Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1019–1026

COHOMOLOGY THEOREMS FOR BOREL-LIKE SOLVABLE LIE ALGEBRAS IN ARBITRARY CHARACTERISTIC

G. LEGER AND E. LUKS

1. Introduction. This paper develops some techniques for the study of derivation algebras and cohomology groups of Lie algebras. We are especially concerned with solvable algebras over arbitrary fields with structural properties like those of the Borel subalgebras of complex semi-simple Lie algebras. In particular, these algebras are semi-direct sums of nilpotent ideals and abelian subalgebras which act on the ideals in a semi-simple fashion. We make strong use, in our discussion, of a cohomology theorem of Hochschild-Serre. This result is stated herein (§ 2) in a modified form which allows us to omit the original hypothesis that the base field have characteristic 0.

Sections 3 and 4 are devoted mainly to the study of $H^1(B, B)$ for our class of algebras B, and to completeness theorems. As one application of these results we are able to obtain $H^1(B, B)$ quickly in the case where B is the Lie algebra, in arbitrary characteristic, of all triangular matrices or that of all trace 0 triangular matrices. This result was derived previously [**7**] only by a rather formidable computation. In addition, we shall apply these results in [**4**] and elsewhere.

In § 5 we turn to $H^2(B, B)$ and more generally to $H(B, B)$ for "Borel-like" solvable algebras. We obtain $H^2(B, B) = 0$ for a wide class of such algebras which exist in any characteristic $\neq 2$. Thus, by a rigidity theorem of Nijenhuis-Richardson [**5**], these form a large collection of solvable rigid Lie algebras. Finally, our technique, coupled with a result of Kostant, gives $H(B, B) = 0$ for Borel subalgebras of semi-simple Lie algebras over fields of characteristic 0.

We shall be considering cochain complexes $C(K, M)$, where K is a Lie algebra, M a K-module, with the usual coboundary [**1**]; in the case when K is a subalgebra of a Lie algebra H, M an ideal in H, K will be understood to act on M via the adjoint representation. Depending on our point of view we shall use either $\Delta(K)$ or $Z^1(K, K)$ to denote the derivation algebra of the Lie algebra K; $I(K)$ will denote the algebra of inner derivations. $\mathfrak{Z}(K)$ will denote the centre of K and, as usual, K will be called complete if $\mathfrak{Z}(K) = 0$ and $\Delta(K) = I(K)$ (i.e., $H^0(K, K) = H^1(K, K) = 0$). An ideal of K is called characteristic if it is invariant under derivations of K. If M is a K-module, $M^K = \{m \text{ in } M | K \cdot m = 0\}$. We recall that an exact sequence $X \rightarrow Y \rightarrow Z$ of

Received March 1, 1971. This research was partially supported by NSF Grants GP11162 and GP11163.

semi-simple K-modules gives rise to the exact sequence $X^K \to Y^K \to Z^K$. A linear Lie algebra is called toroidal if it can be diagonalized over an algebraic closure of the base field; a representation ρ of K is called toroidal if $\rho(K)$ is toroidal. We remark, finally, that all algebras considered will be finite dimensional.

2. A theorem of Hochschild-Serre.

In this section we present a result extracted from [**1**, Theorem 13] of Hochschild-Serre. Our contribution is merely the observation that the proof in [**1**] does not, for our statement, require characteristic 0.

2.1 THEOREM. *Let $B = A + L$ be a semi-direct sum of an abelian subalgebra, A, and an ideal, L, and let M be a B-module. Suppose that $\mathrm{ad}_L A$ and the representation of A on M are toroidal. Then, for $n \geqq 0$,*

$$H^n(B, M) \cong \sum_{i+j=n} H^i(A, F) \otimes H^j(L, M)^A$$

where the base field is considered as a trivial A-module.

We note, for future reference, the obvious consequence:

2.2 *Remark.* $H^i(B, M) = 0$ for $0 \leqq i \leqq n$ if and only if $H^i(L, M)^A = 0$ for $0 \leqq i \leqq n$.

3. $\Delta(B)$ and completeness.

Throughout this section we consider a Lie algebra B, over an arbitrary base field, which is a semi-direct sum $B = A + L$ of an abelian subalgebra A and a (not necessarily nilpotent) ideal L such that $\mathrm{ad}_L A$ is toroidal.

