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ON BOUNDEDNESS OF DIVISORS COMPUTING MINIMAL LOG DISCREPANCIES FOR SURFACES

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Dedicated to Vyacheslav Shokurov with our deepest gratitude on the occasion of his 70th birthday

Abstract Let Γ be a finite set, and $X \ni x$ a fixed kawamata log terminal germ. For any lc germ $(X \ni x, B := \sum_i b_i B_i)$, such that $b_i \in \Gamma$, Nakamura's conjecture, which is equivalent to the ascending chain condition conjecture for minimal log discrepancies for fixed germs, predicts that there always exists a prime divisor E over $X \ni x$, such that $a(E,X,B) = \mathrm{mld}(X \ni x, B)$, and $a(E,X,0)$ is bounded from above. We extend Nakamura's conjecture to the setting that $X \ni x$ is not necessarily fixed and Γ satisfies the descending chain condition, and show it holds for surfaces. We also find some sufficient conditions for the boundedness of $a(E,X,0)$ for any such E.

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1. Introduction

We work over an algebraically closed field of arbitrary characteristic.

The minimal log discrepancy (MLD) introduced by Vyacheslav Shokurov is an important invariant in birational geometry. Shokurov conjectured that the set of MLDs should satisfy the ascending chain condition (ACC) [18, Problem 5], and proved that the conjecture on termination of flips in the minimal model program (MMP) in characteristic 0 follows from two conjectures on MLDs: the ACC conjecture for MLDs and the lower-semicontinuity conjecture for MLDs [21].

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The ACC conjecture for MLDs in dimension 2 was proved by Alexeev [1] and Shokurov [19] independently. We refer readers to [5, Lemma 4.5, Theorems B.1 and B.4] and [9, Theorem 1.5, Appendix A] for detailed proofs following Alexeev's and Shokurov's arguments, respectively, see also [6, Theorem 1.5]. The ACC conjecture for MLDs is still widely open in dimension 3 in general. We refer readers to [8] and references therein for a brief history and related progress.

In order to study this conjecture for the case when $X \ni x$ is a fixed kawamata log terminal (klt) germ, Nakamura proposed conjecture 1.1. It is proved by Mustaţă-Nakamura [17, Theorem 1.5] and Kawakita [13, Theorem 4.6] that Conjecture 1.1 is equivalent to the ACC conjecture for MLDs in this case.

Conjecture 1.1 ([17, Conjecture 1.1]). *Let $\Gamma \subseteq [0,1]$ be a finite set and $X \ni x$ a klt germ. Then there exists an integer N depending only on $X \ni x$ and Γ satisfying the following.*

Let $(X \ni x,B)$ be an lc germ, such that $B \in \Gamma$. Then there exists a prime divisor E over $X \ni x$, such that $a(E,X,B) = \mathrm{mld}(X \ni x,B)$ and $a(E,X,0) \leq N$.

When $\dim X = 2$, Conjecture 1.1 is proved by Mustaţă-Nakamura [17, Theorem 1.3] in characteristic zero. Alexeev proved it when Γ satisfies the descending chain condition (DCC) [1, Lemma 3.7] as one of key steps in his proof of the ACC for MLDs for surfaces. See [5, Theorem B.1] for a proof of Alexeev's result in detail. Very recently, Ishii gave another proof of Conjecture 1.1 in dimension 2 [11, Theorem 1.4] that works in any characteristic. Kawakita proved Conjecture 1.1 for the case when $\dim X = 3$, X is smooth, $\Gamma \subseteq \mathbb{Q}$ and $(X \ni x,B)$ is canonical [13, Theorem 1.3] in characteristic zero.

Naturally, one may ask whether Nakamura's conjecture holds or not when $X \ni x$ is not necessarily fixed and Γ satisfies the DCC. If the answer is yes, not only the ACC conjecture for MLDs will hold for fixed germs $X \ni x$ as we mentioned before, but also both the boundedness conjecture of MLDs and the ACC conjecture for MLDs for terminal threefolds $(X \ni x,B)$ will be immediate corollaries. One of main goals of this paper is to give a positive answer to this question in dimension 2.

Theorem 1.2. *Let $\Gamma \subseteq [0,1]$ be a set which satisfies the DCC. Then there exists an integer N depending only on Γ satisfying the following.*

Let $(X \ni x,B)$ be an lc surface germ, such that $B \in \Gamma$. Then there exists a prime divisor E over $X \ni x$, such that $a(E,X,B) = \mathrm{mld}(X \ni x,B)$ and $a(E,X,0) \leq N$.

Theorem 1.2 implies that for any surface germ $(X \ni x,B)$, such that $B \in \Gamma$, there exists a prime divisor E, which could compute $\mathrm{mld}(X \ni x,B)$ and is 'weakly bounded', in the sense that $a(E,X,0)$ is uniformly bounded from above, among all divisorial valuations.

A natural idea to prove Theorem 1.2 is to take the minimal resolution $\widetilde{f}: \widetilde{X} \to X$, and apply Conjecture 1.1 in dimension 2. However, the coefficients of $B_{\widetilde{X}}$ may not belong to a DCC set anymore, where $K_{\widetilde{X}} + B_{\widetilde{X}} := \widetilde{f}^*(K_X + B)$. In order to resolve this difficulty, we need to show Nakamura's conjecture for the smooth surface germ, while the set of coefficients Γ does not necessarily satisfy the DCC. It would be interesting to ask if similar approaches could be applied to solve some questions in birational geometry in high dimensions, that is, we solve these questions in a more general setting of coefficients on a terminalisation of X.

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Theorem 1.3. *Let $\gamma \in (0,1]$ be a real number. Then $N_0 := \lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \rfloor$ satisfies the following.*

Let $(X \ni x, B := \sum_i b_i B_i)$ be an lc surface germ, where $X \ni x$ is smooth, and B_i are distinct prime divisors. Suppose that $\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty)$. Then there exists a prime divisor E over X, such that $a(E,X,B) = \mathrm{mld}(X \ni x, B)$, and $a(E,X,0) \leq 2^{N_0}$.

We remark that for any DCC set Γ, there exists a positive real number γ, such that $\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, b_i \in \Gamma\} \subseteq [\gamma, +\infty)$ (see Lemma 2.2). However, the converse is not true. For example, the set $\Gamma := [\frac{1}{2} + \frac{\gamma}{2}, 1]$ satisfies our assumption, but it is not a DCC set. It is also worthwhile to remark that all previous works we mentioned before did not give any effective bound even when Γ is a finite set.

Theorem 1.3 indicates that there are some differences between Conjecture 1.1 and the ACC conjecture for MLDs, as the latter does not hold for such kind of sets.

When we strengthen the assumption '$\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty)$' to '$\{\sum_i n_i b_i - 1 \geq 0 \mid n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty)$', we may even give an explicit upper bound for $a(E,X,0)$ for all prime divisors E over $X \ni x$, such that $\mathrm{mld}(X \ni x, B) = a(E,X,B)$. Theorem 1.4 is another main result in this paper.

Theorem 1.4. *Let $\gamma \in (0,1]$ be a real number. Then $N_0 := \lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \rfloor$ satisfies the following.*

Let $(X \ni x, B := \sum_i b_i B_i)$ be an lc surface germ, such that $X \ni x$ is smooth, where B_i are distinct prime divisors. Suppose that $\{\sum_i n_i b_i - 1 \geq 0 \mid n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty)$. Then:

(1) *$|S| \leq N_0$, where $S := \{E \mid E$ is a prime divisor over $X \ni x, a(E,X,B) = \mathrm{mld}(X \ni x, B)\}$ and*

(2) *$a(E,X,0) \leq 2^{N_0}$ for any $E \in S$.*

Theorem 1.4 is a much deeper result than Theorem 1.3. To the best of the authors' knowledge, such types of boundedness that result for $a(E,X,0)$ about all prime divisors which compute MLDs were never formulated in any literature before.

We remark that $\Gamma := [\frac{1}{2} + \frac{\gamma}{2}, 1)$ satisfies the assumption in Theorem 1.4, while $\Gamma := [\frac{1}{2} + \frac{\gamma}{2}, 1]$ does not. It is clear that $(\mathbf{A}^2, B_1 + B_2)$ satisfies the assumptions in Theorem 1.3, but $|S| = +\infty$, where B_1 and B_2 are defined by $x = 0$ and $y = 0$, respectively. For klt germs (X,B), we always have $|S| < +\infty$. In this case, Example 3.3 shows that Theorem 1.3 does not hold for the set $\Gamma := \{\frac{1}{2}\} \cup \{\frac{1}{2} + \frac{1}{k+1} \mid k \in \mathbb{Z}_{\geq 0}\}$, and Example 5.8 shows Theorem 1.4 may fail when $b_i = \frac{1}{2}$ for any i, that is why the equality for '$\sum_i n_i b_i - 1 \geq 0$' is necessary. These examples also indicate that the assumptions in Theorems 1.3 and 1.4 might be optimal.

We also give an effective bound for Theorem 1.3 (respectively, Theorem 1.4) when $X \ni x$ is a fixed lc surface germ (respectively, $X \ni x$ is a fixed klt surface germ), see Theorem 5.4 (respectively, Theorem 5.5).

Postscript. Together with Jihao Liu, the authors proved the ACC for MLDs of terminal threefolds [7, Theorem 1.1]. The proof is intertwined with Theorem 1.2 for terminal threefolds. Bingyi Chen informed the authors that he improved the upper bound 2^{N_0}

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given in Theorem 1.3 when the characteristic of the algebraically field is zero (see [4, Theorem 1.3]).

2. Preliminaries

We adopt the standard notation and definitions in [15], and will freely use them.

2.1. Arithmetic of sets

Definition 2.1 (DCC and ACC sets). We say that $\Gamma \subseteq \mathbb{R}$ satisfies the DCC or Γ is a DCC set if any decreasing sequence $a_1 \geq a_2 \geq \cdots$ in Γ stabilises. We say that Γ satisfies the ACC or Γ is an ACC set if any increasing sequence in Γ stabilises.

Lemma 2.2. *Let $\Gamma \subseteq [0,1]$ be a set which satisfies the DCC, and n a nonnegative integer. There exists a positive real number γ which only depends on n and Γ, such that:*

$$\left\{\sum_i n_i b_i - n > 0 \mid b_i \in \Gamma, n_i \in \mathbb{Z}_{\geq 0}\right\} \subseteq [\gamma, +\infty).$$

Definition 2.3. Let $\epsilon \in \mathbb{R}, I \in \mathbb{R}\backslash\{0\}$ and $\Gamma \subseteq \mathbb{R}$ be a set of real numbers. We define $\Gamma_\epsilon := \cup_{b \in \Gamma}[b-\epsilon, b]$ and $\frac{1}{I}\Gamma := \{\frac{b}{I} \mid b \in \Gamma\}$.

Lemma 2.4. *Let $\Gamma \subseteq [0,1]$ be a set which satisfies the DCC. Then there exist positive real numbers $\epsilon, \delta \leq 1$, such that:*

$$\left\{\sum_i n_i b_i' - 1 > 0 \mid b_i' \in \Gamma_\epsilon \cap [0,1], n_i \in \mathbb{Z}_{\geq 0}\right\} \subseteq [\delta, +\infty).$$

Proof. We may assume that $\Gamma \setminus \{0\} \neq \emptyset$, otherwise we may take $\epsilon = \delta = 1$.

