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STRUCTURE OF PSEUDO-SEMISIMPLE RINGS

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Abstract

A ring R is called right pseudo-semisimple if every right ideal not isomorphic to R is semisimple. Rings of this type in which the right socle S splits off additively were characterized; such a ring has $S^2 = 0$. The existence of right pseudo-semisimple rings with zero right singular ideal Z remained open, except for the trivial examples of semisimple rings and principal right ideal domains. In this work we give a complete characterization of right pseudo-semisimple rings with $S^2 = 0$. We also give examples of non-trivial right pseudo-semisimple rings with $Z = 0$; in fact it is shown that such rings exist as subrings in every infinite-dimensional full linear ring. A structure theorem for non-singular right pseudo-semisimple rings, with homogeneous maximal socle, is given. The general case is still open.

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Throughout this paper, S, Z and J will stand for the right socle, the right singular ideal and the Jacobson radical of a ring R. A *local ring* R will mean one in which $J \neq 0$ and R/J is a division ring. For a subset X of R, X^0 and ${}^0X$ will denote the right and left annihilators in R. It is true in general that $S \leq {}^0J$, and if R/J is semisimple (in particular if R is local), then $S = {}^0J$. We also note that Z and J contain no non-zero idempotents of R; hence a regular ring R has $Z = J = 0$.

The *split extension* $R \rtimes M$ of a ring R by an $(R-R)$-bimodule M, is the ring of all ordered pairs (r, m), $r \in R$ and $m \in M$; with addition defined componentwise and multiplication defined by $(r, m) \cdot (r', m') = (rr', rm' + mr')$.

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The set of positive integers will be denoted by $\mathbb{N}$.

1. General results

LEMMA 1.1. *Let R be a right pseudo-semisimple ring. If $R = A \oplus B$ for right ideals A and B, then A or B is semisimple.*

PROOF. Assume that A is not semisimple; then $A \cong R$. Hence $A = A_1 \oplus B_1$ with $A_1 \cong A \cong R$ and $B_1 \cong B$. Iterating this process, we obtain $R = A_n \oplus B_n \oplus \cdots \oplus B_1$ for every $n \in \mathbb{N}$. Hence R contains the right ideal $\bigoplus_{i \in \mathbb{N}} B_i$ with $B_i \cong B$. This right ideal is not finitely generated, and therefore not isomorphic to R. Thus it is semisimple, and hence B is semisimple.

COROLLARY 1.2. *Let R be a right pseudo-semisimple ring which is not semisimple.*

(1) *If $R = A \oplus B$ for non-zero right ideals A and B, then exactly one of them is semisimple and the other one is isomorphic to R. In particular neither A nor B is an ideal.*

(2) *If e is a non-trivial idempotent of R, then $eR(1-e) \neq 0$.*

LEMMA 1.3. *If R is right pseudo-semisimple, then R/S is a principal right ideal domain.*

PROOF. It is obvious that R/S is a principal right ideal ring. Consider $a, b \in R$ with $a \notin S$ and $ab \in S$. Then $aR \cong R$, and hence $R = a^0 \oplus C$ where $C \cong aR \cong R$. Thus $a^0 \leq S$ by Lemma 1.1. Write $b = x + y$ with $x \in a^0$ and $y \in C$. Since $S \geq abR = ayR \cong yR$, we have $y \in S$. Therefore $b = x + y \in S$.

It follows by Lemma 1.3 that a ring R with zero right socle is right pseudo-semisimple if and only if it is a principal right ideal domain. We call a right pseudo-semisimple ring R *non-trivial* if $0 \neq S \neq R$.

LEMMA 1.4. *Let R be a non-trivial right pseudo-semisimple ring. The following hold in R:*

(1) *S is the smallest essential right ideal;*

(2) ${}^0S = Z \leq S \cap J$;

(3) *$S = Z \oplus I$ where Z and I consist of homogeneous components (hence I is also an ideal);*

(4) *$S = {}^0x$ for every $0 \neq x \in J$, in particular if $J \neq 0$, then $S = {}^0J$;*

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(5) *if* $a \notin S$, *then* $aR \cap S = aS$ *and* $S = aS \oplus K$ *where* K *is isomorphic to a direct summand of* R (*hence finitely generated*);

(6) $SZ = 0$ *and* Z *is torsion-free divisible as a left* R/S*-module.*

PROOF. (1) Consider a non-zero right ideal A of R. Then either $A \le S$ or $A \cong R$, and hence A contains a copy of S. In either case $A \cap S \neq 0$.

(2) $Z = {}^0S$ follows by (1). This also proves that $Z \not\cong R$ and hence $Z \le S$. Then $Z^2 \le ZS = 0$, and consequently $Z \le J$.

(3) If X and Y are minimal right ideals with $X \le Z$ and $Y \nleq Z$, then $XS = 0$ and $YS \neq 0$. Thus $X \not\cong Y$.

(4) That $S \le {}^0J \le {}^0x$ is obvious. Consider an element $a \in R$ such that $a \notin S$. Then $a^0 \le S$ and is a direct summand of R (since $aR \cong R$). Therefore $a^0 \cap J = 0$ and hence $ax \neq 0$. This proves that ${}^0x \le S$, and consequently $S = {}^0J = {}^0x$.

(5) That $aS = aR \cap S$ follows by Lemma 1.3. Let $S = aS \oplus K$. Then $R \cong aR + S = aR \oplus K$.

