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A THEOREM FOR ENUMERATING CERTAIN TYPES OF COLLECTIONS

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Introduction. In this paper, we are concerned with proving a formula for the computation of what is variously called the pattern inventory (e.g., see De Bruijn [**2**]) or the configuration counting series (e.g., see Harary [**3**]). Rather than redeveloping a large number of definitions, we shall assume the reader is already familiar with the terminology used by De Bruijn [**2**].

Polya, in a celebrated paper [**4**], proved a formula for computing the pattern inventory for all functions f defined on a set D (where D is acted on by a permutation group G), and mapping into a set R (which is called the store) for which the "store enumerator" in known. Polya's result assumes that the weight of one of these functions is given by $\sum_{d \in D} w(f(d))$ where w is the weight defined on the objects in the store.

Polya's basic theorem has been adapted and extended in many ways. For example, Polya himself [**4**] describes a formula for computing the pattern inventory for all one-to-one functions defined on a set D (acted upon by a permutation group G), and mapping into R. De Bruijn [**1**] has proved a theorem specifying a formula for computing the pattern inventory for all functions from D into R where D is acted on by the permutation group G, and R is acted upon by the permutation group H.

In this paper we present another "Polya like" result. We assume that D is $\{1, 2, \ldots, t\}$, and consider the class of all one-to-one functions defined on D and mapping into a store R, for which the inventory, or store enumerator, is known. We hypothesize a t-tuple $(n_1, n_2, \ldots, n_t)$, and use it to define the weight of a function to be $\sum_{i=1}^{t} n_i w(f(i))$ (where w is the weight defined on the objects in the store). We stipulate that G, the permutation group defined on $\{1, 2, \ldots, t\}$, be isomorphic to $S_{c_1} \times S_{c_2} \times \ldots \times S_{c_M}$ where $c_i = |\{r | n_r = i\}|$, $M = \max(n_1, n_2, \ldots, n_t)$ and S_k is the symmetric group on k letters. Based on these hypotheses, we prove a formula for computing the pattern inventory for this class of functions. This pattern inventory proves useful in solving an interesting class of enumeration problems.

As an example of the type of problem which can be solved using this result, consider the following. Suppose R, the store, is literally a retail store (or more precisely the collection of items sold in a particular retail store). Let $C(x) = \sum_{i=1}^{\infty} c_i x^i$ be the store inventory, defined such that c_i is the number of items in

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the store priced at i cents per unit. A one-to-one function from $\{1, 2, \ldots, t\}$ into R would thus correspond to a shopper's selection of t distinct items. Specifying the t-tuple $(n_1, n_2, \ldots, n_t)$ and using it to compute the weight of such a function, we see that this weight must correspond to the cost of buying n_1 units of item 1, n_2 units of item 2, ... Hence the pattern inventory for this class of functions tells us the number of ways in which the cost of buying n_1 units of one item, n_2 units of a different item, . . . , n_t units of a tth item, will come to i cents, for all values of i.

1. In order to state our theorem and some subsequent results precisely, we develop some notation and definitions of our own.

1.1. *Definition.* Let $p(x)$ be a polynomial in x. The notation $s_i{}^j[p(x)]$ is defined by $s_i{}^j[p(x)] = p^j(x^i)$, and is referred to as the *composition* of $s_i{}^j$ over $p(x)$. Moreover, the composition is extended to polynomials in the variables $s_1, s_2, s_3, \ldots$ with rational coefficients by requiring it to be left distributive over addition and multiplication. Thus

$$\begin{aligned}(as_{i_1}{}^{j_1} + bs_{i_2}{}^{j_2}s_{i_3}{}^{j_3})[p(x)] &= as_{i_1}{}^{j_1}[p(x)] + bs_{i_2}{}^{j_2}[p(x)]s_{i_3}{}^{j_3}[p(x)] \\ &= ap^{j_1}(x^{i_1}) + bp^{j_2}(x^{i_2})p^{j_3}(x^{i_3}).\end{aligned}$$

In general we shall be interested in performing compositions over counting polynomials. Hence we now carefully define what a counting polynomial is.

1.2. *Definition.* A *weight function*, $w : C \to \mathbf{Z}^+$, is a function mapping C, a class of objects, into $\mathbf{Z}^+$, the positive integers, such that

$$j_z = \{c \in C | w(c) = z\}$$

if finite, for all $z \in \mathbf{Z}^+$. If $c \in C$, then we often refer to $w(c)$ as the *weight of* c.