Since Γ satisfies the DCC, by Lemma 2.2, there exists a real number $\gamma \in (0,1]$, such that $\Gamma \setminus \{0\} \subseteq (\gamma, 1]$, and $\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, b_i \in \Gamma\} \subseteq [\gamma, +\infty)$. It suffices to prove that there exist $0 < \epsilon, \delta < \frac{\gamma}{2}$, such that the set $\{\sum_i n_i b_i' - 1 \in (0,1] \mid b_i' \in \Gamma_\epsilon \cap [0,1], n_i \in \mathbb{Z}_{\geq 0}\}$ is bounded from below by δ, or equivalently:

$$\left\{\sum_i n_i b_i' - 1 > 0 \mid b_i' \in \Gamma_\epsilon \cap [0,1], n_i \in \mathbb{Z}_{\geq 0}, \sum_i n_i \leq \frac{4}{\gamma}\right\} \subseteq [\delta, +\infty).$$

We claim that $\epsilon = \frac{\gamma^2}{8}, \delta = \frac{\gamma}{2}$ have the desired property. Let $b_i' \in \Gamma_\epsilon \cap [0,1]$ and $n_i \in \mathbb{Z}_{\geq 0}$, such that $\sum_i n_i b_i' - 1 > 0$ and $\sum_i n_i \leq \frac{4}{\gamma}$. We may find $b_i \in \Gamma$, such that $0 \leq b_i - b_i' \leq \epsilon$ for any i. In particular, $\sum_i n_i b_i - 1 > 0$. By the choice of γ, $\sum_i n_i b_i - 1 \geq \gamma$. Thus:

$$\sum_i n_i b_i' - 1 = \left(\sum_i n_i b_i - 1\right) - \sum_i n_i (b_i - b_i') \geq \gamma - \frac{4}{\gamma}\epsilon = \frac{\gamma}{2}. \qquad \square$$

We will use the following lemma frequently without citing it in this paper.

Lemma 2.5. *Let $\Gamma \subseteq [0,1]$ be a set, and $\gamma \in (0,1]$ be a real number. If $\{\sum_i n_i b_i - 1 > 0 \mid b_i \in \Gamma, n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty)$, then $\Gamma \setminus \{0\} \subseteq [\gamma, 1]$.*

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Proof. Otherwise, we may find $b\in\Gamma$, such that $0<b<\gamma$. Then $0<(\lfloor\frac{1}{b}\rfloor+1)\cdot b-1\leq b<\gamma$, a contradiction. $\square$

2.2. Singularities of pairs

Definition 2.6. A pair (X,B) consists of a normal quasiprojective variety X and an $\mathbb{R}$-divisor $B\geq 0$, such that K_X+B is $\mathbb{R}$-Cartier. A *germ* $(X\ni x,B:=\sum_i b_iB_i)$ consists of a pair (X,B), and a closed point $x\in X$, such that $b_i>0$, and B_i are distinct prime divisors on X with $x\in\cap_i\operatorname{Supp}B_i$. We call it a *surface germ* if $\dim X=2$. $(X\ni x,B)$ is called an lc (respectively, klt, canonical, purely log terminal (plt), terminal) surface germ if $(X\ni x,B)$ is a surface germ, and (X,B) is lc (respectively, klt, canonical, plt, terminal) near x.

Let $X\ni x$ be a germ. A birational morphism $f:Y\to X\ni x$ is a birational morphism from Y to X, such that all the exceptional divisors on Y are centered at x.

Definition 2.7. Let X be a normal quasiprojective surface and $x\in X$ a closed point. Then a birational morphism $f:Y\to X$ (respectively, $f:Y\to X\ni x$) is called a *minimal resolution* of X (respectively, $X\ni x$) if Y is smooth (respectively, smooth over a neighborhood of $x\in X$) and there is no (-1)-curve on Y (respectively, over a neighborhood of $x\in X$).

Note that the existence of resolutions of singularities for surfaces (see [16]) and the minimal model program for surfaces (see [22] and [24]) are all known in positive characteristic. In particular, for any surface X (respectively, surface germ $X\ni x$), we can construct a minimal resolution $\tilde{f}:\widetilde{X}\to X$ (respectively, $\widetilde{f}:\widetilde{X}\to X\ni x$).

Definition 2.8. Let $(X\ni x,B)$ be an lc germ. We say a prime divisor E is over $X\ni x$ if E is over X and $\operatorname{center}_X E=x$.

The *minimal log discrepancy* of $(X\ni x,B)$ is defined as:

$$\operatorname{mld}(X\ni x,B):=\min\{a(E,X,B)\mid E\text{ is a prime divisor over }X\ni x\}.$$

Let $f:Y\to X$ be a projective birational morphism, we denote it by $f:Y\to X\ni x$ if $\operatorname{center}_X E=x$ for any f-exceptional divisor E.

Let $\widetilde{f}:\widetilde{X}\to X\ni x$ be the minimal resolution of $X\ni x$, and we may write $K_{\widetilde{X}}+B_{\widetilde{X}}+\sum_i(1-a_i)E_i=\widetilde{f}^*(K_X+B)$, where $B_{\widetilde{X}}$ is the strict transform of B, E_i are $\widetilde{f}$-exceptional prime divisors and $a_i:=a(E_i,X,B)$ for all i. The *partial log discrepancy* (PLD) of $(X\ni x,B)$, $\operatorname{pld}(X\ni x,B)$, is defined as follows.

$$\operatorname{pld}(X\ni x,B):=\begin{cases}\min_i\{a_i\} & \text{if } x\in X \text{ is a singular point,}\\ +\infty & \text{if } x\in X \text{ is a smooth point.}\end{cases}$$

The following result is known as the ACC for PLDs (for surfaces).

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Theorem 2.9 ([1, Theorem 3.2],[9, Theorem 2.2]). *Let $\Gamma \subseteq [0,1]$ be a set which satisfies the DCC. Then:*

$$\mathrm{Pld}(2,\Gamma) := \{\mathrm{pld}(X \ni x, B) \mid (X \ni x, B) \text{ is lc}, \dim X = 2, B \in \Gamma\},$$

satisfies the ACC.

2.3. Dual graphs

Definition 2.10 (c.f. [15, Definition 4.6]). Let $C = \cup_i C_i$ be a collection of proper curves on a smooth surface U. We define the *dual graph* $\mathcal{DG}$ of C as follows.

(1) The vertices of $\mathcal{DG}$ are the curves C_j.

(2) Each vertex is labelled by the negative self intersection of the corresponding curve on U, we call it the *weight* of the vertex (curve).

(3) The vertices C_i, C_j are connected with $C_i \cdot C_j$ edges.

Let $f : Y \to X \ni x$ be a projective birational morphism with exceptional divisors $\{E_i\}_{1\le i\le m}$, such that Y is smooth. Then the dual graph $\mathcal{DG}$ of f is defined as the dual graph of $E = \cup_{1\le i\le m} E_i$. In particular, $\mathcal{DG}$ is a connected graph.

Definition 2.11. A *cycle* is a graph whose vertices and edges can be ordered $v_1,\dots,v_m$ and $e_1,\dots,e_m (m \ge 2)$, such that e_i connects v_i and v_{i+1} for $1 \le i \le m$, where $v_{m+1} = v_1$.

Let $\mathcal{DG}$ be a dual graph with vertices $\{C_i\}_{1\le i\le m}$. We call $\mathcal{DG}$ a *tree* if:

(1) $\mathcal{DG}$ does not contain a subgraph which is a cycle and

(2) $C_i \cdot C_j \le 1$ for all $1 \le i \ne j \le m$.

Moreover, if C is a vertex of $\mathcal{DG}$ that is adjacent to more than three vertices, then we call C a *fork* of $\mathcal{DG}$. If $\mathcal{DG}$ contains no fork, then we call it a *chain*.

Lemma 2.12. *Let $X \ni x$ be a surface germ. Let Y, Y' be smooth surfaces, and let $f : Y \to X \ni x$ and $f' : Y' \to X \ni x$ be two projective birational morphisms, such that f' factors through f. If the dual graph of f is a tree whose vertices are all smooth rational curves, then the dual graph of f' is a tree whose vertices are all smooth rational curves.*

Proof. Let $g : Y' \to Y$ be the projective birational morphism, such that $f \circ g = f'$. Since g is a composition of blow-ups at smooth closed points, by induction on the number of blow-ups, we may assume that g is a single blow-up of Y at a smooth closed point $y \in Y$.

Let E' be the g-exceptional divisor on Y', $\{E_i\}_{1\le i\le m}$ the set of distinct exceptional curves of f on Y and $\{E_i'\}_{1\le i\le m}$ their strict transforms on Y'. By assumption, $E_i' \cdot E_j' \le g^*E_i \cdot E_j' = E_i \cdot E_j \le 1$ for $1 \le i \ne j \le m$. Since E_i is smooth, $0 = g^*E_i \cdot E' \ge (E_i' + E') \cdot E'$. It follows that $E' \cdot E_i' \le 1$ for $1 \le i \le m$.

If the dual graph of f' contains a cycle, then E' must be a vertex of this cycle. Let $E', E_{i_1}', \dots, E_{i_k}'$ be the vertices of this cycle, $1 \le k \le m$. Then the vertex-induced subgraph by $E_{i_1},\dots,E_{i_k}$ of the dual graph of f is a cycle, a contradiction. $\square$

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The following lemma maybe well known to experts (c.f. [5, Lemma 4.2]). For the reader's convenience, we include the proof here.

Lemma 2.13. *Let $\epsilon_0 \in (0,1]$ be a real number. Let $(X \ni x, B)$ be an lc surface germ, Y a smooth surface and $f: Y \to X \ni x$ a projective birational morphism with the dual graph $\mathcal{DG}$. Let $\{E_k\}_{1\le k\le m}$ be the set of vertices of $\mathcal{DG}$, and $w_k := -E_k \cdot E_k$, $a_k := a(E_k, X, B)$ for each k. Suppose that $a_k \le 1$ for any $1 \le k \le m$, then we have the following:*

1. *$w_k \le \frac{2}{a_k}$ if $a_k > 0$, and in particular, $w_k \le \frac{2}{\epsilon_0}$ for $1 \le k \le m$ if $\mathrm{mld}(X \ni x, B) \ge \epsilon_0$.*
2. *If $w_k \ge 2$ for some k, then for any E_{k_1}, E_{k_2} which are adjacent to E_k, we have $2a_k \le a_{k_1} + a_{k_2}$. Moreover, if the equality holds, then $f_*^{-1}B \cdot E_k = 0$, and either $w_k = 2$ or $a_k = a_{k_1} = a_{k_2} = 0$.*
3. *If E_{k_0} is a fork, then for any $E_{k_1}, E_{k_2}, E_{k_3}$ which are adjacent to E_{k_0} with $w_{k_i} \ge 2$ for $0 \le i \le 2$, $a_{k_3} \ge a_{k_0}$. Moreover, if the equality holds, then $w_{k_i} = 2$ and $f_*^{-1}B \cdot E_{k_i} = 0$ for $0 \le i \le 2$.*
4. *Let $E_{k_0}, E_{k_1}, E_{k_2}$ be three vertices, such that E_{k_1}, E_{k_2} are adjacent to E_{k_0}. Assume that $a_{k_1} \ge a_{k_2}$, $a_{k_1} \ge \epsilon_0$, and $w_{k_0} \ge 3$, then $a_{k_1} - a_{k_0} \ge \frac{\epsilon_0}{3}$.*
5. *If E_{k_0} is a fork, and there exist three vertices $E_{k_1}, E_{k_2}, E_{k_3}$ which are adjacent to E_{k_0} with $w_{k_i} \ge 2$ for $0 \le i \le 3$, then $a(E, X, B) \ge a_{k_0}$ for any vertex E of $\mathcal{DG}$.*
6. *Let $\{E_{k_i}\}_{0 \le i \le m'}$ be a set of distinct vertices, such that E_{k_i} is adjacent to $E_{k_{i+1}}$ for $0 \le i \le m' - 1$, where $m' \ge 2$. If $a_{k_0} = a_{k_{m'}} = \mathrm{mld}(X \ni x, B) > 0$ and $w_{k_i} \ge 2$ for $1 \le i \le m' - 1$, then $a_{k_0} = a_{k_1} = \cdots = a_{k_{m'}}$ and $w_{k_i} = 2$ for $1 \le i \le m' - 1$.*

Proof. For (1), we may write $K_Y + f_*^{-1}B + \sum_{1\le i\le m}(1-a_i)E_i = f^*(K_X + B)$. For each $1 \le k \le m$, we have $0 = (K_Y + f_*^{-1}B + \sum_{1\le i\le m}(1-a_i)E_i) \cdot E_k$, or equivalently,

$$a_k w_k = 2 - 2p_a(E_k) - \sum_{i\ne k}(1-a_i)E_i \cdot E_k - f_*^{-1}B \cdot E_k. \tag{2.1}$$

So $a_k w_k \le 2$, and $w_k \le \frac{2}{a_k}$.