We have an injection

$$Z^1(L, B)^A \xrightarrow{e} \Delta(B)$$

which extends a cocycle z in $Z^1(L, B)^A$ so that $e(z)(A) = 0$ and $e(z)|L = z$. We shall often identify $Z^1(L, B)^A$ with its image under e in $\Delta(B)$. We note also that if L is a characteristic ideal and $L^A = 0$ then $Z^1(L, B)^A \cong Z^1(L, L)^A$. Further, we may clearly identify $\mathrm{Hom}(A, \mathfrak{Z}(B))$ with the set of derivations of B which vanish on L and map A into $\mathfrak{Z}(B)$. Since $B = A \cdot B \oplus B^A$ as an A-module and $\mathfrak{Z}(B) \cap B^A$ we have $\mathrm{Hom}(A, \mathfrak{Z}(B)) \cap I(B) = 0$. As our first application of Theorem 2.1 we see that to determine $\Delta(B)$ it suffices to know these derivations.

3.1 PROPOSITION. $\Delta(B) = Z^1(L, B)^A + \mathrm{Hom}(A, \mathfrak{Z}(B)) + I(B)$.

Proof. Letting $\overline{}$ denote cohomology class in $H^1(B, B)$, we show that

$$\overline{Z^1(L, B)^A} \oplus \overline{\mathrm{Hom}(A, \mathfrak{Z}(B))} = H^1(B, B)$$

This result is essentially contained in 2.1 and may be verified by tracing the actual identifications involved in the theorems leading to [**1**, Theorem 13].

However, for the reader's convenience we verify the equality by checking dimensions. Note that $\overline{Z^1(L, B)^A} = H^1(L, B)^A$ and $\dim \overline{\mathrm{Hom}(A, \mathfrak{Z}(B))} = \dim \mathrm{Hom}(A, \mathfrak{Z}(B))$. Then, applying Theorem 2.1 with $M = B$ and $n = 1$ and noting that $H^0(L, B)^A \cong \mathfrak{Z}(B)$, we see that both spaces have dimension $= \dim H^1(L, B)^A + (\dim A)(\dim \mathfrak{Z}(B))$.

3.2 COROLLARY. *Suppose L is a characteristic ideal in $B = A + L$, $\mathfrak{Z}(B) = 0$, and the derivations of L which commute with* $\mathrm{ad}_L A$ *are contained in* $\mathrm{ad}_L A + I(L)$. *Then B is complete.*

Proof. It suffices to show that $H^1(L, B)^A = 0$. But from the short exact sequence

$$0 \to L \xrightarrow{i} B \xrightarrow{p} B/L \to 0$$

of B-modules we obtain the exact sequence

$$H^1(L, L)^A \xrightarrow{i^*} H^1(L, B)^A \xrightarrow{p^*} H^1(L, B/L)^A.$$

By hypothesis the image of $Z^1(L, L)^A$ in $C^1(L, B)^A$ is in $B^1(L, B)$. Thus $i^* = 0$. Since L is characteristic, an element of $Z^1(L, B)^A$ maps L into L and hence $p^* = 0$.

Our search for the following proposition was motivated by the Schenkman-Wielandt Tower Theorem: In a finite number of steps the derivation tower, K, $\Delta(K)$, $\Delta(\Delta(K))$, . . . , etc., of a centreless Lie algebra K yields a complete algebra. Proposition 3.3 illustrates that for a large class of algebras $\Delta(K)$ is already complete. We first make the following observation.

Remark. If K is a Lie algebra with $\mathfrak{Z}(K) = 0$ then, with the usual identifications, $K \subset \Delta(K) \subset \Delta(\Delta(K))$. Note that $\Delta(K)$ is complete if and only if K is an ideal in $\Delta(\Delta(K))$.

3.3 PROPOSITION. *Suppose the centre of $B = A + L$ is* 0 *and $B^2 = L$. Then $\Delta(B)$ is complete.*

Proof. We show that B is an ideal in $\Delta(\Delta(B))$. Take c in $\Delta(\Delta(B))$. Since $L = B^2$, $[c, L] \subset B$. $\Delta(\Delta(B))$ is a semi-simple $\mathrm{ad}\, A$ module and so we can write $c = c_1 + c_2$ with $[c_1, A] = 0$ and $c_2 \in [A, \Delta(\Delta(B))] \subset \Delta(B)$. Then $[c, A] = [c_2, A] \subset B$.

