| \geqslant bk$, a contradiction. Hence, $T \neq \varnothing$. Let

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

According to the definition of T, we have

$$0 \leqslant h \leqslant a-1.$$

We shall consider various cases according to the value of h and derive contradictions. ▯

CASE 1. $h = 0$.

At first, we prove the following claim.

CLAIM 1. $\big(bn - (a+b) - bk + 2\big)/(n-1) > 1$.

PROOF: Since

$$n \geqslant \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1},$$

then we have

$$\begin{aligned} bn - (a+b) - bk + 2 - (n-1) &= (b-1)n - (a+b) - bk + 3 \\ &\geqslant (b-1)\Big(\frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1}\Big) \\ &\qquad -(a+b) - bk + 3 \\ &= \frac{(b-1)(a+b-1)(a+b-2)}{b} - (a+b) + 3 \\ &\geqslant (a+b-2) - (a+b) + 3 > 0 \end{aligned}$$

Thus, we have

$$\frac{bn - (a+b) - bk + 2}{n-1} > 1.$$

Let $m = \Big|\{x : x \in T,\ d_{G-S}(x) = 0\}\Big|$, and let $Y = V(G) \setminus S$. Then $N_G(Y) \neq V(G)$ since $h = 0$. In view of the definition of the binding number $\mathrm{bind}(G)$, we get that

$$|N_G(Y)| \geqslant \mathrm{bind}(G)|Y|.$$

Thus, we obtain

$$n - m \geqslant |N_G(Y)| \geqslant \mathrm{bind}(G)|Y| = \mathrm{bind}(G)\big(n - |S|\big),$$

that is,

$$|S| \geqslant n - \frac{n-m}{\mathrm{bind}(G)}. \tag{2}$$

Using $|S| + |T| \leqslant n$ and (1) and (2) and Claim 1, we have

$$\begin{aligned} bk - 1 \geqslant \delta_G(S,T) &= b|S| + d_{G-S}(T) - a|T| \\ &\geqslant b|S| - (a-1)|T| - m \\ &\geqslant b|S| - (a-1)\big(n - |S|\big) - m \end{aligned}$$

$$\begin{aligned}
&= (a+b-1)|S| - (a-1)n - m \\
&\geqslant (a+b-1)\Big(n - \frac{n-m}{\operatorname{bind}(G)}\Big) - (a-1)n - m \\
&= bn - (a+b-1)\Big(\frac{n-m}{\operatorname{bind}(G)}\Big) - m \\
&> bn - (a+b-1)\Big(\frac{(n-m)(bn-(a+b)-bk+2)}{(a+b-1)(n-1)}\Big) - m \\
&= bn - \Big(\frac{(n-m)(bn-(a+b)-bk+2)}{n-1}\Big) - m \\
&\geqslant bn - \Big(\frac{(n-1)(bn-(a+b)-bk+2)}{n-1}\Big) - 1 \\
&= bk + (a+b) - 3 \\
&\geqslant bk,
\end{aligned}$$

which is a contradiction. □

CASE 2. $1 \leqslant h \leqslant a-1$.

Let x_1 be a vertex in T such that $d_{G-S}(x_1) = h$, and let $Y = (V(G) \setminus S) \setminus N_{G-S}(x_1)$. Then $x_1 \in Y \setminus N_G(Y)$, so $Y \neq \varnothing$ and $N_G(Y) \neq V(G)$. In view of the definition of the binding number $\operatorname{bind}(G)$, we obtain

$$\frac{|N_G(Y)|}{|Y|} \geqslant \operatorname{bind}(G).$$

Thus, we get that

$$n-1 \geqslant \big|N_G(Y)\big| \geqslant \operatorname{bind}(G)|Y| = \operatorname{bind}(G)\big(n-h-|S|\big),$$

that is,

$$|S| \geqslant n - h - \frac{n-1}{\operatorname{bind}(G)}. \tag{3}$$

By $|S| + |T| \leqslant n$ and (1) and (3), we obtain

$$\begin{aligned}
bk-1 \geqslant \delta_G(S,T) &= b|S| + d_{G-S}(T) - a|T| \\
&\geqslant b|S| - (a-h)|T| \\
&\geqslant b|S| - (a-h)\big(n-|S|\big) \\
&= (a+b-h)|S| - (a-h)n \\
&\geqslant (a+b-h)\Big(n-h-\frac{n-1}{\operatorname{bind}(G)}\Big) - (a-h)n \\
&> (a+b-h)\Big(n-h-\frac{bn-(a+b)-bk+2}{a+b-1}\Big) - (a-h)n. \qquad (4)
\end{aligned}$$

Let

$$f(h) = (a+b-h)\Big(n-h-\frac{bn-(a+b)-bk+2}{a+b-1}\Big) - (a-h)n.$$

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In fact, the function $f(h)$ attains its minimum value at $h = 1$ since $1 \leqslant h \leqslant a - 1$ is an integer. Then, we have

$$f(h) \geqslant f(1).$$

Combining this with (4), we obtain

$$\begin{aligned} bk - 1 > f(1) &= (a+b-1)\Big(n - 1 - \frac{bn - (a+b) - bk + 2}{a+b-1}\Big) - (a-1)n \\ &= (a+b-1)(n-1) - (bn - (a+b) - bk + 2) - (a-1)n \\ &= bk - 1, \end{aligned}$$

that is a contradiction.