For (2), by (2.1), $2a_k \le a_k w_k \le a_{k_1} + a_{k_2} - f_*^{-1}B \cdot E_k \le a_{k_1} + a_{k_2}$. If $2a_k = a_{k_1} + a_{k_2}$, then $f_*^{-1}B \cdot E_k = 0$, and either $w_k = 2$ or $a_k = a_{k_1} = a_{k_2} = 0$.

For (3), let $k = k_i$ in (2.1) for $i = 1,2$,

$$a_{k_i} w_{k_i} \le 1 + a_{k_0} - \left(\sum_{j\ne k_0, k_i}(1-a_j)E_j \cdot E_{k_i} + f_*^{-1}B \cdot E_{k_i}\right) \le 1 + a_{k_0},$$

or $a_{k_i} \le \frac{1+a_{k_0}}{w_{k_i}}$. Thus, let $k = k_0$ in (2.1), we have:

$$\begin{aligned} a_{k_3} \ge a_{k_0}w_{k_0} + 1 - a_{k_1} - a_{k_2} + f_*^{-1}B \cdot E_{k_0} &\ge a_{k_0}\left(w_{k_0} - \frac{1}{w_{k_1}} - \frac{1}{w_{k_2}}\right) + \left(1 - \frac{1}{w_{k_1}} - \frac{1}{w_{k_2}}\right) \\ &\ge a_{k_0}. \end{aligned}$$

If the equality holds, then $w_{k_i} = 2$ and $f_*^{-1}B \cdot E_i = 0$ for $0 \le i \le 2$.

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For (4), by (2.1), we have $a_{k_0}w_{k_0} \le a_{k_1} + a_{k_2} - d$, where $d := f_*^{-1}B \cdot E_{k_0} + \sum_{j \ne k_1,k_2,k_0}(1-a_j)E_j \cdot E_{k_0}$. Hence:

$$a_{k_1} - a_{k_0} \ge \frac{(w_{k_0}-1)a_{k_1} - a_{k_2} + d}{w_{k_0}} \ge \frac{(w_{k_0}-2)a_{k_1}}{w_{k_0}} \ge \frac{\epsilon_0}{3}.$$

For (5), we may assume that $E \ne E_{k_0}$. There exist $m'+1$ distinct vertices $\{F_i\}_{0 \le i \le m'}$ of $\mathcal{DG}$, such that:

- $F_0 = E_{k_0}$, $F_{m'} = E$ and
- F_i is adjacent to F_{i+1} for $0 \le i \le m'-1$.

Denote $a'_i := a(F_i,X,B)$ for $0 \le i \le m'$. By (3), we have $a'_1 \ge a'_0$, and by (2), $a'_{i+1} - a'_i \ge a'_i - a'_{i-1}$ for $1 \le i \le m'-1$. Thus, $a'_m - a'_0 \ge 0$.

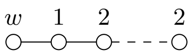
For (6), by (2) $a_{k_0} \le a_{k_1} \le \ldots \le a_{k_{m'-1}} \le a_{k_{m'}}$. Thus, $a_{k_0} = \ldots = a_{k_{m'}}$. By (2) again, $w_{k_i} = 2$ for $1 \le i \le m'-1$. □

Lemma 2.14. *Let $(X \ni x, B)$ be an lc surface germ. Let Y be a smooth surface and $f : Y \to X \ni x$ a birational morphism with the dual graph $\mathcal{DG}$. If $\mathcal{DG}$ contains a (-1)-curve E_0, then:*

(1) *E_0 can not be adjacent to two (-2)-curves in $\mathcal{DG}$,*

(2) *if either $\mathrm{mld}(X \ni x, B) \ne \mathrm{pld}(X \ni x, B)$ or $\mathrm{mld}(X \ni x, B) > 0$, then E_0 is not a fork in $\mathcal{DG}$ and*

(3) *if $E, E_0, \ldots, E_m$ are distinct vertices of $\mathcal{DG}$, such that E is adjacent to E_0, E_i is adjacent to E_{i+1} for $0 \le i \le m-1$, and $-E_i \cdot E_i = 2$ for $1 \le i \le m$, then $m+1 < -E \cdot E = w$.*



Proof. For (1), if E_0 is adjacent to two (-2)-curves E_{k_1} and E_{k_2} in $\mathcal{DG}$, then we may contract E_0 and get a smooth model $f' : Y' \to X \ni x$ over X, whose dual graph contains two adjacent (-1)-curves, this contradicts the negativity lemma.

By [15, Theorem 4.7] and the assumptions in (2), the dual graph of the minimal resolution of $X \ni x$ is a tree. If E_0 is a fork, we may contract E_0 and get a smooth model $f' : Y' \to X \ni x$, whose dual graph contains a cycle, this contradicts Lemma 2.12.

For (3), we will construct a sequence of contractions of (-1)-curve $X_0 := X \to X_1 \to \ldots X_m \to X_{m+1}$ inductively. Let E_{X_k} be the strict transform of E on X_k, and $w_{X_k} := -E_{X_k} \cdot E_{X_k}$. For simplicity, we will always denote the strict transform of E_k on X_j by E_k for all k,j. Let $f_1 : X_0 \to X_1$ be the contraction of E_0 on X_0, then $w_{X_1} = w-1$, and $E_1 \cdot E_1 = -1$ on X_1. Let $f_2 : X_1 \to X_2$ be the contraction of E_1 on X_1, then $w_{X_2} = w_{X_1} - 1 = w-2$, and $E_2 \cdot E_2 = -1$ on X_2. Repeating this procedure, we have $f_k : X_{k-1} \to X_k$ the contraction of E_{k-1}, and $w_{X_k} = w-k$, $E_k \cdot E_k = -1$ on X_k for $1 \le k \le m+1$. By the negativity lemma, $w_{X_{m+1}} = w-(m+1) > 0$, and we are done. □

Lemma 2.15. *Let $\gamma \in (0,1]$ be a real number. Let $(X \ni x, B := \sum_i b_i B_i)$ be an lc surface germ, where B_i are distinct prime divisors. Let Y be a smooth surface and $f : Y \to X \ni x$ a* *birational morphism with the dual graph $\mathcal{DG}$. Let $\{E_k\}_{0\le k\le m}$ be a vertex-induced subchain of $\mathcal{DG}$, such that E_k is adjacent to E_{k+1} for $0\le k\le m-1$, and let $w_k := -E_k\cdot E_k$, $a_k := a(E_k,X,B)$ for all k. Suppose that $w_0 = 1$, E_0 is adjacent to only one vertex E_1 of $\mathcal{DG}$, $a_k\le 1$, and $w_k\ge 2$ for each $k\ge 1$, then:*

1. *if $\{\sum_i n_ib_i - 1 > 0 \mid n_i\in\mathbb{Z}_{\ge 0}\}\subseteq[\gamma,+\infty)$, and $a_0 < a_1$, then $m\le \frac{1}{\gamma}$,*
2. *if $\sum_i n_ib_i - 1\ne 0$ for all $n_i\in\mathbb{Z}_{\ge 0}$, and $a_0\le a_1$, then $a_0 < a_1$ and*
3. *if $\{\sum_i n_ib_i - 1 \ge 0 \mid n_i\in\mathbb{Z}_{\ge 0}\}\subseteq[\gamma,+\infty)$, and $a_0 \le a_1$, then $m\le \frac{1}{\gamma}$.*

Proof. We may write $K_Y + f_*^{-1}B + \sum_i(1-a_i)E_i = f^*(K_X+B)$, then:

$$-2 + f_*^{-1}B\cdot E_0 + w_0a_0 + \sum_{i\ne 0}(1-a_i)E_i\cdot E_0 = 0. \tag{2.2}$$

Since E_0 is adjacent to only one vertex E_1 of $\mathcal{DG}$, by (2.2), we have:

$$a_1 - a_0 = f_*^{-1}B\cdot E_0 - 1.$$

For (1), since $1\ge a_1 > a_0$, it follows that $f_*^{-1}B\cdot E_0 - 1\in\{\sum_i n_ib_i - 1 > 0\mid n_i\in\mathbb{Z}_{\ge 0}\}\subseteq[\gamma,+\infty)$. Thus, $a_1 - a_0\ge\gamma$. By Lemma 2.13(2), we have $a_{i+1} - a_i \ge a_1 - a_0\ge\gamma$ for any $0\le i\le m-1$, and $1\ge a_m\ge\gamma m$. So $m\le\frac{1}{\gamma}$.

For (2), since $f_*^{-1}B\cdot E_0 - 1 = \sum_i n_ib_i - 1\ne 0$ for some $n_i\in\mathbb{Z}_{\ge 0}$, $a_0 < a_1$,

(3) follows immediately from (1) and (2). □

2.4. Sequence of blow-ups

Definition 2.16. Let $X\ni x$ be a smooth surface germ. We say $X_n\to X_{n-1}\to\cdots\to X_1\to X_0 := X$ is a *sequence of blow-ups with the data* $(f_i,F_i,x_i\in X_i)$ if:

- $f_i : X_i\to X_{i-1}$ is the blow-up of X_{i-1} at a closed point $x_{i-1}\in X_{i-1}$, with the exceptional divisor F_i for any $1\le i\le n$, where $x_0 := x$ and
- $x_i\in F_i$ for any $1\le i\le n-1$.

In particular, F_n is the only exceptional (-1)-curve over X.

For convenience, we will always denote the strict transform of F_i on X_j by F_i for any $n\ge j\ge i$.

The following lemma is well known. For a proof, see for example, [9, Lemma 3.15].

Lemma 2.17. *Let $(X\ni x,B)$ be an lc surface germ, such that* $\mathrm{mld}(X\ni x,B) > 1$, *then* $\mathrm{mld}(X\ni x,B) = 2 - \mathrm{mult}_x B$, *and there is exactly one prime divisor E over $X\ni x$, such that $a(E,X,B) = \mathrm{mld}(X\ni x,B)$.*

Lemma 2.18 ([17, Lemma 4.2]). *Let $X\ni x$ be a smooth surface germ, and $X_{l_0}\to\cdots\to X_1\to X_0 := X$ a sequence of blow-ups with the data $(f_i,E_i,x_i\in X_i)$, then $a(E_{l_0},X,0)\le 2^{l_0}$.*

2.5. Extracting divisors computing MLDs

Lemma 2.19. *Let $(X\ni x,B)$ be an lc surface germ. Let $h : W\to(X,B)$ be a log resolution, and $S = \{E_j\}$ a finite set of valuations of h-exceptional prime divisors over*

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$X \ni x$, such that $a(E_j,X,B) \le 1$ for all j. Then there exist a smooth surface Y and a projective birational morphism $f: Y \to X \ni x$ with the following properties.

(1) *$K_Y + B_Y = f^*(K_X + B)$ for some $\mathbb{R}$-divisor $B_Y \ge 0$ on Y,*

(2) *each valuation in S corresponds to some f-exceptional divisor on Y and*

(3) *each f-exceptional (-1)-curve corresponds to some valuation in S.*

Proof.

$$K_W + B_W = h^*(K_X + B) + F_W,$$

where $B_W \ge 0$ and $F_W \ge 0$ are $\mathbb{R}$-divisors with no common components. We construct a sequence of $(K_W + B_W)$-MMP over X as follows. Each time we will contract a (-1)-curve whose support is contained in F_W. Suppose that $K_W + B_W$ is not nef over X, then $F_W \neq 0$. By the negativity lemma, there exists a h-exceptional irreducible curve $C \subseteq \operatorname{Supp} F_W$, such that $F_W \cdot C = (K_W + B_W) \cdot C < 0$. Since $B_W \cdot C \ge 0$, $K_W \cdot C < 0$. Thus, C is a h-exceptional (-1)-curve. We may contract C and get a smooth surface $Y_0 := W \to Y_1$ over X. We may continue this process, and finally reach a smooth model Y_k on which $K_{Y_k} + B_{Y_k}$ is nef over X, where B_{Y_k} is the strict transform of B_W on Y_k. By the negativity lemma, F_W is contracted in the MMP, thus $K_{Y_k} + B_{Y_k} = h_k^*(K_X + B)$, where $h_k: Y_k \to X$, since $a(E_j,X,B) \le 1$, E_j is not contracted in the MMP for any $E_j \in S$.