3.4 *Example.* We demonstrate the necessity of the hypothesis $B^2 = L$ in Proposition 3.3; e.g., it does not suffice just to assume that L is a characteristic ideal in B. Let L be the 5-dimensional nilpotent algebra over any field F, of characteristic not 2 or 3, with basis x_1, x_2, x_3, x_4, x_5 such that

$$[x_1, x_3] = x_4; [x_1, x_4] = [x_2, x_3] = x_5$$

and $[x_i, x_j] = 0$ for $i < j$, otherwise.

https://doi.org/10.4153/CJM-1972-103-1 Published online by Cambridge University Press

The linear transformation, a, of L given by $a = \mathrm{diag}(0, 0, 1, 1, 1)$ with respect to this basis is a derivation of L. Let $A = Fa$ and form the semi-direct sum $B = A + L$. Proposition 3.1 and consideration of the weight spaces for A quickly reveal that $\dim H^1(B, B) = 2$. We choose representative outer derivations b and y in $Z^1(L, B)^A$ where $b(x_1) = x_1$, $b(x_2) = 2x_2$, $b(x_3) = 0$, $b(x_4) = x_4$, $b(x_5) = 2x_5$ and $y(x_1) = x_2$, $y(x_2) = 0$, $y(x_3) = 0$, $y(x_4) = x_5$, $y(x_5) = 0$. Thus $\Delta(B) = \hat{B} = \hat{A} + \hat{L}$ where $\hat{A} = Fb \oplus A$, $\hat{L} = Fy + L$. $\hat{B}$ being of the "right" form, Proposition 3.1 leads now to a single representative outer derivation, c, in $Z^1(L, \hat{B})^{\hat{A}}$ where $c(y) = y$, $c(x_1) = y$, $c(x_2) = x_2$, $c(x_3) = 0$, $c(x_4) = 0$, $c(x_5) = x_5$. Thus, in particular, $\Delta(B)$, is not complete. Note, finally, that $\Delta(\Delta(B)) = (Fc \oplus \hat{A}) + \hat{L}$ *is* complete by Corollary 3.2 (This also follows from Proposition 3.3 since $\hat{B}^2 = \hat{L}$.)

4. Applications. The following proposition is useful in the application of the results of section 3.

4.1 PROPOSITION. *Let L be a Lie algebra, A a maximal toroidal subalgebra of $\Delta(L)$ and A' a subalgebra of A of codimension m. Suppose the weight spaces (after passing to an algebraic closure of the base field) of A' are one dimensional and that zero is not a weight of A'. Then* $\dim H^1(A' + L, A' + L) = m$. *In particular, $A + L$ is complete.*

Proof. Since zero is not a weight, $\mathfrak{Z}(A' + L) = 0$. Then Proposition 3.1 yields $\Delta(A' + L) = Z^1(L, A' + L)^{A'} + I(A' + L)$. Since $(A' + L)^2 = L$, L is a characteristic ideal and so $Z^1(L, A' + L)^{A'} \cong Z^1(L, L)^{A'}$. But the hypothesis on the weight spaces implies that $Z^1(L, L)^{A'}$ is toroidal. Thus, because A is maximal, $Z^1(L, L)^{A'} = A$. Note that A is the unique maximal toroidal extension of A'. Since the elements of A' induce inner derivations of $A' + L$, the conclusion follows.

4.2 *Remark.* In the situations herein to which we apply Proposition 4.1 the maximality of A will be assured by the fact that L will be nilpotent and $\dim A = \dim(L/L^2)$.

4.3 COROLLARY. *A Borel subalgebra of a semi-simple Lie algebra over a field of characteristic* 0 *is complete.*

In the particular case of a Borel subalgebra of A_l this result appears in [**2**, Exercise 3.3]. More generally, the derivation algebra of the Lie algebra of trace zero triangular matrices has been obtained in arbitrary characteristic by Tôgô [**7**] via a lengthy computation. We shall show how our methods facilitate the latter author's results. There are actually three algebras of interest: T, the Lie algebra of all triangular matrices, T_0 the trace zero matrices in T, and $T_1 = T/\mathfrak{Z}(T)$. We introduce some notation: let N be the set of strictly triangular $n \times n$ matrices (i.e., with only zero entries on or below the diagonal) over a field of arbitrary characteristic p; D the set of diagonal