1.3. *Definition.* A weighted class K, is defined to be a class of objects C, together with a weight function on C.

Thus we see that a store is an example of a weighted class.

1.4. *Definition.* Let K be a weighted class with w its weight function; $p(x)$, the *counting polynomial for* K, is defined by $p(x) = \sum_{j=1}^{\infty} a_j x^j$, where

$$a_j = |\{c \in C | w(c) = j\}|.$$

1.5. *Definition.* Let K be the weighted class consisting of C, a store, along with its weight function. Suppose $t \in \mathbf{Z}^+$. We shall call f a *t-function into* K if f is one-to-one and maps $\{1, 2, \ldots, t\}$ into C.

Our goal is to compute the pattern inventory for the t-functions into K. We define the weight of f, a t-function into K, as follows. Let $(n_1, n_2, \ldots, n_t)$ be a finite sequence of positive integers. Then $W(f)$, the weight of f, is defined by $W(f) = \sum_{i=1}^{t} n_i w(f(i))$.

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We now need one more definition.

1.6. *Definition.* Let K be a weighted class with weight function w, and let $(n_1, n_2, \ldots, n_t)$ be a t-tuple of non-negative integers. The *class of functions into K of type* $(n_1, n_2, \ldots, n_t)$ is defined to be a weighted class whose objects are all the t-functions into K, and whose weight function is given by

$$W(f) = \sum_{i=1}^{t} n_i w\{f(i)\}.$$

2. We can now state our first enumeration theorem:

2.1. THEOREM. *Let K be a weighted class whose counting polynomial is $p(x)$. If we denote the function inventory for the class of functions into K of type $(n_1, n_2, \ldots, n_t)$ by $Q(n_1, n_2, \ldots, n_t)$, then*

$$Q(n_1, n_2, \ldots n_t) = \left[\sum_{P\in\phi(t)} \prod_{B\in\beta(P)} (-1)^{|B|-1}(|B|-1)!s_{\Delta_B}\right][p(x)],$$

where $\phi(t)$ denotes the set of all partitions of $\{1, 2, \ldots, t\}$, $\beta(P)$ the set of all blocks of any such partition, and $\Delta_B = \sum_{i\in B} n_i$.

Proof. In this proof we shall talk about functions into K of type $(n_1, \ldots, n_t)$. Technically, these are just t-functions; intuitively, they represent possible ways of choosing an ordered sequence of t distinct objects from K, then repeating the ith one n_i times. In this proof we refer, unambiguously, to functions into K of type $(n_1, n_2, \ldots, n_t)$ as functions of type $(n_1, n_2, \ldots, n_t)$.

The proof is by induction on t. Assume $t = 1$. We wish to determine $Q(n)$, the function inventory for functions f of type (n), where the store enumerator for the store C is given by $p(x)$. There is a clear one-to-one correspondence between the objects in C and all possible functions of type (n). For if $c \in C$, then we associate the function of type (n) for which $f(1) = c$. For this function, $W(f) = \sum_{i=1}^{1} n_i w(f(i)) = nw(c)$. Hence it is clear that $Q(n) = p(x^n)$.

According to the statement of the theorem,

$$Q(n) = (-1)^{1-1}(1-1)!s_n[p(x)] = 1 \cdot s_n[p(x)] = p(x^n).$$

Hence the assertion is true for $t = 1$.

Now assume the assertion is correct for $t - 1$. We wish to verify it for t.

We recall that if f is a function of type $(n_1, n_2, \ldots, n_t)$, then it must be one-to-one. If this is so, then:

$$(2.1.1) \quad Q(n_1, n_2, \ldots, n_t) = s_{n_1}s_{n_2}\ldots s_{n_t}[p(x)] - \sum_{P\in\psi(t)} Q\left(\sum_{c1,i\in C_1} n_{c1,i}, \sum_{c2,i\in C_2} n_{c2,i}, \ldots, \sum_{ck,i\in C_k} n_{ck,i}\right)$$

where $\psi(t)$ denotes the set of all partitions $\{C_1, C_2, \ldots, C_u, \ldots, C_k\}$ of $\{1, 2, \ldots, t\}$ which are not the discrete partition $\{1\}, \{2\}, \ldots, \{t\}$.