We now construct a sequence of smooth models over X, $Y_k \to Y_{k+1} \to \cdots$, by contracting a curve C' satisfying the following conditions in each step.

- C' is an exceptional (-1)-curve over X and
- $C' \notin S$.

Since each time the Picard number of the variety will drop by one, after finitely many steps, we will reach a smooth model Y over X, such that $f: Y \to X$ and (Y, B_Y) satisfy (1)–(3), where B_Y is the strict transform of B_{Y_k} on Y. □

We will need Lemma 2.20 to prove our main results. It may be well known to experts. Lemma 2.20(1)–(4) could be proved by constructing a sequence of blow-ups (c.f. [5, Lemma 4.3]). We give another proof here.

We remark that Lemma 2.20(5) will only be applied to prove Theorem 1.2.

Lemma 2.20. *Let $(X \ni x, B)$ be an lc surface germ, such that $1 \ge \operatorname{mld}(X \ni x, B) \neq \operatorname{pld}(X \ni x, B)$. There exist a smooth surface Y and a projective birational morphism $f: Y \to X$ with the dual graph $\mathcal{DG}$, such that:*

(1) *$K_Y + B_Y = f^*(K_X + B)$ for some $\mathbb{R}$-divisor $B_Y \ge 0$ on Y,*

(2) *there is only one f-exceptional divisor E_0, such that $a(E_0,X,B) = \operatorname{mld}(X \ni x, B)$,*

(3) *E_0 is the only (-1)-curve of $\mathcal{DG}$ and*

(4) *$\mathcal{DG}$ is a chain.*

Moreover, if $X \ni x$ is not smooth, let $\widetilde{f}: \widetilde{X} \to X \ni x$ be the minimal resolution of $X \ni x$, and let $g: Y \to \widetilde{X}$ be the morphism, such that $\widetilde{f} \circ g = f$, then:

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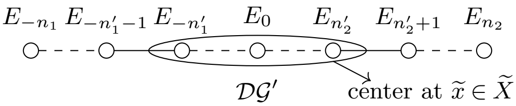


Figure 1. The dual graph of *f*.

(5) *there exist a $\widetilde{f}$-exceptional prime divisor $\widetilde{E}$ on $\widetilde{X}$ and a closed point $\widetilde{x} \in \widetilde{E}$, such that $a(\widetilde{E},X,B) = \mathrm{pld}(X \ni x,B)$, and* $\mathrm{center}_{\widetilde{X}} E = \widetilde{x}$ *for all g-exceptional divisors E.*

Proof. By Lemma 2.19, we can find a smooth surface Y_0 and a birational morphism $h: Y_0 \to X \ni x$, such that $a(E_0', X, B) = \mathrm{mld}(X \ni x, B)$ for some h-exceptional divisor E_0', and $K_{Y_0} + B_{Y_0} = h^*(K_X + B)$ for some $B_{Y_0} \geq 0$ on Y_0.

We now construct a sequence of smooth models over X, $Y_0 \to Y_1 \to \cdots$, by contracting a curve C' satisfying the following conditions in each step.

- C' is an exceptional (-1)-curve over X and
- there exists $C'' \neq C'$ over X, such that $a(C'', X, B) = \mathrm{mld}(X \ni x, B)$.

Since each time the Picard number of the variety will drop by one, after finitely many steps, we will reach a smooth model Y over X, such that $f: Y \to X$ and (Y, B_Y) satisfy (1), where B_Y is the strict transform of B_{Y_0} on Y. Since $\mathrm{mld}(X \ni x, B) \neq \mathrm{pld}(X \ni x, B)$, by the construction of Y, there exists a curve E_0 on Y satisfying (2)–(3).

For (4), by [15, Theorem 4.7], the dual graph of the minimal resolution $\widetilde{f}: \widetilde{X} \to X \ni x$ is a tree whose vertices are smooth rational curves. Since Y is smooth, f factors through $\widetilde{f}$. By Lemma 2.12, the dual graph $\mathcal{DG}$ of f is a tree whose vertices are smooth rational curves. It suffices to show that there is no fork in $\mathcal{DG}$. By Lemma 2.14(2), E_0 is not a fork. Suppose that $\mathcal{DG}$ contains a fork $E' \neq E_0$, by (3) and (5) of Lemma 2.13, we have $a(E', X, B) \leq a(E_0, X, B)$, this contradicts (2). Thus, $\mathcal{DG}$ is a chain.

For (5), since there exists only one f-exceptional (-1)-curve, there is at most one closed point $\widetilde{x} \in \widetilde{X}$, such that $\mathrm{center}_{\widetilde{X}} E = \widetilde{x}$ for all g-exceptional divisors E. Thus, the dual graph of g, which is denoted by $\mathcal{DG}'$, is a vertex-induced connected subchain of $\mathcal{DG}$ by all g-exceptional divisors. Since $\mathrm{mld}(X \ni x, B) \neq \mathrm{pld}(X \ni x, B)$, we have $\mathcal{DG}' \subsetneq \mathcal{DG}$.

We may index the vertices of $\mathcal{DG}$ as $\{E_i\}_{-n_1 \leq i \leq n_2}$ for $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, such that E_i is adjacent to E_{i+1}, and $a_i := a(E_i, X, B)$ for all possible i. We may assume that the set of vertices of $\mathcal{DG}'$ is $\{E_j\}_{-n_1' \leq j \leq n_2'}$, where $0 \leq n_1' \leq n_1$ and $0 \leq n_2' \leq n_2$ (see Figure 1). If $n_1 > n_1'$, then by Lemma 2.13(2), $a_k - a_{-n_1'-1} \geq \min\{0, a_{-1} - a_0\} \geq 0$ for all $-n_1 \leq k < -n_1'$. If $n_2 > n_2'$, then again by Lemma 2.13(2), $a_{k'} - a_{n_2'+1} \geq \min\{0, a_1 - a_0\} \geq 0$ for all $n_2' < k' \leq n_2$. Set $a_{-n_1-1} = 1, E_{-n_1-1} = E_{-n_1}$ if $n_1 = n_1'$, and set $a_{n_2+1} = 1, E_{n_2+1} = E_{n_2}$ if $n_2 = n_2'$. Then $\min\{a_{n_2+1}, a_{-n_1-1}\} = \mathrm{pld}(X \ni x, B)$, and $\widetilde{x} = g(E_{-n_1'-1}) \cap g(E_{n_2+1}) \in \widetilde{E}$, where $a(\widetilde{E}, X, B) = \mathrm{pld}(X \ni x, B)$. □

The following lemma gives an upper bound for number of vertices of a certain kind of $\mathcal{DG}$ constructed in Lemma 2.20, with the additional assumption that $\mathrm{mld}(X \ni x, B)$ is bounded from below by a positive real number.

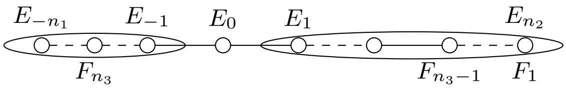


Figure 2. The dual graph of g.

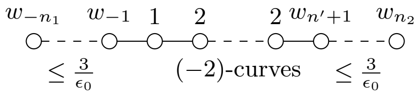
Lemma 2.21. *Let $\epsilon_0 \in (0,1]$ be a real number. Then $N_0' := \lfloor \frac{8}{\epsilon_0} \rfloor$ satisfies the following properties.*

Let $(X \ni x, B := \sum b_i B_i)$ be an lc surface germ, such that $\mathrm{mld}(X \ni x, B) \geq \epsilon_0$, where B_i are distinct prime divisors. Let Y be a smooth surface, and $f: Y \to X \ni x$ a birational morphism with the dual graph $\mathcal{DG}$, such that:

- *$K_Y + B_Y = f^*(K_X + B)$ for some $\mathbb{R}$-divisor $B_Y \geq 0$ on Y,*
- *$\mathcal{DG}$ is a chain with only one (-1)-curve E_0,*
- *$a(E_0, X, B) = \mathrm{mld}(X \ni x, B)$ and*
- *E_0 is adjacent to two vertices of $\mathcal{DG}$.*

Then the number of vertices of $\mathcal{DG}$ is bounded from above by N_0'.

Proof. Let $\{E_i\}_{-n_1 \leq i \leq n_2}$ be the vertices of $\mathcal{DG}$, such that E_i is adjacent to E_{i+1} for $-n_1 \leq i \leq n_2 - 1$, and $w_i := -(E_i \cdot E_i), a_i := a(E_i, X, B)$ for all i. We may assume that E_0 is adjacent to two vertices E_{-1}, E_1 of $\mathcal{DG}$.



By Lemma 2.14(1), we may assume that $w_{-1} > 2$. By (2) and (4) of Lemma 2.13, $a_{i-1} - a_i \geq \frac{\epsilon_0}{3}$ for any $-n_1 + 1 \leq i \leq -1$, and $a_{-1} \geq \frac{\epsilon_0}{3}$. Since $a_{-n_1} \leq 1$, $n_1 \leq \frac{3}{\epsilon_0}$. Similarly, $n_2 - n' \leq \frac{3}{\epsilon_0}$, where n' is the largest nonnegative integer, such that $w_i = 2$ for any $1 \leq i \leq n'$. By Lemma 2.13(1), $w_{-1} \leq \frac{2}{\epsilon_0}$, and by Lemma 2.14(3), $n' < \frac{2}{\epsilon_0} - 1$. Hence, $n_1 + n_2 + 1$, the number of vertices of $\mathcal{DG}$ is bounded from above by $\frac{8}{\epsilon_0}$. □

3. Proof of Theorem 1.3

Lemma 3.1 is crucial in the proof of Theorem 1.3. Before providing the proof, we introduce some notations first.

Notation $(\star)$. Let $X \ni x$ be a smooth surface germ, and let $g: X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 := X$ be a sequence of blow-ups with the data $(f_i, F_i, x_i \in X_i)$. Let $\mathcal{DG}$ be the dual graph of g, and assume that $\mathcal{DG}$ is a chain.

Let $n_3 \geq 2$ be the largest integer, such that $x_i \in F_i \setminus F_{i-1}$ for any $1 \leq i \leq n_3 - 1$, where we set $F_0 := \emptyset$. Let $\{E_j\}_{-n_1 \leq j \leq n_2}$ be the vertices of $\mathcal{DG}$, such that $E_0 := F_n$ is the only g-exceptional (-1)-curve on X_n, $E_{n_2} := F_1$, and E_i is adjacent to E_{i+1} for any $-n_1 \leq i \leq n_2 - 1$ (see Figure 2).

We define $n_i(g) := n_i$ for $1 \leq i \leq 3$, $n(g) = n$, $w_j(g) := -E_j \cdot E_j$ for all j and $W_1(g) := \sum_{j<0} w_j(g)$ and $W_2(g) := \sum_{j>0} w_j(g)$.

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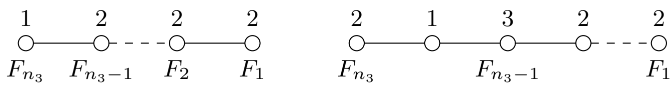

Figure 3. The dual graph for the case $n=n_3$ and $n=n_3+1$.

Lemma 3.1. *With* ***Notation*** $(\star)$. *Then:*

$$(W_1(g)-n_1(g))+n_3(g)-1=W_2(g)-n_2(g). \tag{3.1}$$

In particular, $n(g)=n_1(g)+n_2(g)+1\le n_3(g)+\min\{W_1(g),W_2(g)\}$.

Proof. For simplicity, let $n:=n(g)$, $n_i:=n_i(g)$ for $1\le i\le 3$, $w_j:=w_j(g)=-E_j\cdot E_j$ for all j and $W_j:=W_j(g)$ for $j=1,2$.