https://doi.org/10.4153/CJM-1972-103-1 Published online by Cambridge University Press

matrices; D_0 the trace zero matrices in D; D_1 any fixed vector space complement in D to the subspace spanned by the identity, ι; e_{ij} the $n \times n$ matrix with 1 in the ith row and jth column and zero elsewhere. Now $T = D + N$, $T_0 = D_0 + N$ and it is clear that $\mathfrak{Z}(T) = F\iota$ so $T_1 \cong D_1 + N$; also, $T_1 \cong T_0$ if and only if $n \not\equiv 0 \bmod p$.

The next lemma seems to pinpoint the reason for the existence of exceptional cases in [**7**, Theorem 1(ii)] or our Proposition 4.5 (iii).

4.4 Lemma. *The weight spaces for* $\mathrm{ad}_N D$ *and* $\mathrm{ad}_N D_1$ *are one dimensional. Further, if* $n \neq 3$ *or* 4 *the weight spaces for* $\mathrm{ad}_N D_0$ *are one dimensional.*

Proof. Clearly, the weight spaces for $\mathrm{ad}_N D = \mathrm{ad}_N D_1$ are precisely the Fe_{ij} with $i < j$. Suppose that the weight spaces for $\mathrm{ad}_N D_0$ are *not* one dimensional (then, of course, $n \equiv 0 \bmod p$). Since $D_0 \subset D$ there is a pair e_{ij}, e_{kl} with $(i, j) \neq (k, l)$ belonging to the same weight space for $\mathrm{ad}_N D_0$. We may choose r in the set $\{i, j\}$ and not in $\{k, l\}$; but then, if there were an s, $1 \leqq s \leqq n$, not in $\{i, j, k, l\}$, e_{ij} and e_{kl} would be in different weight spaces for $\mathrm{ad}_N(e_{rr} - e_{ss})$. Thus $n = 3$ or 4.

Proposition (Tôgô). (i) $T_1 (\cong T_0$ *if* $n \not\equiv 0 \bmod p)$ *is complete.*

(ii) *The outer derivation algebra of* T *has dimension* n.

(iii) *If* $n \equiv 0 \bmod p$ *and* $n \neq 2, 3, 4$, *the outer derivation algebra of* T_0 *has dimension* n.

Proof. By 4.2 and 4.4 we see that we may apply 4.1 to conclude that $T_1 = D_1 + N$ is complete; thus (i) is proved. Because of (i) and the decomposition $T = F\iota \oplus (D_1 + N)$ the outer derivation algebra of T may be identified with the space $\mathrm{Hom}(D, F\iota)$; this proves (ii). With T_0 as in (iii), 4.1, 4.2 and 4.4 yield $\dim H^1(D_0 \cap D_1 + N, D_0 \cap D_1 + N) = 1$ ($n \neq 2$ is required to assure that zero is not a weight). Thus (iii) follows from the observation $T_0 = F\iota \oplus (D_0 \cap D_1 + N)$ (when $n \equiv 0 \bmod p$).

5. $H^2(B, B)$ and higher cohomology.

5.1 In the ensuing proposition and its corollary, L will denote a nilpotent Lie algebra over a field, F, of characteristic not 2, A a subalgebra of $\Delta(L)$, B the semi-direct sum $B = A + L$ and we shall assume the following properties (i)–(iv) hold.

(i) A is diagonalizable over F and $\dim A = \dim(L/L^2) = n$.
Let W denote the set of weights of A in L and, for each α in W, denote by L_α the weight space for α.

(ii) For α in W, $\dim L_\alpha = 1$ and, if $\alpha, \beta, \alpha + \beta$ are all in W, $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.
We fix $\alpha_1, \alpha_2, \ldots, \alpha_n$ in W and $x_1, x_2, \ldots, x_n$ in L so that $L = \sum Fx_i + L^2$ (vector space direct sum) and $a \cdot x_j = \alpha_j(a)x_j$ for a in A. The weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ will be called primitive. Every weight in W has the form $\sum_{i=1}^n r_i\alpha_i$ where the r_i are non-negative integers.

(iii) If the characteristic of F is $p \neq 0$ we assume further, for every α in W, that $0 \leqq r_i < p/2$ for each i.