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Each term of the summation computes the weight of a different non-one-to-one function mapping $\{1, 2, \ldots, t\}$ into the objects enumerated by $p(x)$. The $s_{n_1}s_{n_2} \ldots s_{n_t}[p(x)]$ term computes the sum of the weights of all functions from $\{1, 2, \ldots, t\}$ into the objects enumerated by $p(x)$. Thus, since the summation includes all such non-one-to-one functions, the expression indeed represents the effects of all one-to-one functions from $\{1, \ldots, t\}$ into C, and hence computes the sum of the weights of all functions of type $(n_1, \ldots, n_t)$. Since no P of the summation in (2.1.1) is the discrete one, $k \leqq t - 1$. Hence by our inductive assumption, the assertion of the theorem is true for the summands, and we can write:

$$(2.1.2) \quad Q\left(\sum_{c_1, i \in C_1} n_{c_1, i}, \sum_{c_2, i \in C_2} n_{c_2, i}, \ldots, \sum_{c_k, i \in C_k} n_{c_k, i}\right) = \sum_{R \in \theta(C_1, \ldots, C_k)} \prod_{B \in \beta(R)} (-1)^{|B|-1}(|B| - 1)! s_{\Gamma_B}[p(x)],$$

where $\theta(C_1, C_2, \ldots, C_k)$ denotes the set of all partitions $\{B_1, \ldots, B_e\}$ of $\{C_1, \ldots, C_k\}$, $\beta(R)$ the set of all blocks of any such partition, and

$$\Gamma_B = \sum_{C_i \in B} \sum_{j \in C_i} n_{c_i, j}.$$

Using (2.1.2) we can rewrite (2.1.1) as:

$$(2.1.3) \quad Q(n_1, n_2, \ldots, n_t) = \left(s_{n_1}s_{n_2} \ldots s_{n_t} - \sum_{P \in \phi(t)} \sum_{R \in \theta(C_1, C_2, \ldots, C_k)} \prod_{B \in \beta(R)} (-1)^{|B|-1}(|B| - 1)! s_{\Gamma_B}\right) [p(x)].$$

We must now verify that the coefficients of all terms to the right of the equal sign equal coefficients yielded by the assertion of the theorem.

The theorem asserts that the coefficient of $s_{n_1}s_{n_2} \ldots s_{n_t}$ is $\prod_{i=1}^{t} (-1)^0(0)! = 1^t = 1$. We note that on the right of the equality $s_{n_1}s_{n_2} \ldots s_{n_t}$ appears only once, as the result of the discrete partition, and has coefficient 1. So the equality holds for the $s_{n_1}s_{n_2} \ldots s_{n_t}$ term.

We now show that the equality also holds for the $s_{n_1+n_2+\ldots+n_t}$ term. The assertion of the theorem is that the $s_{n_1+n_2+\ldots+n_t}$ term of the polynomial $Q(n_1, n_2, \ldots, n_t)$ arises only from the partition C_1 (where $C_1 = \{1, 2, \ldots, t\}$). Hence we are interested in the value:

$$Q(n_1 + n_2 + \ldots + n_t) = (-1)^{t-1}(t - 1)! s_{n_1+\ldots+n_t}[p(x)].$$

We must thus verify that the coefficient of the $s_{n_1+n_2+\ldots+n_t}[p(x)]$ term yielded by the summation on the right of the equal sign in (2.1.3) is $(-1)^{t-1}(t - 1)!$

Clearly the only sources of $s_{n_1+n_2+\ldots+n_t}[p(x)]$ terms in (2.1.3) result from the cases where $R = \{B_1\}$. Hence, any non-discrete partition $P = \{C_1, \ldots, C_k\}$ of $\{1, 2, \ldots, t\}$ followed by the partition $R = \{B_1\}$ of the $\{C_1, C_2, \ldots, C_k\}$

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will yield an $s_{n_1+n_2+\ldots+n_t}$ term. Hence the sum of the coefficients of these terms is:

$$\sum_{k=1}^{t-1} \sum_{\nu(t,k)} (-1)^{k-1}(k-1)!$$

where $\nu(t, k)$ denotes all the ways of dividing t objects into k non-void subsets.