We prove (3.1) by induction on the nonnegative integer $n-n_3$.

If $n=n_3$, then $n_1=W_1=0$, $n_2=n_3-1$ and $W_2=2n_3-2$, thus, (3.1) holds (see Figure 3). If $n=n_3+1$, then $x_{n_3}\in F_{n_3}\cap F_{n_3-1}$. In this case, $n_1=1$, $W_1=2$, $n_2=n_3-1$ and $W_2=2n_3-1$, thus, (3.1) holds (see Figure 3).

In general, suppose (3.1) holds for any sequence of blow-ups g, as in **Notation** $(\star)$ with positive integers n,n_3 satisfying $1\le n-n_3\le k$. For the case when $n-n_3=k+1$, we may contract the (-1)-curve on X_n, and consider $g':X_{n-1}\to\cdots\to X_0:=X$, a subsequence of blow-ups of g with the data $(f_i,F_i,x_i\in X_i)$ for $0\le i\le n-1$. Denote $n_i':=n_i(g')$ for any $1\le i\le 3$, and $W_j':=W_j(g')$ for any $1\le j\le 2$. By Lemma 2.14(1), either $w_{-1}=2$ or w_1=2. In the former case, $W_1'=W_1-2,W_2'=W_2-1,n_1'=n_1-1$, $n_2'=n_2$ and $n_3'=n_3$. In the latter case, $W_1'=W_1-1,W_2'=W_2-2,n_1'=n_1,n_2'=n_2-1$ and $n_3'=n_3$. In both cases, by induction,

$$W_2'-n_2'-(W_1'-n_1')=(W_2-n_2)-(W_1-n_1)=n_3-1.$$

Hence, we finish the induction, and (3.1) is proved.

Since $w_j\ge 2$ for $j\ne 0$, we have $W_1=\sum_{-n_1\le j\le -1}w_j\ge 2n_1$ and $W_2=\sum_{1\le j\le n_2}w_j\ge 2n_2$. By (3.1),

$$n_1+n_2+1\le n_1+W_2-n_2+1=W_1+n_3,$$

and

$$n_1+n_2+1\le W_1-n_1+n_2+n_3-1=W_2,$$

which imply that $n=n_1+n_2+1\le n_3+\min\{W_1,W_2\}$. □

We will need Lemma 3.2 to prove Theorems 1.3 and 1.4.

Lemma 3.2. *Let* $\gamma\in(0,1]$ *be a real number. Let* $N_0:=\lfloor 1+\frac{32}{\gamma^2}+\frac{1}{\gamma}\rfloor$, *then we have the following.*

Let $(X\ni x,B:=\sum_i b_iB_i)$ *be an lc surface germ, such that* $X\ni x$ *is smooth and* B_i *are distinct prime divisors. Suppose that* $\{\sum_i n_ib_i-1>0\mid n_i\in\mathbb{Z}_{\ge 0}\}\subseteq[\gamma,+\infty)$. *Let* Y *be a*

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smooth surface and $f : Y \to X \ni x$ be a birational morphism with the dual graph $\mathcal{DG}$, such that:

- *$K_Y + B_Y = f^*(K_X + B)$ for some $B_Y \geq 0$ on Y,*
- *$\mathcal{DG}$ is a chain that contains only one (-1)-curve E_0,*
- *E_0 is adjacent to two vertices of $\mathcal{DG}$ and*
- *either E_0 is the only vertex of $\mathcal{DG}$, such that $a(E_0, X, B) = \mathrm{mld}(X \ni x, B)$, or $a(E_0, X, B) = \mathrm{mld}(X \ni x, B) > 0$ and $\sum_i n_i b_i \neq 1$ for all $n_i \in \mathbb{Z}_{\geq 0}$.*

Then the number of vertices of $\mathcal{DG}$ is bounded from above by N_0.

Proof. By Lemma 2.5, $b_i \geq \gamma$ for all i.

If $\mathrm{mld}(X \ni x, B) \geq \frac{\gamma}{2}$, then by Lemma 2.21 with $\epsilon_0 = \frac{\gamma}{2}$, the number of vertices of $\mathcal{DG}$ is bounded from above by $\frac{16}{\gamma}$.

Thus, we may assume that $0 \leq \mathrm{mld}(X \ni x, B) \leq \frac{\gamma}{2}$. We may index the vertices of $\mathcal{DG}$ as $\{E_j\}_{-n_1 \leq j \leq n_2}$ for some positive integer n_1, n_2, where E_j is adjacent to E_{j+1} for $-n_1 \leq j \leq n_2 - 1$. Let $w_j := -E_j \cdot E_j$ and $a_j := a(E_j, X \ni x, B)$ for all j.

For all $-n_1 \leq k \leq n_2$, we have:

$$\left(K_Y + f_*^{-1}B + \sum_j (1 - a_j)E_j\right) \cdot E_k = f^*(K_X + B) \cdot E_k = 0. \tag{3.2}$$

Let $k = 0$, (3.2) becomes $0 = -2 + f_*^{-1}B \cdot E_0 + (1 - a_{-1}) + (1 - a_1) + w_0 a_0$, thus:

$$(a_1 - a_0) + (a_{-1} - a_0) = f_*^{-1}B \cdot E_0 - a_0.$$

By the last assumption in the lemma, either $(a_{-1} - a_0) + (a_1 - a_0) > 0$ or $a_0 > 0$, thus, $f_*^{-1}B \cdot E_0 > 0$ in both cases. Hence, $f_*^{-1}B \cdot E_0 - a_0 \geq \gamma - \frac{\gamma}{2} = \frac{\gamma}{2}$. Possibly switching $E_j (j < 0)$ with $E_j (j > 0)$, we may assume that $a_{-1} - a_0 \geq \frac{\gamma}{4}$.

By Lemma 2.13(2), $a_{-j} - a_{-j+1} \geq a_{-1} - a_0 \geq \frac{\gamma}{4}$ for $1 \leq j \leq n_1$, thus, $n_1 \cdot \frac{\gamma}{4} \leq a_{-n_1} \leq 1$, and $n_1 \leq \frac{4}{\gamma}$. Since $a_j \geq \frac{\gamma}{4}$ for all $-n_1 \leq j \leq -1$, by Lemma 2.13(1), $w_j \leq \frac{8}{\gamma}$ for all $-n_1 \leq j \leq -1$. Thus, $\sum_{j=-1}^{-n_1} w_j \leq n_1 \cdot \frac{8}{\gamma} \leq \frac{32}{\gamma^2}$. Note that $X \ni x$ is smooth and $\mathcal{DG}$ has only one (-1)-curve, thus, $f : Y \to X$ is a sequence of blow-ups as in Definition 2.16. Moreover, $\mathcal{DG}$ is a chain, thus, by Lemma 3.1, $1 + n_1 + n_2 \leq n_3 + \frac{32}{\gamma^2}$, where $n_3 = n_3(f)$ is defined as in **Notation** $(\star)$.

It suffices to show that n_3 is bounded, we may assume that $n_3 > 2$. By the definition of n_3, there exists a sequence of blow-ups $X_{n_3} \to \ldots X_1 \to X_0 := X$ with the data $(f_i, F_i, x_i \in X_i)$, such that $x_i \in F_i \setminus F_{i-1}$ for any $1 \leq i \leq n_3 - 1$. Here, $F_0 := \emptyset$.

Let B_{X_i} be the strict transform of B on X_i for $0 \leq i \leq n_3$, and let $a'_i := a(F_i, X, B)$ for $1 \leq i \leq n_3$, and $a'_0 := 1$. Since $x_i \in F_i \setminus F_{i-1}$, $a'_i - a'_{i+1} = \mathrm{mult}_{x_i} B_{X_i} - 1$ for any $n_3 - 1 \geq i \geq 0$. By Lemma 2.13(2), $a'_i - a'_{i+1} \geq \min\{a_1 - a_0, a_{-1} - a_0\} \geq 0$ for $1 \leq i \leq n_3 - 2$ (see Figure 2). Thus, by the last assumption in the lemma, either $\min\{a_1 - a_0, a_{-1} - a_0\} > 0$ or $\mathrm{mult}_{x_i} B_{X_i} - 1 > 0$, in both cases, we have $a'_i - a'_{i+1} = \mathrm{mult}_{x_i} B_{X_i} - 1 > 0$. Hence, $a'_i - a'_{i+1} =$

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$\mathrm{mult}_{x_i} B_{X_i} - 1 \geq \gamma$ for any $1 \leq i \leq n_3 - 2$ as $\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\gamma, +\infty)$. Therefore,

$$0 \leq a'_{n_3-1} = a'_0 + \sum_{i=0}^{n_3-2} (a'_{i+1} - a'_i) \leq 1 - (n_3 - 1)\gamma,$$

and $n_3 \leq 1 + \frac{1}{\gamma}$.

To sum up, the number of vertices of $\mathcal{DG}$ is bounded from above by $\lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \rfloor$. $\square$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 2.17, we may assume that $\mathrm{mld}(X \ni x, B) \leq 1$.

Let $f : Y \to X \ni x$ be the birational morphism constructed in Lemma 2.20 with the dual graph $\mathcal{DG}$. We claim that the number of vertices of $\mathcal{DG}$ is bounded from above by $N_0 := \lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \rfloor$.

Assume the claim holds, then by Lemma 2.18, $a(E, X, 0) \leq 2^{N_0}$ for some exceptional divisor E, such that $a(E, X, B) = \mathrm{mld}(X \ni X, B)$, we are done. It suffices to show the claim.

If the f-exceptional (-1)-curve is adjacent to only one vertex of $\mathcal{DG}$, then by Lemma 2.15(1), the number of vertices of $\mathcal{DG}$ is bounded from above by $1 + \frac{1}{\gamma}$.

If the f-exceptional (-1)-curve is adjacent to two vertices of $\mathcal{DG}$, then by Lemma 3.2, the number of vertices of $\mathcal{DG}$ is bounded from above by $\lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \rfloor$. Thus, we finish the proof. $\square$

To end this section, we provide an example which shows that Theorem 1.3 does not hold if we do not assume that $\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}\}$ is bounded from below by a positive real number.

Example 3.3. Let $\{(\mathbf{A}^2 \ni 0, B_k)\}_{k \geq 2}$ be a sequence of klt surface germs, such that $B_k := \frac{1}{2} B_{k,1} + (\frac{1}{2} + \frac{1}{k+1}) B_{k,2}$, and $B_{k,1}$ (respectively, $B_{k,2}$) is defined by the equation $x = 0$ (respectively, $x - y^k = 0$) at $0 \in \mathbf{A}^2$.

For each k, we may construct a sequence of blow-ups $X_k \to X_{k-1} \to \cdots \to X_1 \to X_0 := X$ with the data $(f_i, F_i, x_i \in X_i)$, such that $x_{i-1} \in F_{i-1}$ is the intersection of the strict transforms of $B_{k,1}$ and $B_{k,2}$ on F_{i-1} for $1 \leq i \leq k$. Let $g_k : X_k \to X$ be the natural morphism induced by $\{f_i\}_{1 \leq i \leq k}$, we have $K_{X_k} + B_{X_k} = g_k^*(K_X + B_k)$ for some snc divisor $B_{X_k} \geq 0$ on X_k, and the coefficients of B_{X_k} are no more than $\frac{k}{k+1}$. Thus, we will need at least k blow-ups as constructed above to extract an exceptional divisor F_k, such that $a(F_k, \mathbf{A}^2, B_k) = \mathrm{mld}(\mathbf{A}^2 \ni 0, B_k) = \frac{1}{k+1}$, and $a(F_k, \mathbf{A}^2, 0) \geq k$.

4. Proof of Theorem 1.2

Proof of Theorem 1.2. We may assume that $\Gamma \setminus \{0\} \neq \emptyset$.

Let $(X \ni x, B)$ be an lc surface germ with $B \in \Gamma$. By Lemma 2.17, we may assume that $\mathrm{mld}(X \ni x, B) \leq 1$. By Theorem 1.3, it suffices to show the case when $X \ni x$ is not smooth.