Since each weight in L^2 is the sum of two weights, it follows from (iii) that zero is not in W.

(iv) If $\alpha, \beta, \gamma, \delta, \alpha + \gamma, \beta + \delta$ are all in W with α, β primitive and unequal and with $\alpha + \gamma = \beta + \delta$, then there is some μ in W such that $\delta = \alpha + \mu$, $\gamma = \beta + \mu$ and at least one of the following is satisfied:

Case 1. $\alpha + \beta$ is not in W.

Case 2. $\alpha + \beta$ is in W but $\alpha + 2\beta$ is not in W and $\mu = \beta + \nu$ for some ν in W.

Case 3. $\alpha + \beta$ is in W but $2\alpha + \beta$ is not in W and $\mu = \alpha + \nu$ for some ν in W.

5.2 *Remark*. Properties (i)–(iv) are abstracted from two large classes of solvable algebras. The first is the class of Borel subalgebras of semi-simple Lie algebras in characteristic 0. The second is a class of algebras, introduced in [**4**], in characteristic not 2, in which the weights are in one-one correspondence with the subtrees of any tree graph, the primitive weights corresponding to the vertices. In all these examples, cases 2 and 3 of (iv) occur only in the presence of Borel subalgebras of the simple algebras C_l and F_4.

5.3 Proposition. *Suppose A, L are as in* 5.1, (i)–(iv). *Let M be an $A + L$ module such that the representation of A on M is toroidal and the weights of A in M are in W. Then $H^2(L, M)^A = 0$.*

Proof. Since $L_\alpha \cdot M_\beta \subset M_{\alpha+\beta}$, L operates on M by means of nilpotent transformation and so M has a one dimensional $A + L$ submodule. Since one always has the exact cohomology sequence

$$H^2(L, M')^A \rightarrow H^2(L, M)^A \rightarrow H^2(L, M/M')^A$$

arising from an $A + L$ submodule M' of M, it suffices to consider $\dim M = 1$.

Suppose then $M = Fm$ and denote the weight of A on M by Λ. Choose, for each α in W, a non-zero x_α in L_α. Let the scalar $c(\alpha, \beta)$, for α, β in $\mathrm{Hom}(A, F)$, be defined by

$$\begin{aligned} [x_\alpha, x_\beta] &= c(\alpha, \beta)x_{\alpha+\beta}, \text{ if } \alpha, \beta, \alpha + \beta \text{ are all in } W,\\ c(\alpha, \beta) &= 0, \text{ otherwise.} \end{aligned}$$

Let $\bar{f} \in H^2(L, M)^A$. We choose a representative cocycle $f \in \bar{f}$ such that $A \cdot f = 0$. We shall show f cobounds. Define the scalar $\Phi(\alpha, \beta)$, for α, β in $\mathrm{Hom}(A, F)$ by

$$\begin{aligned} f(x_\alpha, x_\beta) &= \Phi(\alpha, \beta)m, \text{ if } \alpha, \beta \text{ are in } W,\\ \Phi(\alpha, \beta) &= 0, \text{ otherwise.} \end{aligned}$$

Since $A \cdot f = 0$, $\Phi(\alpha, \beta) \neq 0$ only if $\alpha + \beta = \Lambda$.

Let ρ, σ, τ be in W. The Jacoby identity and the cocycle condition applied to x_ρ, x_σ, x_τ give:

$$\text{(a)} \quad \begin{aligned} &c(\rho, \sigma)c(\rho + \sigma, \tau) + c(\sigma, \tau)c(\sigma + \tau, \rho) + c(\tau, \rho)c(\tau + \rho, \sigma) = 0,\\ &c(\rho, \sigma)\Phi(\rho + \sigma, \tau) + c(\sigma, \tau)\Phi(\sigma + \tau, \rho) + c(\tau, \rho)\Phi(\tau + \rho, \sigma) = 0. \end{aligned}$$

Suppose, in particular, that $\rho + \sigma$, $\sigma + \tau$ are in W and $\tau + \rho$ is not in W. Then $c(\tau + \rho, \sigma) = \Phi(\tau + \rho, \sigma) = 0$, and (a) yields:

$$\text{(b)} \qquad c(\sigma + \tau, \rho)\Phi(\rho + \sigma, \tau) = c(\rho + \sigma, \tau)\Phi(\sigma + \tau, \rho).$$

