Similarly $$\triangle AFE = S\cos^2 A$$

and $$\triangle BFD = S\cos^2 B$$

$$\therefore \quad \triangle DEF = S(1 - \cos^2 A - \cos^2 B - \cos^2 C).$$

But $$\frac{FE^2}{CB^2} = \frac{\triangle AFE}{\triangle ABC} = \cos^2 A$$

$\therefore$ $FE = a\cos A$. Similarly $DE = c\cos C$, and angle

$$FED = 180^\circ - 2B.$$

$$\begin{aligned} \therefore \quad \triangle DEF &= \tfrac{1}{2}FE \,.\, ED \sin FED \\ &= \tfrac{1}{2}a\cos A \,.\, c\cos C \,.\, \sin(180^\circ - 2B) \\ &= \tfrac{1}{2}ac\cos A\cos C \,.\, 2\sin B\cos B \\ &= \tfrac{1}{2}ac\sin B \,.\, 2\cos A\cos B\cos C \\ &= S \,.\, 2\cos A\cos B\cos C. \end{aligned}$$

The result follows by equating the two values found for $\triangle DEF$.

A. G. BURGESS.

**Proof of some Triangle Formulae.** — Let $I$ be the incentre of $\triangle ABC$, and let the excentre opposite $A$ be $I_1$. Draw perpendiculars $IF$ and $I_1F_1$ to $AB$. $\angle IBI_1 = 90^\circ$.

$$\therefore \quad \angle FBI = 90^\circ - \angle F_1BI_1 = \angle F_1I_1B.$$

Hence $\triangle FBI$ is similar to $\triangle F_1I_1B$.

$$\therefore \quad \frac{IF}{FB} = \frac{BF_1}{F_1I_1}$$

$$\therefore \quad IF \,.\, F_1I_1 = FB \,.\, BF_1.$$

Again $$\angle AI_1B = \tfrac{1}{2}(180^\circ - B) - \frac{A}{2} = \frac{C}{2}$$

$\therefore$ $\triangle BAI_1$ is similar to $\triangle IAC$.

$$\therefore \quad \frac{AI}{AC} = \frac{AB}{AI_1}$$

$$\therefore \quad AI \,.\, AI_1 = AB \,.\, AC.$$



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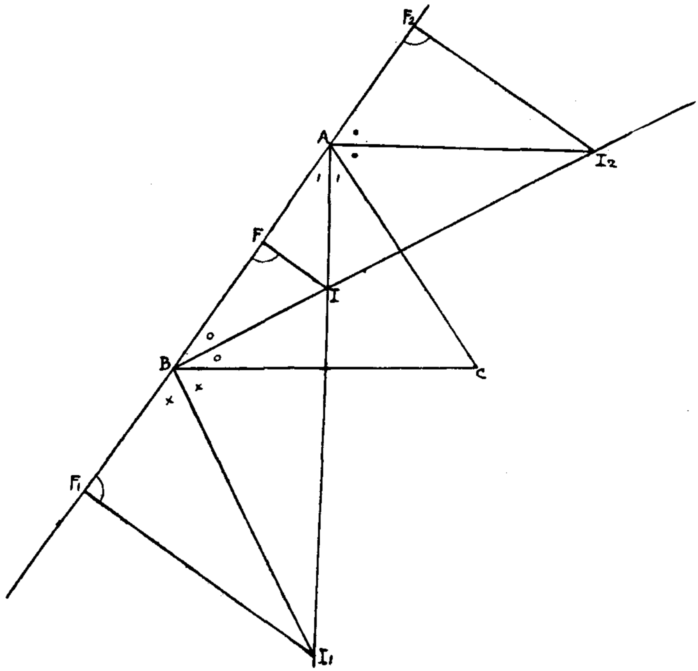


(1) $\tan\frac{A}{2}$.

$$IF \,.\, F_1I_1 = FB \,.\, BF_1$$

$$\therefore \quad AF\tan\frac{A}{2} \,.\, AF_1\tan\frac{A}{2} = (s-b)(s-c)$$

$$\therefore \quad (s-a)\tan\frac{A}{2} \,.\, s\tan\frac{A}{2} = (s-b)(s-c)$$

$$\therefore \quad \tan^2\frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)}.$$

(2) $\cos\frac{A}{2}$.

$$AI \,.\, AI_1 = AC \,.\, AB$$

$$\therefore \quad AF\sec\frac{A}{2} \,.\, AF_1\sec\frac{A}{2} = b\,c$$

$$\therefore \quad (s-a)\sec\frac{A}{2} \,.\, s\sec\frac{A}{2} = b\,c$$

$$\therefore \quad \cos^2\frac{A}{2} = \frac{s(s-a)}{b\,c}.$$



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(3) $\sin\frac{A}{2}$.

$$AI \,.\, AI_1 = AC \,.\, AB$$

$$\therefore \quad IF \operatorname{cosec}\frac{A}{2} \,.\, I_1F_1 \operatorname{cosec}\frac{A}{2} = b\,c$$

$$\therefore \quad FB \,.\, BF_1 \operatorname{cosec}^2\frac{A}{2} = b\,c$$

$$\therefore \quad (s-b)(s-c)\operatorname{cosec}^2\frac{A}{2} = b\,c$$

$$\therefore \quad \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{b\,c}.$$