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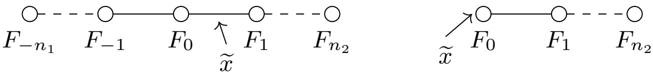

Figure 4. Cases when $\widetilde{x} \in F_0 \cap F_1$ and when $\widetilde{x} \notin F_i$ for $i \neq 0$.

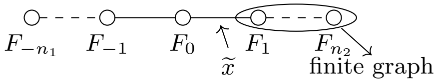

Figure 5. Cases when $a_1 - a_0 \geq \epsilon$.

If $\mathrm{mld}(X \ni x, B) = \mathrm{pld}(X \ni x, B)$, then $a(E, X, 0) \leq 1$ for some prime divisor E over $X \ni x$, such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$. So we may assume that $\mathrm{mld}(X \ni x, B) \neq \mathrm{pld}(X \ni x, B)$.

By Lemma 2.20, there exists a birational morphism $f : Y \to X \ni x$ which satisfies Lemma 2.20(1)–(5). Let $\widetilde{f} : \widetilde{X} \to X$ be the minimal resolution of $X \ni x$, $g : Y \to \widetilde{X} \ni \widetilde{x}$ the birational morphism, such that $\widetilde{f} \circ g = f$, where $\widetilde{x} \in \widetilde{X}$ is chosen as in Lemma 2.20(5), and there exists an $\widetilde{f}$-exceptional prime divisor $\widetilde{E}$ over $X \ni x$, such that $a(\widetilde{E}, X, B) = \mathrm{pld}(X \ni x, B)$ and $\widetilde{x} \in \widetilde{E}$. Moreover, there is at most one other vertex $\widetilde{E'}$ of $\widetilde{\mathcal{DG}}$, such that $\widetilde{x} \in \widetilde{E'}$.

Let $\widetilde{\mathcal{DG}}$ be the dual graph of $\widetilde{f}$ and $\{F_i\}_{-n_1 \leq i \leq n_2}$ the vertices of $\widetilde{\mathcal{DG}}$, such that $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, F_i is adjacent to F_{i+1}, $w_i := -F_i \cdot F_i, a_i := a(F_i, X, B)$ for all i, and $F_0 := \widetilde{E}, F_1 := \widetilde{E'}$ (see Figure 4). We may write $K_{\widetilde{X}} + B_{\widetilde{X}} = \widetilde{f}^*(K_X + B)$, where $B_{\widetilde{X}} := \widetilde{f}^{-1}_* B + \sum_i (1 - a_i) F_i$, and we define $\widetilde{B} := \widetilde{f}^{-1}_* B + \sum_{\widetilde{x} \in F_i} (1 - a_i) F_i$.

If $\widetilde{x} \notin F_i$ for all $i \neq 0$, then we consider the surface germ $(\widetilde{X} \ni \widetilde{x}, \widetilde{B} = \widetilde{f}^{-1}_* B + (1 - a_0) F_0)$, where $\widetilde{B} \in \Gamma' := \Gamma \cup \{1 - a \mid a \in \mathrm{Pld}(2, \Gamma)\}$. By Lemma 2.9, Γ' satisfies the DCC. Thus, by Theorem 1.3, we may find a positive integer N_1 which only depends on Γ, and a prime divisor E over $\widetilde{X} \ni \widetilde{x}$, such that $a(E, \widetilde{X}, \widetilde{B}) = a(E, X, B) = \mathrm{mld}(X \ni x, B)$, and $a(E, X, 0) \leq a(E, \widetilde{X}, 0) \leq N_1$.

So we may assume that $\widetilde{x} = F_0 \cap F_1$. By Lemma 2.4, there exist positive real numbers $\epsilon, \delta \leq 1$ depending only on Γ, such that $\{\sum_i n_i b_i - 1 > 0 \mid b_i \in \Gamma'_\epsilon \cap [0,1], n_i \in \mathbb{Z}_{\geq 0}\} \subseteq [\delta, +\infty)$. Recall that $\Gamma'_\epsilon = \cup_{b' \in \Gamma'} [b' - \epsilon, b']$.

If $a_1 - a_0 \leq \epsilon$, then we consider the surface germ $(\widetilde{X} \ni \widetilde{x}, \widetilde{B} = \widetilde{f}^{-1}_* B + (1 - a_0) F_0 + (1 - a_1) F_1)$, where $\widetilde{B} \in \Gamma'_\epsilon \cap [0,1]$. By Theorem 1.3, there exist a positive integer N_2 which only depends on Γ, and a prime divisor E over $\widetilde{X} \ni \widetilde{x}$, such that $a(E, \widetilde{X}, \widetilde{B}) = a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(E, X, 0) \leq a(E, \widetilde{X}, 0) \leq N_2$.

If $a_1 - a_0 \geq \epsilon$, then we claim that there exists a DCC set Γ'' depending only on Γ, such that $1 - a_1 \in \Gamma''$.

Assume the claim holds, then we consider the surface germ $(\widetilde{X} \ni \widetilde{x}, \widetilde{B} = \widetilde{f}^{-1}_* B + (1 - a_0) F_0 + (1 - a_1) F_1)$, where $\widetilde{B} \in \Gamma'' \cup \Gamma'$. By Theorem 1.3, we may find a positive integer N_3 which only depends on Γ, and a prime divisor E over $\widetilde{X} \ni \widetilde{x}$, such that $a(E, \widetilde{X}, \widetilde{B}) = a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and $a(E, X, 0) \leq a(E, \widetilde{X}, 0) \leq N_3$. Let $N := \max\{N_1, N_2, N_3\}$, and we are done.

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It suffices to show the claim. By Lemma 2.13(1), $w_i \le \frac{2}{\epsilon}$ for any $0 < i \le n_2$. Since $1 \ge a_{n_2} = a_0 + \sum_{i=0}^{n_2-1}(a_{i+1}-a_i) \ge n_2\epsilon$, $n_2 \le \frac{1}{\epsilon}$. We may write:

$$K_{\widetilde{X}} + \widetilde{f}_*^{-1}B + \sum_{-n_1 \le i \le n_2}(1-a_i)F_i = \widetilde{f}^*(K_X+B).$$

For each $1 \le j \le n_2$, we have:

$$(K_{\widetilde{X}} + \widetilde{f}_*^{-1}B + \sum_{-n_1 \le i \le n_2}(1-a_i)F_i)\cdot F_j = 0,$$

which implies $\sum_{-n_1 \le i \le n_2}(a_i-1)F_i \cdot F_j = -{F_j}^2 - 2 + \widetilde{f}_*^{-1}B\cdot F_j$, or equivalently,

$$\begin{pmatrix} F_1\cdot F_1 & \cdots & F_{n_2}\cdot F_1 \\ \vdots & \ddots & \vdots \\ F_1\cdot F_{n_2} & \cdots & F_{n_2}\cdot F_{n_2}\end{pmatrix}\begin{pmatrix} a_1-1 \\ \vdots \\ a_{n_2}-1\end{pmatrix} = \begin{pmatrix} w_1 - 2 + \widetilde{f}_*^{-1}B\cdot F_1 + (1-a_0) \\ \vdots \\ w_{n_2} - 2 + \widetilde{f}_*^{-1}B\cdot F_{n_2}\end{pmatrix}.$$

By assumption, $w_j - 2 + \widetilde{f}_*^{-1}B\cdot F_j$ belongs to a DCC set, and by Lemma 2.9, $1-a_0$ belongs to the DCC set $\{1-a \mid a \in \mathrm{Pld}(2,\Gamma)\}$.

By [15, Lemma 3.40], $(F_i\cdot F_j)_{1\le i,j\le n_2}$ is a negative definite matrix. Let $(s_{ij})_{n_2\times n_2}$ be the inverse matrix of $(F_i\cdot F_j)_{1\le i,j\le n_2}$. By [15, Lemma 3.41], $s_{ij} < 0$ for any $1 \le i,j \le n_2$, thus:

$$1-a_1 = -s_{11}(w_1 - 2 + \widetilde{f}_*^{-1}B\cdot F_1 + (1-a_0)) - \sum_{j=2}^{n_2} s_{1j}(w_j - 2 + \widetilde{f}_*^{-1}B\cdot F_j)$$

belongs to a DCC set. □

5. Proof of Theorem 1.4

In this section, we will first show Theorem 1.4, then we will generalise Theorems 1.3 and 1.4 (see Theorems 5.4 and 5.5).

Lemma 5.1. *Let $(X \ni x, B := \sum_i b_iB_i)$ be an lc surface germ, where B_i are distinct prime divisors. Let $h: W \to X \ni x$ be a log resolution of $(X \ni x, B)$, and let $S = \{E_j\}$ be a finite set of valuations of h-exceptional prime divisors, such that $a(E_j,X,B) = \mathrm{mld}(X \ni x,B)$ for all j. Suppose that $\sum_i n_ib_i \ne 1$ for any $n_i \in \mathbb{Z}_{\ge 0}$, $\mathrm{mld}(X \ni x,B) \in (0,1]$, and E_j is exceptional over $\tilde{X}$ for some j, where $\tilde{X} \to X$ is the minimal resolution of $X \ni x$. Then there exist a smooth surface Y and a birational morphism $f: Y \to X \ni x$ with the dual graph $\mathcal{DG}$, such that:*

(1) *$K_Y + B_Y = f^*(K_X+B)$ for some $\mathbb{R}$-divisor $B_Y \ge 0$ on Y,*

(2) *each valuation in S corresponds to some vertex of $\mathcal{DG}$,*

(3) *$\mathcal{DG}$ contains only one (-1)-curve E_0, and it corresponds to a valuation in S and*

(4) *$\mathcal{DG}$ is a chain.*

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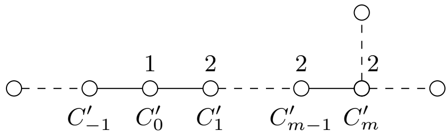

Figure 6. C'_0 is adjacent to C'_{-1} and C'_1.

Proof. By Lemma 2.19, there exist a smooth surface Y and a birational morphism $f:Y\to X\ni x$ with the dual graph of $\mathcal{DG}$ which satisfy (1)–(2), and each f-exceptional (-1)-curve corresponds to some valuation in S.

For (3), by the assumption on S, $\mathcal{DG}$ contains at least one (-1)-curve E_0. If there exist two (-1)-curves $E'_0\neq E_0$, then $a(E_0,X,B)=a(E'_0,X,B)=\operatorname{mld}(X\ni x,B)>0$ by construction, and there exists a set of distinct vertices $\{C_k\}_{0\le k\le n}$ of $\mathcal{DG}$, such that $n\ge 2$, $C_n:=E'_0$, C_0 is a (-1)-curve, $-C_k\cdot C_k\ge -2$ for $1\le k\le n-1$ and C_k is adjacent to C_{k+1} for $0\le k\le n-1$. Since $a(C_0,X,B)=a(C_n,X,B)=\operatorname{mld}(X\ni x,B)>0$, by Lemma 2.13(6), $-C_k\cdot C_k=2$ for $1\le k\le n-1$. Let $E:=\sum_{k=0}^n C_k$, then $E\cdot E=0$, which contradicts the negativity lemma.

For (4), suppose that $\mathcal{DG}$ contains a fork F. Since E_0 is a (-1)-curve, by Lemma 2.14(2), $E_0\neq F$. By (3) and (5) of Lemma 2.13, we have $a(E_0,X,B)\ge a(F,X,B)\ge \operatorname{mld}(X\ni x,B)$, thus, $a(E_0,X,B)=a(F,X,B)=\operatorname{mld}(X\ni x,B)>0$. There exists a set of distinct vertices $\{C'_i\}_{0\le i\le m}$, such that $C'_0:=E_0$, $C'_m:=F$, and C'_i is adjacent to C'_{i+1} for $0\le i\le m-1$. We may denote $w'_i:=-C'_i\cdot C'_i$ and $a'_i:=a(C'_i,X,B)$ for $0\le i\le m$. By Lemma 2.13(6), we have $a'_0=\cdots=a'_m$, and $w'_k=2$ for $1\le k\le m-1$. Since C'_m is a fork and $a'_{m-1}=a'_m$, by Lemma 2.13(3), $w'_m=2$.