From the argument above, we deduce the contradictions, so the hypothesis can not hold. Hence, G is (a, b, k)-critical.

Completing the proof of Theorem 8.

REMARK. Let us show that the condition

$$\text{bind}(G) > \frac{(a+b-1)(n-1)}{bn - (a+b) - bk + 2}$$

in Theorem 8 can not be replaced by

$$\text{bind}(G) \geqslant \frac{(a+b-1)(n-1)}{bn - (a+b) - bk + 2}.$$

Let $b > a \geqslant 2, k \geqslant 0$ be three integers such that $a + b + k$ is odd, and let

$$n = \frac{(a+b-1)(a+b-2) + (a+b-2) + (a+2b-1)k}{b}$$

is an integer, and let $l = (a+b+k-1)/2$ and

$$m = n - 2l = n - (a+b+k-1) = \frac{(a+b-1)(a-2) + (a+b-2) + (a+b-1)k}{b}.$$

Clearly, m is an integer. Let $H = K_m \bigvee lK_2$. Let $X = V(lK_2)$, for any $x \in X$, then $|N_H(X \setminus x)| = n - 1$. By the definition of bind(H),

$$\text{bind}(H) = \frac{|N_H(X \setminus x)|}{|X \setminus x|} = \frac{n-1}{2l-1} = \frac{n-1}{a+b+k-2} = \frac{(a+b-1)(n-1)}{bn - (a+b) - bk + 2}.$$

Let $S = V(K_m) \subseteq V(H)$, $T = V(lK_2) \subseteq V(H)$, then $|S| = m \geqslant k$, $|T| = 2l$. Thus, we get

$$\begin{aligned} \delta_H(S,T) &= b|S| - a|T| + d_{H-S}(T) \\ &= b|S| - a|T| + |T| = b|S| - (a-1)|T| \\ &= b\frac{(a+b-1)(a-2) + (a+b-2) + (a+b-1)k}{b} \\ &\qquad - (a-1)(a+b+k-1) \\ &= bk - 1 < bk. \end{aligned}$$

By Lemma 2.1, H is not an (a, b, k)-critical graph. In the above sense, Theorem 8 is best possible.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph theory with applications* (The Macmillan Press, London, 1976).

[2] C. Chen, 'Binding number and minimum degree for $[a, b]$-factor', *J. Systems Sci. Math. Sci.* **6** (1993), 179–185.

[3] M. Kano, 'A sufficient condition for a graph to have $[a, b]$-factors', *Graphs Combin.* **6** (1990), 245–251.

[4] P. Katerinis and D.R. Woodall, 'Binding numbers of graphs and the existence of k-factors', *Quart. J. Math. Oxford* **38** (1987), 221–228.

[5] J. Li, 'Sufficient conditions for graphs to be (a, b, n)-critical graphs', *Mathematica Applicata (Wuhan)* **17** (2004), 450–455.

[6] Y. Li and M. Cai, 'A degree condition for a graph to have $[a, b]$-factors', *J. Graph Theory* **27** (1998), 1–6.

[7] G. Liu, '$(g < f)$-factors of graphs', *Acta Math. Sci. (Chinese)* **14** (1994), 285–290.

[8] G. Liu and J. Wang, '(a, b, k)-critical graphs', *Adv. Math. (China)* **27** (1998), 536–540.

[9] H. Matsuda, 'Fan-type results for the existence of $[a, b]$-factors', *Discrete Math.* **306** (2006), 688–693.

[10] S. Zhou, 'Some sufficient conditions for graphs to have (g, f)-factors', *Bull. Austral. Math. Soc.* **75** (2007), 447–452.

[11] S. Zhou and X. Xue, '(g, f)-factors of graphs with prescribed properties', *J. Systems Sci. Math. Sci. (Chinese)* (to appear).

[12] S. Zhou and X. Xue, 'Complete-factors and (g, f)-covered graphs', *Australas. J. Combin.* **37** (2007), 265–269.

[13] S. Zhou, 'Sufficient conditions for (a, b, k)-critical graphs', *J. Jilin Univ. Sci.* **43** (2005), 607–609.

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