We recognize that the number of ways of partitioning t objects into k non-void subsets is given by $S_2(t, k)$, the Stirling number of type 2. Moreover, $(k-1)! = S_1(k, 1)$, the Stirling number of type 1. Hence the coefficient of $s_{n_1+n_2+\ldots n_t}[p(x)]$ yielded by the right side of (2.1.3) is now seen to be

$$-\sum_{k=1}^{t-1} (-1)^{k-1}S_2(t, k)S_1(k, 1).$$

A famous combinatorial identity (see, for example, [**5**]) is

$$\sum_{k=1}^{t} (-1)^{k-1}S_2(t, k)S_1(k, 1) = \begin{cases} 0, \text{ if } t > 1 \\ 1, \text{ if } t = 1. \end{cases}$$

We began the proof by showing that the theorem is true for $t = 1$. Hence we need only consider the case where $t > 1$. In this case

$$-\sum_{k=1}^{t-1} (-1)^{k-1}S_2(t, k)S_1(k, 1) = -[0 - (-1)^{t-1}S_2(t, t)S_1(t, 1)] = (-1)^{t-1}(t-1)!$$

as required.

Let us now consider whether the equality in (2.1.3) holds for the general term

(2.1.4) $$s_{\Omega_1} s_{\Omega_2} \ldots s_{\Omega_f}$$

where $\Omega_j = \sum_{g_j,i \in G_j} n_{g_j,i}$ and the G_j are disjoint with $\bigcup_{j=1}^{f} G_j = \{1, 2, \ldots, t\}$.

Clearly terms such as (2.1.4) can arise only from those terms of the summation on the right side of (2.1.3) for which $R = \{B_1, B_2, \ldots, B_f\}$, where each $B_i = \{A_{i1}, A_{i2}, \ldots, A_{iz}\}$ and the A_{ij} are such that $\bigcup_{j=1}^{z} A_{ij} = G_i$.

The sum of the coefficients of all terms described above can be seen, by reasoning analogous to the above, to be

(2.1.5) $$-\sum_{k_1=1}^{|G_1|} \sum_{k_2=1}^{|G_2|} \ldots \sum_{k_f=1}^{|G_f|} ((-1)^{k_1-1}S_2(|G_1|, k_1)S_1(k_1, 1) \ldots (-1)^{k_f-1}S_2(|G_f|, k_f)S_1(k_f, 1)) + (-1)^{|G_1|-1}S_2(|G_1|, |G_1|)S_1(|G_1|, 1) \ldots (-1)^{|G_f|}S_2(|G_f|, |G_f|)S_1(|G_f|, 1).$$

Note that the f-fold summation in (2.1.5) takes into account all partitions, *including* the discrete partition, of $\{1, \ldots t\}$ which are subsequently subpartitioned into $\{B_1, B_2, \ldots, B_f\}$. The additive term is included to exclude the effect of the $\{B_1, B_2, \ldots, B_f\}$ partition of the discrete partition of $\{1, 2, \ldots t\}$.

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We rewrite (2.1.5) as

$$(2.1.6)\quad -\sum_{k_1=1}^{|G_1|}(-1)^{k_1-1}S_2(|G_1|,k_1)S_1(k_1,1)\ldots\sum_{k_f=1}^{|G_f|}(-1)^{k_f-1}S_2(|G_f|,k_f)S_1(k_f,1)$$
$$+(-1)^{|G_1|-1}S_2(|G_1|,|G_1|)S_1(|G_1|,1)\ldots(-1)^{|G_f|-1}S_2(|G_f|,|G_f|)S_1(|G_f|,1)$$
$$= -\delta(|G_1|,1)\ldots\delta(|G_f|,1)+(-1)^{|G_1|-1}(|G_1|-1)!\ldots(-1)^{|G_f|-1}(|G_f|-1)!$$

where $\delta(i,j)$ is the "Kronecker Delta" defined by $\delta(i,j)=\begin{cases}0 \text{ if } i\neq j\\ 1 \text{ if } i=j.\end{cases}$

We need only consider the case where $|G_i|>1$ for some $1\leqq i\leqq f$, for if $|G_i|=1$ for all i, $1\leqq i\leqq f$, then $\{G_1,G_2,\ldots,G_f\}$ would represent the discrete partition of $\{1,2,\ldots,t\}$, and we would be considering the term $s_{n_1}\ldots s_{n_t}$, for which the equality in (2.1.3) was already verified. Hence, assuming $|G_i|>1$ for some i, $1\leqq i\leqq f$, then $\prod_{i=1}^{f}\delta(|G_1|,1)=0$, and the sum of the coefficients of all terms such as described in (2.1.4) which arise from the summation on the right side of (2.1.3) is seen to be

$$\sum_{i=1}^{f}(-1)^{|G_1|-1}(|G_1|-1)!.$$

This is seen to be the assertion made by the theorem. Hence (2.1.3) is verified to be an equality. The coefficient of each term in the righthand summation is equal to the coefficient yielded by applying the assertion of the theorem. Hence the induction has been verified.