If C'_0 is adjacent to only one vertex of $\mathcal{DG}$, which is C'_1 by our construction, then since $\sum_i n_ib_i\neq 1$ for any $n_i\in\mathbb{Z}_{\ge 0}$, $a'_0\neq a'_1$ by Lemma 2.15(2), a contradiction.

Thus, we may assume that C'_0 is adjacent to a vertex C'_{-1} of $\mathcal{DG}$ other than C'_1 (see Figure 6). We may contract C'_k for $0\le k\le m-1$ step by step, and will end up with a fork which is a (-1)-curve, this contradicts Lemma 2.14(2). $\square$

Lemma 5.2. *Let $(X\ni x,B:=\sum_i b_iB_i)$ be an lc surface germ, where B_i are distinct prime divisors. Suppose that $X\ni x$ is a klt surface germ,* $\operatorname{mld}(X\ni x,B)=0$, *and $\sum_i n_ib_i\neq 1$ for any $n_i\in\mathbb{Z}_{\ge 0}$. Then there is only one prime divisor E over $X\ni x$, such that $a(E,X,B)=0$.*

Proof. By [2, Lemma 2.7] (this holds for surfaces in any characteristic), we can find a plt blow up $g:Y\to X\ni x$, such that:

- $K_Y+B_Y=g^*(K_X+B)$ for some $\mathbb{R}$-divisor $B_Y\ge 0$ on Y,
- there is only one g-exceptional prime divisor E,
- $\operatorname{Supp}E\subseteq\lfloor B_Y\rfloor$ and
- (Y,E) is plt.

Since the relative Kawamata-Viehweg vanishing theorem holds for birational morphisms between surfaces in any characteristic (see [23, Theorem 0.5]), by a similar argument

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as in [10, Proposition 4.1], E is normal. By the adjunction formula ([20, §3], [14, §16]), $K_E+B_E:=(K_Y+B_Y)|_E$, where:

$$B_E=\sum_i \frac{m_i-1+\sum_j n_{i,j}b_j}{m_i}p_i$$

for some distinct closed points p_i on E and some $m_i\in\mathbb{Z}_{>0}$ and $n_{i,j}\in\mathbb{Z}_{\geq 0}$, such that $\sum_j n_{i,j}b_j\leq 1$ for all i. By assumption, $\sum_j n_{i,j}b_j\neq 1$, thus, $\sum_j n_{i,j}b_j<1$ for all i, which implies $\lfloor B_E\rfloor=0$, thus, (E,B_E) is klt. By the inversion of adjunction for surfaces (which is well known in any characteristic), (Y,B_Y) is plt, and we are done. □

Proof of Theorem 1.4. By Lemma 2.17, we may assume that $\mathrm{mld}(X\ni x,B)\leq 1$.

If $\mathrm{mld}(X\ni x,B)=0$, since $X\ni x$ is smooth, then by Lemma 5.2, there exists a unique prime divisor E over $X\ni x$, such that $a(E,X,B)=0$. By Theorem 1.3, $a(E,X,0)\leq 2^{N_0}$. Thus, we may assume that $\mathrm{mld}(X\ni x,B)>0$.

We may apply Lemma 5.1 to S, there exist a smooth surface Y and a birational morphism $f:Y\to X\ni x$ with the dual graph $\mathcal{DG}$, which satisfy Lemma 5.1(1)–(4). Let E_0 be the unique (-1)-curve in $\mathcal{DG}$. It suffices to give an upper bound for the number of vertices of $\mathcal{DG}$.

If E_0 is adjacent to only one vertex of $\mathcal{DG}$, then by Lemma 2.15(2)–(3), the number of vertices of $\mathcal{DG}$ is bounded from above by $1+\frac{1}{\gamma}$, and $|S|=1$.

If E_0 is adjacent to two vertices of $\mathcal{DG}$, then by Lemma 3.2, the number of vertices of $\mathcal{DG}$ is bounded from above by $1+\frac{32}{\gamma^2}+\frac{1}{\gamma}$. Thus, $|S|\leq N_0$, and by Lemma 2.18, $a(E,X,0)\leq 2^{N_0}$ for any $E\in S$. □

Now we are going to introduce and prove Theorems 5.4 and 5.5. First, we need to introduce the following definition.

Definition 5.3. Let $X\ni x$ be a klt surface germ. Let $\widetilde{f}:\widetilde{X}\to X\ni x$ be the minimal resolution and $\{E_i\}_{1\leq i\leq n}$ the set of $\widetilde{f}$-exceptional prime divisors. The *determinant* of $X\ni x$ is defined by:

$$\det(X\ni x):=\begin{cases}|\det(E_i\cdot E_j)_{1\leq i,j\leq n}| & \text{if } x\in X \text{ is a singular point,}\\ 1 & \text{if } x\in X \text{ is a smooth point.}\end{cases}$$

Theorem 5.4. *Let $\gamma\in(0,1]$ be a real number and I a positive integer. Then $N_0:=\lfloor 1+\frac{32I^2}{\gamma^2}+\frac{I}{\gamma}\rfloor$ satisfies the following.*

Let $(X\ni x,B:=\sum_i b_iB_i)$ be an lc surface germ, where B_i are distinct prime divisors. Suppose that $\det(X\ni x)\mid I$, and $\{\sum_i n_ib_i-t>0\mid n_i\in\mathbb{Z}_{\geq 0},t\in\mathbb{Z}\cap[1,I]\}\subseteq[\gamma,+\infty)$. Then there exists a prime divisor E over $X\ni x$, such that $a(E,X,B)=\mathrm{mld}(X\ni x,B)$ and $a(E,X,0)\leq 2^{N_0}$.

Theorem 5.5. *Let $\gamma\in(0,1]$ be a real number and $X\ni x$ a klt surface germ. Let N be the number of vertices of the dual graph of the minimal resolution of $X\ni x$ and $I:=\det(X\ni x)$. Then $N_0:=\lfloor 1+\frac{32I^2}{\gamma^2}+\frac{I}{\gamma}\rfloor+N$ satisfies the following.*

Let $(X\ni x,B:=\sum_i b_iB_i)$ be an lc surface germ, where B_i are distinct prime divisors. Let $S:=\{E\mid E$ is a prime divisor over $X\ni x,a(E,X,B)=\mathrm{mld}(X\ni x,B)\}$. Suppose that

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$\{\sum_i n_i b_i - t \geq 0 \mid n_i \in \mathbb{Z}_{\geq 0}, t \in \mathbb{Z} \cap [1,I]\} \subseteq [\gamma, +\infty)$. *Then* $|S| \leq N_0$, *and* $a(E,X,0) \leq 2^{N_0}$ *for any* $E \in S$.

Remark 5.6. It is easy to see that Theorem 5.5 does not hold if $X \ni x$ is not klt. Also it does not hold if we only bound $\det(X \ni x)$ as in Theorem 5.4 (see Example 5.9).

Theorem 5.4 follows from Theorem 1.3, and Theorem 5.5 follows from Theorems 1.4 and 5.4. We will need the following lemma to prove the above theorems.

Lemma 5.7. *Let* $(X \ni x, B := \sum_i b_i B_i)$ *be an lc surface germ, where* B_i *are distinct prime divisors. Let* $\widetilde{f} : \widetilde{X} \to X \ni x$ *be the minimal resolution, and we may write* $K_{\widetilde{X}} + B_{\widetilde{X}} = \widetilde{f}^*(K_X + B)$ *for some* $\mathbb{R}$*-divisor* $B_{\widetilde{X}} := \sum_j \widetilde{b}_j \widetilde{B}_j \geq 0$, *where* $\widetilde{B}_j$ *are distinct prime divisors on* $\widetilde{X}$. *Let* $I \in \mathbb{Z}_{>0}$, *such that* $\det(X \ni x) \mid I$, *then:*

(1) $\{\sum_j n'_j \widetilde{b}_j - 1 > 0 \mid n'_j \in \mathbb{Z}_{\geq 0}\} \subseteq \frac{1}{I}\{\sum_i n_i b_i - t > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, t \leq I\}$ *and*

(2) $\{\sum_j n'_j \widetilde{b}_j - 1 \geq 0 \mid n'_j \in \mathbb{Z}_{\geq 0}\} \subseteq \frac{1}{I}\{\sum_i n_i b_i - t \geq 0 \mid n_i \in \mathbb{Z}_{\geq 0}, \sum_i n_i > 0, t \leq I\}$.

Proof. Let $\mathcal{DG}$ be the dual graph of $\widetilde{f}$. Let $\{E_i\}_{1 \leq i \leq m}$ be the set of $\widetilde{f}$-exceptional divisors and $w_i := -E_i \cdot E_i$, $a_i := a(E_i, X \ni x, B)$ for all i.

For each $1 \leq j \leq m$, $(K_Y + f_*^{-1}B + \sum_{i=1}^m (1-a_i)E_i) \cdot E_j = 0$, which implies that $\sum_{i=1}^m (a_i - 1)E_i \cdot E_j = -E_j{}^2 - 2 + \widetilde{f}_*^{-1}B \cdot E_j$.

By [15, Lemma 3.40], $(E_i \cdot E_j)_{1 \leq i,j \leq m}$ is a negative definite matrix. Let $(s_{ij})_{m \times m}$ be the inverse matrix of $(E_i \cdot E_j)_{1 \leq i,j \leq m}$. By [15, Lemma 3.41] and the assumption on I, we have $Is_{ij} \in \mathbb{Z}_{<0}$ for $1 \leq i,j \leq m$. Thus, for all i,

$$1 - a_i = \frac{1}{I} \sum_{j=1}^m (-Is_{ij}) \cdot (w_j - 2 + \widetilde{f}_*^{-1}B \cdot E_j). \tag{5.1}$$

Since $w_j \geq 2$ for all j, (1) and (2) follow immediately from (5.1) and the equation $B_{\widetilde{X}} = \sum_j \widetilde{b}_j \widetilde{B}_j = \widetilde{f}_*^{-1}B + \sum_{i=1}^m (1-a_i)E_i$. □

Proof of Theorem 5.4. By Lemma 2.17, we may assume that $\mathrm{mld}(X \ni x, B) \leq 1$.

If $\mathrm{mld}(X \ni x, B) = \mathrm{pld}(X \ni x, B)$, then $a(E,X,0) \leq 1$ for some prime divisor E over $X \ni x$, such that $a(E,X,B) = \mathrm{mld}(X \ni x, B)$. So we may assume that $\mathrm{mld}(X \ni x, B) \neq \mathrm{pld}(X \ni x, B)$.

By Lemma 2.20, there exists a birational morphism $f : Y \to X \ni x$, which satisfies Lemma 2.20(1)–(4). Let $\widetilde{f} : \widetilde{X} \to X$ be the minimal resolution of $X \ni x$ and $g : Y \to \widetilde{X}$ the natural morphism induced by f. We may write $K_{\widetilde{X}} + B_{\widetilde{X}} = \widetilde{f}^*(K_X + B)$ for some $\mathbb{R}$-divisor $B_{\widetilde{X}} := \sum_j \widetilde{b}_j \widetilde{B}_j \geq 0$ on $\widetilde{X}$, where $\widetilde{B}_j$ are distinct prime divisors.

By Lemma 5.7(1), we have $\{\sum_j n'_j \widetilde{b}_j - 1 > 0 \mid n'_j \in \mathbb{Z}_{\geq 0}\} \subseteq \frac{1}{I}\{\sum_i n_i b_i - t > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, t \leq I\} \subseteq [\frac{\gamma}{I}, \infty)$ by the assumption. Since the dual graph $\mathcal{DG}$ of f contains only one (-1)-curve, there exists a closed point $\widetilde{x} \in \widetilde{X}$, such that $\mathrm{center}_{\widetilde{X}} E = \widetilde{x}$ for any g-exceptional divisor E. Apply Theorem 1.3 to the surface germ $(\widetilde{X} \ni \widetilde{x}, B_{\widetilde{X}})$, we are done. □

Proof of Theorem 5.5. By Lemma 2.17, we may assume that $\mathrm{mld}(X \ni x, B) \leq 1$.