(4) $\Delta$.

$$\Delta = r\,s = (s-a)\tan\frac{A}{2} \,.\, s$$

$$= s(s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

(5) $\Delta$.

$$IF \,.\, F_1I_1 = FB \,.\, BF_1$$

$$\therefore \quad r\,r_1 = (s-b)(s-c).$$

If $I_2F_2$ be drawn perpendicular to $AB$, $\triangle I_2F_2B$ is similar to $\triangle IFB$ and hence to $\triangle BF_1I_1$.

$$\therefore \quad \frac{I_2F_2}{F_2B} = \frac{BF_1}{F_1I_1}$$

$$\therefore \quad I_2F_2 \,.\, F_1I_1 = F_2B \,.\, BF_1$$

$$\therefore \quad r_2\,r_1 = (s-c)\,s$$

Hence by symmetry $\quad r_2r_3 = s(s-a)$

$$\therefore \quad \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{r\,r_1}{r_2r_3}}$$

and $\quad \Delta = \sqrt{r\,r_1\,r_2\,r_3}\,.$



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(6) By means of these relations many others which occur in Elementary Trigonometry can be proved for the case in which $A$, $B$, $C$ are angles of a triangle. Appended are some examples.

(i) $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$.

$$\Delta = \tfrac{1}{2} AB \,.\, AC \sin A \text{ (proved as usual)}$$
$$= \tfrac{1}{2} AI \,.\, AI_1 \sin A$$

But $$\Delta = rs = IF \,.\, AF_1 = AI \sin \frac{A}{2} \,.\, AI_1 \cos \frac{A}{2}$$

$$\therefore \quad \tfrac{1}{2} \sin A = \sin \frac{A}{2} \cos \frac{A}{2}.$$

(ii) $\cos A = 1 - 2 \sin^2 \frac{A}{2}$.

$$\cos^2 A = 1 - \sin^2 A = 1 - 4 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}$$
$$= 1 - 4 \sin^2 \frac{A}{2} + 4 \sin^4 \frac{A}{2} = \left(1 - 2 \sin^2 \frac{A}{2}\right)^2.$$

(iii) $\cos\left(\frac{B+C}{2}\right) = \cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2}$.

$$\cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2}$$
$$= \sqrt{\frac{s(s-b) \,.\, s(s-c)}{ac \,.\, ab}} - \sqrt{\frac{(s-a)(s-c) \,.\, (s-a)(s-b)}{ac \,.\, ab}}$$
$$= \sqrt{\frac{(s-b)(s-c)}{bc}} \left\{\frac{s-(s-a)}{a}\right\}$$
$$= \sqrt{\frac{(s-b)(s-c)}{bc}} = \sin \frac{A}{2} = \cos\left(\frac{B+C}{2}\right).$$

Similarly for $\sin \frac{B+C}{2}$, $\cos \frac{B-C}{2}$, $\sin \frac{B-C}{2}$, and hence $\tan \frac{B+C}{2}$, $\tan \frac{B-C}{2}$ can be derived.



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(7) The triangle formulae for $\cos A$ and $\tan\dfrac{B-C}{2}$ are derivable in the same way. Also, by using the formulae corresponding to $\tan\dfrac{A}{2}=\sqrt{\dfrac{rr_1}{r_2r_3}}$, $\tan\dfrac{B-C}{2}$ can be shown equal to $\dfrac{r_2-r_3}{r_1+r}\tan\dfrac{A}{2}$.

As a final example,

$$\Sigma\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)=\Sigma\sqrt{\frac{(s-a)(s-c)}{s(s-b)}\cdot\frac{(s-a)(s-b)}{s(s-c)}}=\Sigma\left(\frac{s-a}{s}\right)=1$$

or $$\Sigma\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)=\Sigma\sqrt{\frac{r\,r_2}{r_1r_3}\cdot\frac{r\,r_3}{r_1r_2}}=\Sigma\left(\frac{r}{r_1}\right)$$

$$\therefore\quad\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}.$$

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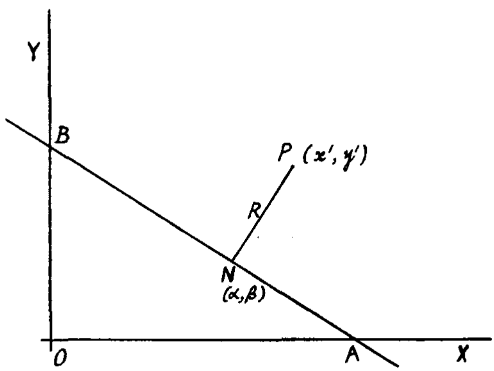
**The Distance of a given Point from a given Line.**—If $P(x', y')$ is the given point and $ax+by+c=0$ the equation of the given line, the expression for the distance of $P$ from the line and the coordinates of the foot of the perpendicular from $P$ to the line can be obtained by projections as follows:—



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