Thus we have now established a formula for the computation of the function inventory for the class of functions into K of type $(n_1,n_2,\ldots,n_t)$. We seek a formula for the pattern inventory.

The pattern inventory is an enumeration of the equivalence classes of the functions enumerated by the function inventory. Hence a pattern inventory is always relative to a particular equivalence relation on the functions. It is usual for the equivalence relation to be induced by a permutation group on the elements of the domain D in the following way.

Let G be a permutation group acting on the elements of D. Let f_1,f_2 be two functions defined on D. We say that f_1 is equivalent to f_2 provided that there exists $g\in G$ such that $f_1(g(d))=f_2(d)$ for all $d\in D$.

Hence in our case it would seem that we could allow G to be any permutation group on t symbols, and compute the pattern inventory for the induced equivalence classes of functions. A problem arises, however, due to our choice of weight for the functions of type $(n_1,n_2,\ldots,n_t)$ into K. If we allow G to

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be any group of t-permutations, then it is conceivable that equivalent functions might have different weights. It is difficult to see what meaning the pattern inventory might have under such circumstances.

For this reason, we shall take G to be $S_{c1} \times S_{c2} \times \ldots \times S_{c_M}$ the product of the symmetric groups on $c_1, c_2, \ldots, c_M$ elements, where the c_i are defined by

$$c_i = |\{r|n_r = i\}|, \quad \text{and} \quad M = \max\{n_1, n_2, \ldots, n_t\}.$$

Thus two functions cannot be equivalent unless their weights are equivalent. We shall now compute the pattern inventory of the $(n_1, n_2, \ldots, n_t)$ functions into K relative to the permutation group $S_{c1} \times S_{c2} \times S_{c_M}$. We suggest what these patterns intuitively represent by the following definition.

2.2. *Definition*. A *collection of K of type* $(n_1, n_2, \ldots, n_t)$ is an equivalence class of functions into K of type $(n_1, n_2, \ldots, n_t)$, where the equivalence relation on the functions is induced by the permutation group $S_{c1} \times S_{c2} \times \ldots \times S_{c_M}$ acting on $\{1, 2, \ldots, t\}$.

2.3. THEOREM. *Let $N(n_1, n_2, \ldots, n_t)$ be the counting polynomial for the collections of K of type $(n_1, n_2, \ldots, n_t)$. Then*

$$N(n_1, n_2, \ldots, n_t) = (c_1!c_2! \ldots c_M!)^{-1}Q(n_1, n_2, \ldots, n_t)$$

where $c_i = |\{r|n_r = i\}|$, $M = \max(n_1, n_2, \ldots, n_t)$.

Proof. The proof easily follows from the observation that each equivalence class must consist of exactly $c_1!c_2! \ldots c_M!$ functions.

3. A fairly immediate and straightforward application of 2.3 is the following:

A tree is commonly defined to be a connected graph containing no circuits. The term forest is often used to denote a collection of trees. We now define a forest of type $(n_1, n_2, \ldots, n_t)$, or more simply a $(n_1, n_2, \ldots, n_t)$-forest, to be a forest consisting of n_1 isomorphic copies of some tree, n_2 isomorphic copies of a different tree, ..., and n_t isomorphic copies of still another tree.

The application of this theorem to the problem of enumerating $(n_1, \ldots, n_t)$-forests is clear. We take C to be the collection of all trees, and w to be the function which assigns to each tree the number of points it contains; $p(x)$ is taken to be the corresponding counting polynomial for trees.

This counting polynomial for trees is known to be given by

$$t(x) = x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + \ldots .$$

According to 2.3, the counting polynomial for $(n_1, n_2, \ldots, n_t)$-forests is given by $N(n_1, n_2, \ldots, n_t)$.