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We may assume that $B \neq 0$, otherwise, we may take $\widetilde{f}: \widetilde{X} \to X \ni x$ as the minimal resolution, and $K_{\widetilde{X}} + B_{\widetilde{X}} = \widetilde{f}^* K_X$, where $B_{\widetilde{X}}$ is an snc divisor. Since $X \ni x$ is klt, all elements in S are on $\widetilde{X}$, thus, $|S| \le N$, and $a(E,X,0) \le 1$ for all $E \in S$. For the same reason, we may assume that not all elements of S are on the minimal resolution $\widetilde{X}$ of $X \ni x$.

If $\mathrm{mld}(X \ni x, B) = 0$, then by Lemma 5.2, there exists a unique exceptional divisor E over $X \ni x$, such that $a(E,X,B) = 0$. By Theorem 5.4, $a(E,X,0) \le 2^{\lfloor 1 + \frac{32I^2}{\gamma^2} + \frac{I}{\gamma} \rfloor}$. Thus, we may assume that $\mathrm{mld}(X \ni x, B) > 0$ from now on. Since $\sum_i n_i b_i \neq 1$ for any $n_i \in \mathbb{Z}_{\ge 0}$, we have $\lfloor B \rfloor = 0$, and (X,B) is klt near $x \in X$, hence, by [15, Proposition 2.36], S is a finite set.

Now, we may apply Lemma 5.1 to the set S, there exists a birational morphism $f: Y \to X \ni x$ that satisfies Lemma 5.1(1)–(4). Let $\widetilde{f}: \widetilde{X} \to X$ be the minimal resolution of $X \ni x$ and $g: Y \to \widetilde{X}$ the natural morphism induced by f. We may write $K_{\widetilde{X}} + B_{\widetilde{X}} = \widetilde{f}^*(K_X + B)$ for some $\mathbb{R}$-divisor $B_{\widetilde{X}} := \sum_j \widetilde{b_j} \widetilde{B_j} \ge 0$ on $\widetilde{X}$, where $\widetilde{B_j}$ are distinct prime divisors on $\widetilde{X}$.

By Lemma 5.7(2), we have $\{\sum_j n'_j \widetilde{b_j} - 1 \ge 0 \mid n'_j \in \mathbb{Z}_{\ge 0}\} \subseteq \frac{1}{I}\{\sum_i n_i b_i - t \ge 0 \mid n_i \in \mathbb{Z}_{\ge 0}, \sum_i n_i > 0, t \le I\} \subseteq [\frac{\gamma}{I}, \infty)$. Since the dual graph $\mathcal{DG}$ of f contains only one (-1)-curve, there exists a closed point $\widetilde{x} \in \widetilde{X}$, such that $\mathrm{center}_{\widetilde{X}} E = \widetilde{x}$ for any g-exceptional divisor E. Apply Theorem 1.4 to the surface germ $(\widetilde{X} \ni \widetilde{x}, B_{\widetilde{X}})$, we are done. $\square$

To end this section, we provide some examples.

The following example shows that Theorem 1.4 does not hold if $\sum_i n_i b_i = 1$ for some $n_i \in \mathbb{Z}_{\ge 0}$.

Example 5.8. Let $\{b_i\}_{1 \le i \le m} \subseteq (0,1]$, such that $\sum_{i=1}^m n_i b_i = 1$ for some $n_i \in \mathbb{Z}_{>0}$.

Let $(X \ni x) := (\mathbf{A}^2 \ni 0)$, $D_{m,n}$ the Cartier divisor, which is defined by the equation $x - y^m - y^{m+n} = 0$ and $B_k := \sum_{i=1}^m b_i \sum_{j=1}^{n_i} D_{k+i,j}$ for any positive integer k, where $D_{k+i,j}$ is defined by $x - y^{k+i} - y^{k+i+j} = 0$. Then $(X \ni x, B_k)$ is canonical near x for any k as $\mathrm{mult}_x B_k \le 1$.

For each k, we may construct a sequence of blow-ups $X_k \to X_{k-1} \to \cdots \to X_1 \to X_0 := X$ with the data $(f_i, F_i, x_i \in X_i)$, such that $x_{i-1} \in F_{i-1}$ is the intersection of the strict transforms of these $D_{k+i,j}$ on F_{i-1} for all i,j. Then $a(F_i, X, B_k) = 1 = \mathrm{mld}(X \ni x, B_k)$ for any k and $1 \le i \le k$. Thus, both $|S_k|$ and $\sup_{E \in S_k}\{a(E,X,0)\}$ are not bounded from above as $|S_k| \ge k$, and $a(F_k, X, 0) \ge k$ for each k.

The following example shows that Theorem 5.5 does not hold if we do not fix the germ, even when we bound the determinant of the surface germs.

Example 5.9. Let $\{(X_k \ni x_k)\}_{k \ge 1}$ be a sequence of surface germs, such that each $X_k \ni x_k$ is a Du Val singularity of type D_{k+3} ([15, Theorem 4.22]). We have $\det(X_k \ni x_k) = 4$ for all $k \ge 4$, but $|S_k| = k+3$ for $k \ge 1$ is not bounded from above.

6. An equivalent conjecture for MLDs on a fixed germ

Definition 6.1. Let X be a normal variety and $B := \sum b_i B_i$ an $\mathbb{R}$-divisor on X, where B_i are distinct prime divisors. We define $||B|| := \max_i\{|b_i|\}$. Let E be a prime divisor

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over X and $Y \to X$ a birational model, such that E is on Y. We define $\operatorname{mult}_E B_i$ to be the multiplicity of the strict transform of B_i on Y along E for each i, and $\operatorname{mult}_E B := \sum_i b_i \operatorname{mult}_E B_i$.

Conjecture 6.2 could be regarded as an inversion of stability type conjecture for divisors which compute MLDs, see [3, Main Proposition 2.1], [5, Theorem 5.10] for other inversion of stability type results.

Conjecture 6.2. *Let $\Gamma \subseteq [0,1]$ be a finite set, X a normal quasiprojective variety and $x \in X$ a closed point. Then there exists a positive real number τ depending only on Γ and $x \in X$ satisfying the following.*

Assume that $(X \ni x, B)$ and $(X \ni x, B')$ are two lc germs, such that:

(1) *$B' \le B$, $||B - B'|| < \tau, B \in \Gamma$ and*

(2) *$a(E, X, B') = \operatorname{mld}(X \ni x, B')$ for some prime divisor E over $X \ni x$.*

Then $a(E, X, B) = \operatorname{mld}(X \ni x, B)$.

Conjecture 6.2 does not hold for surface germs $(X \ni x, B)$, in general, if either Γ is a DCC set (see [9, Example 7.5]) or we do not fix $X \ni x$ (see [9, Example 7.6]).

Proposition 6.3. *For any fixed $\mathbb{Q}$-Gorenstein germ $X \ni x$ over an algebraically closed field of characteristic zero, Conjecture 6.2 is equivalent to Conjecture 1.1, thus, Conjecture 6.2 is equivalent to the ACC conjecture for MLDs.*

Remark 6.4. We need to work in characteristic zero since we need to apply [12, Theorem 1.1].

Proof. Suppose that Conjecture 6.2 holds. Let $t := \min\{1, \tau\} > 0$. Then $||(1-t)B - B|| < \tau$. Let E be a prime divisor over $X \ni x$, such that $a(E, X, (1-t)B) = \operatorname{mld}(X \ni x, (1-t)B)$. Then:

$$\operatorname{mld}(X \ni x, 0) \ge a(E, X, (1-t)B) = a(E, X, B) + t \operatorname{mult}_E B \ge t \operatorname{mult}_E B,$$

and $\operatorname{mult}_E B \le \frac{1}{t} \operatorname{mld}(X \ni x, 0)$.

By assumption, $\operatorname{mld}(X \ni x, 0) \ge \operatorname{mld}(X \ni x, B) = a(E, X, B) = a(E, X, 0) - \operatorname{mult}_E B$. Hence:

$$a(E, X, 0) \le \left(1 + \frac{1}{t}\right) \operatorname{mld}(X \ni x, 0),$$

and Conjecture 1.1 holds.

Suppose that Conjecture 1.1 holds, then there exists a positive real number N which only depends on Γ and $X \ni x$, such that $a(E_0, X, 0) \le N$ for some E_0 satisfying $a(E_0, X, B) = \operatorname{mld}(X \ni x, B)$. In particular, $\operatorname{mult}_{E_0} B = -a(E_0, X, B) + a(E_0, X, 0) \le N$.

By [12, Theorem 1.1], there exists a positive real number δ, which only depends on Γ and $X \ni x$, such that $a(E, X, B) \ge \operatorname{mld}(X \ni x, B) + \delta$ for any $(X \ni x, B)$ and prime divisor E over $X \ni x$, such that $B \in \Gamma$ and $a(E, X, B) > \operatorname{mld}(X \ni x, B)$.

We may assume that $\Gamma \setminus \{0\} \ne \emptyset$. Let $t := \frac{\delta}{2N}$ and $\tau := t \cdot \min\{\Gamma \setminus \{0\}\}$. We claim that τ has the required properties.

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For any $\mathbb{R}$-divisor B', such that $||B'-B|| < \tau$, we have $B - B' < tB$. Let E be a prime divisor over $X \ni x$, such that $a(E,X,B') = \mathrm{mld}(X \ni x, B')$. Suppose that $a(E,X,B) > \mathrm{mld}(X \ni x, B)$. Then $a(E,X,B) \geq \mathrm{mld}(X \ni x, B) + \delta = a(E_0,X,B) + \delta$. Thus:

$$\begin{aligned} a(E_0,X,0) - \mathrm{mult}_{E_0} B' = a(E_0,X,B') \geq a(E,X,B') \geq a(E,X,B) \\ \geq a(E_0,X,B) + \delta = a(E_0,X,0) - \mathrm{mult}_{E_0} B + \delta. \end{aligned}$$

Hence, $\delta \leq \mathrm{mult}_{E_0}(B-B') \leq t\,\mathrm{mult}_{E_0} B \leq tN = \frac{\delta}{2}$, a contradiction. □

Proposition 6.5. *For any fixed $\mathbb{Q}$-Gorenstein germ $X \ni x$ over an algebraically closed field of characteristic zero, Conjecture 1.1 implies Conjecture 6.2 holds for any DCC set Γ when B has only one component.*

Proof. Otherwise, there exist two sequence of germs $\{(X \ni x, t_iB_i)\}_i$ and $\{(X \ni x, t_i'B_i)\}_i$, and prime divisors E_i over $X \ni x$, such that:

- for each i, B_i is a prime divisor on X, $t_i \in \Gamma$ is increasing and $t_i' < t_i$,
- $t := \lim_{i\to\infty} t_i = \lim_{i\to\infty} t_i'$,
- $a(E_i,X,t_i'B_i) = \mathrm{mld}(X \ni x, t_i'B_i)$ and
- $a(E_i,X,t_iB_i) \neq \mathrm{mld}(X \ni x, t_iB_i)$.

By Proposition 6.3, there exists a positive real number τ, which only depends on t and $X \ni x$, such that if $|t - t_i'| < \tau$, then $a(E_i,X,tB_i) = \mathrm{mld}(X \ni x, tB_i)$. Since $a(E_i,X,t_*B_i)$ is a linear function with respect to the variable t_*, we have $a(E_i,X,t_iB_i) = \mathrm{mld}(X \ni x, t_iB_i)$, a contradiction. □

To end this section, we ask the following:

Conjecture 6.6. *Let $\Gamma \subseteq [0,1]$ be a finite set and $X \ni x$ a klt germ. Then there exists a positive real number τ depending only on Γ and $X \ni x$ satisfying the following.*

Assume that:

(1) $B' \leq B$, $||B-B'|| < \tau, B \in \Gamma$ *and*

(2) $K_X + B'$ *is* $\mathbb{R}$*-Cartier.*

Then $K_X + B$ *is* $\mathbb{R}$*-Cartier.*

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