Now let $\alpha, \beta, \gamma, \delta$ be in W with α, β primitive and $\alpha + \gamma = \beta + \delta = \Lambda$. We claim that

$$(*) \qquad c(\alpha, \gamma)\Phi(\beta, \delta) = c(\beta, \delta)\Phi(\alpha, \gamma).$$

By 5.1 (iv), $\gamma = \beta + \mu$, $\delta = \alpha + \mu$ for some μ in W. In case 1 of (iv), (b) applied with $\rho = \alpha$, $\sigma = \mu$, $\tau = \beta$ is precisely $(*)$. In case 2, (a) applied with $\rho = \alpha$, $\sigma = \mu$, $\tau = \beta$ and (b) applied with $\rho = \beta$, $\sigma = \nu$, $\tau = \alpha + \beta$ yield the claim; case 3 is similar. Considering $(*)$ we may then consistently define g in $C^1(L, M)$ so that

$$g(x_\rho) = 0 \text{ for } \rho \neq \Lambda$$

and

$$c(\alpha, \gamma)g(x_\Lambda) = -\Phi(\alpha, \gamma)m$$

for any α, γ in W such that α is primitive and $\alpha + \gamma = \Lambda$.

Let $\hat{f} = f - \delta g$. By construction, $\hat{f}(x_\alpha, L) = 0$ if α is primitive. But then, by the cocycle condition, $\hat{f}(x_\rho, L) = 0$ implies, for α primitive, that $f([x_\alpha, x_\rho], L) = 0$. Thus one sees inductively that $\hat{f} = 0$, i.e., $f = \delta g$.

5.4 Corollary. *Let $B = A + L$ as described in* 5.1, (i)–(iv). *Then* $H^2(B, B) = 0$.

Proof. Since, by Proposition 4.1, B is complete, it suffices by 2.2 to show that $H^2(L, B)^A = 0$. For this we need only consider the exact cohomology sequence

$$H^2(L, L)^A \to H^2(L, B)^A \to H^2(L, B/L)^A$$

Now $H^2(L, L)^A = 0$ by Proposition 5.3; and $C^2(L, B/L)^A = 0$ since B/L is a trivial A-module while, by 5.1 (iii), zero is never the sum of two weights of A in L.

5.5 *Remark.* One consequence of the corollary is the existence of a large class of solvable rigid Lie algebras. This seems unexpected in view of the remarks in the introduction of [**6**].

The essential point in the proof of Corollary 5.4 was the fact that $H^2(L, L)^A = 0$. Now, a result of Kostant [**3**, Corollary 5.3.3] includes, in the particular case of a Borel subalgebra, $B = A + L$, of a semi-simple Lie algebra in characteristic 0 the more general statement, $H^i(L, L)^A = 0$ for $i \geqq 2$. With this, an analogous proof to that of 5.4 (using 2.2 and induction) yields:

5.6 Proposition. *Let B be a Borel subalgebra of a semi-simple Lie algebra over a field of characteristic* 0. *Then* $H(B, B) = 0$.

Remark. It is shown similarly in [**8**] that, in characteristic 0, the tree algebras mentioned in Remark 5.2 have the same property.

References

1. G. Hochschild and J.-P. Serre, *Cohomology of Lie algebras*, Ann. of Math. *57* (1953), 591–603.
2. N. Jacobson, *Lie algebras* (Interscience, New York, 1962).
3. B. Kostant, *Lie algebra cohomology and generalized Schubert cells*, Ann. of Math. *77* (1963), 72–144.
4. G. Leger and E. Luks, *Lie algebras and trees* (to appear).
5, A. Nijenhuis and R. Richardson, *Deformations of Lie algebra structures*, J. Math. Mech. *17* (1967), 89–105.
6. R. Richardson, *On the rigidity of semi-direct products of Lie algebras*, Pacific J. Math. *22* (1967), 339–344.
7. S. Tôgô, *On some properties of* $t(n, \Phi)$ *and* $st(n, \Phi)$, J. Sci. Hiroshima Univ. Sec. A–1 Math. *31* (1967), 35–58.

Tufts University,
Medford, Massachusetts;
Bucknell University,
Lewisburg, Pennsylvania

https://doi.org/10.4153/CJM-1972-103-1 Published online by Cambridge University Press