Hence, as an illustration, let us enumerate (1, 1, 2)-forests:

$$\begin{aligned} N(1, 1, 2) &= (2!1!0!)^{-1}((-1)^2 2!t(x^4) + \{(-1)^1(1)!(-1)^0(0)!\}\{t(x^3)t(x) \\ &\quad + t^2(x^2) + t(x)t(x^3)\} + \{(-1)^0(0)!\}^3 t(x^2)t^2(x)) \\ &= \tfrac{1}{2}(2t(x^4) - 2t(x^3)t(x) - t^2(x^2) + t(x^2)t^2(x)). \end{aligned}$$

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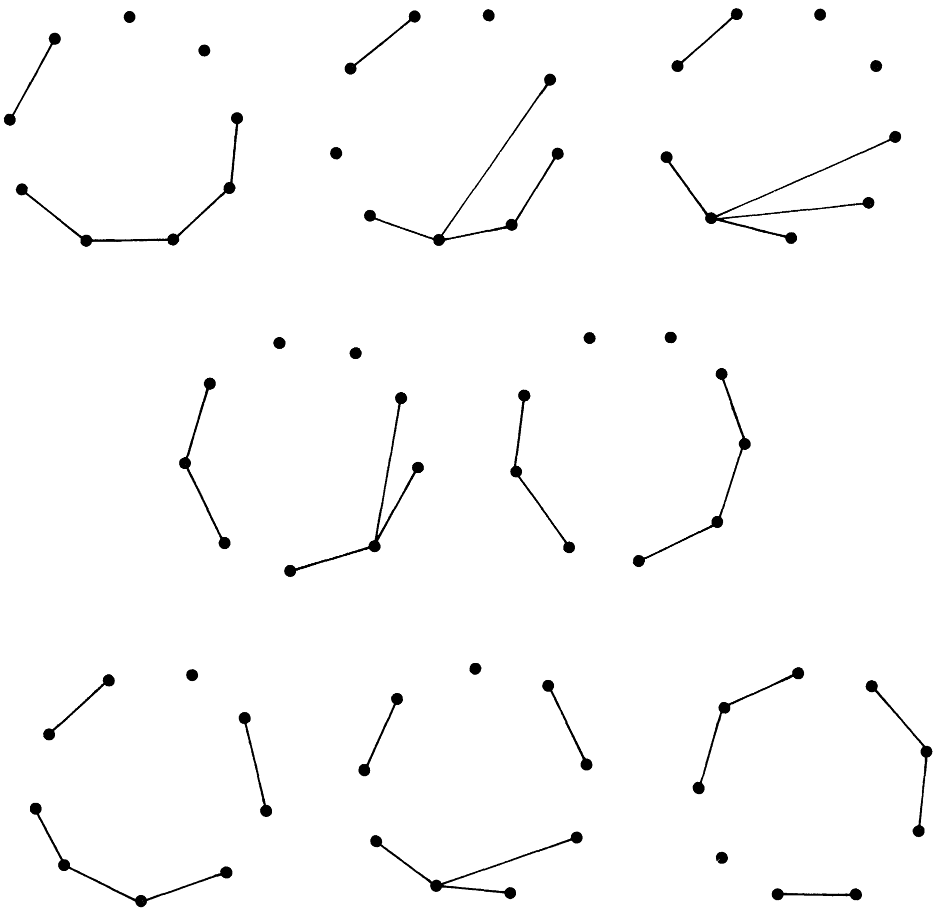
We compute:

$$\begin{aligned} t(x) &= x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + \dots . \\ t(x^2) &= x^2 + x^4 + x^6 + \dots . \\ t(x^3) &= x^3 + x^6 + \dots . \\ t(x^4) &= x^4 + \dots . \\ t(x^3)t(x) &= x^4 + x^5 + x^6 + 3x^7 + 4x^8 + 7x^9 + \dots . \\ t^2(x^2) &= x^4 + 2x^6 + 3x^8 + \dots . \\ t(x^2)t^2(x) &= x^4 + 2x^5 + 4x^6 + 8x^7 + 15x^8 + 30x^9 + \dots . \end{aligned}$$

Hence:

$$\begin{aligned} N(1, 1, 2) &= \tfrac{1}{2}(2x^7 + 6x^8 + 16x^9 + \dots .) \\ &= x^7 + 3x^8 + 8x^9 + \dots . \end{aligned}$$

As a check, we show the eight (1, 1, 2)-forests on nine points:



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