(6) The result is trivial in the case $Z = 0$. Assume that $Z \neq 0$. Since $Z \le J$, $S = {}^0Z$ by (4). Hence Z is a torsion-free R/S-module.

Next we prove that $Z \le A$ for every right ideal A not contained in S. Write $Z = (A \cap Z) \oplus B$. Then $A \oplus B \cong R$. Since $B \le Z$, $B^2 = 0$ by (2), and hence $B = 0$. Thus $Z = A \cap Z$ and $Z \le A$.

Now consider an element $a \in R$ such that $a \notin S$. Since $aR \cong R$, $aR = bR$ with $b^0 = 0$. If $br \in Z$, then $brS = 0$, and hence $rS = 0$ and $r \in Z$ by (2). Therefore $bR \cap Z = bZ$. Now

$$aZ = aRZ = bRZ = bZ = bR \cap Z = Z.$$

Hence Z is divisible as a left R/S-module.

LEMMA 1.5. *Let* R *be a non-trivial right pseudo-semisimple ring.*

(1) $Z = S$ *if and only if* $S^2 = 0$ *and* $S \le J \le S^0$.

(2) $Z \neq S$ *if and only if* S *contains a countable set of non-zero orthogonal idempotents and* $S^0 = J < S$.

PROOF. (1) is obvious.

(2) Assume that $Z \neq S$. Then $S^2 \neq 0$, and hence S contains a non-zero idempotent. By an argument similar to that given in Lemma 1.1, we conclude that S contains a countable set of non-zero orthogonal idempotents $\{e_n\}$.

Write $S = (J \cap S) \oplus X$. It is clear that the projections of the e_n into X are still non-zero orthogonal idempotents, and hence X is not finitely generated. It then follows by Lemma 1.4(5) that $J \not\cong R$. Thus $J \le S$, and since $S^2 \neq 0$, $J < S$.

Let $B = S^0$. Then $J \leq B$ and so $J \leq B \cap S$. Moreover $(B \cap S)^2 = 0$, and hence $B \cap S \leq J$. Thus $B \cap S = J$ and $S = (B \cap S) \oplus X$. Again by Lemma 1.4(5) we get $B \not\leq R$ and hence $B \leq S$. Therefore $B = B \cap S = J$.

The converse is obvious.

COROLLARY 1.6. *If R is a right pseudo-semisimple ring, then the right socle of R contains the left socle.*

PROOF. The result is obvious in the trivial cases. So assume that R is non-trivial and consider the two cases of Lemma 1.5.

If $Z = S$, study a minimal left ideal A. Assume that $A^2 \neq 0$. Then $A = Re$ for some idempotent e such that $0 \neq e \neq 1$. However, by Lemma 1.1, either $e \in S$ or $1 - e \in S$, in contradiction to $S^2 = 0$. Hence $A^2 = 0$, and $A \leq S$ by Lemma 1.3.

If $Z \neq S$, then $S^0 < S$ by Lemma 1.5. It follows readily that S is essential as a left ideal, and therefore contains the left socle.

The following generalization of our theorem in [3] characterizes right pseudo-semisimple rings with socle square zero.

THEOREM 1.7. *Let R be a ring with $S^2 = 0$. Then R is right pseudo-semisimple if and only if R/S is a principal right ideal domain and S is torsion-free divisible as a left R/S-module.*

PROOF. The 'only if' part follows from Lemma 1.3 and Lemma 1.4 ((2) and (6)).

Conversely, assume that R satisfies the conditions. Consider a right ideal A of R which is not contained in S, and select $x \in A - S$. Since ${}_{R/S}S$ is divisible, $S = xS \leq A$. Now R/S is a principal right ideal domain implies $A = aR$ with $a^0 \leq S$. But then $a^0 = 0$ as ${}_{R/S}S$ is torsion-free. Hence $A = aR \cong R$.

According to Lemma 1.5, a non-trivial right pseudo-semisimple ring with $S^2 = 0$ satisfies $0 \neq S \leq J \leq S^0$. We list examples of the four possible cases.

EXAMPLES 1.8. (1) $S = J = S^0$: any local ring R with $J^2 = 0$.

(2) $S = J < S^0$: $R = F[X] \rtimes F(X)$, the split extension of the polynomial ring over a field F by the rational function field, made into an $F[X]$-bimodule via the natural multiplication on the left and multiplication by the constant coefficient on the right; compare [3].

(3) $S < J = S^0$: the localization of (2) at $\langle X \rangle$.

(4) $S < J < S^0$: the localization of (2) at $\langle X \rangle \cap \langle X + 1 \rangle$.

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THEOREM 1.9. *Let R be a ring with $S \neq 0$ and $S^2 = 0$. Then R is right and left pseudo-semisimple if and only if R is a local ring with radical square zero.*

PROOF. The 'if' part is obvious. Conversely, assume that R is right and left pseudo-semisimple. Then the left-right analogue of Corollary 1.6 implies that S is the left socle of R. Consider a minimal left ideal A. Since $A^2 = 0$, $A = Rx$ with $x \in J$, and $A \cong R/{}^0x$. Since $S = {}^0x$ by Lemma 1.4(4), S is a maximal left ideal. Then $S = J$ as $S \leq J$ and the result follows.

PROPOSITION 1.10. *Let R be a non-trivial right pseudo-semisimple ring. Then R/Z is right pseudo-semisimple with $Z(R/Z) = 0$. Moreover R/Z is semisimple if and only if R is a local ring with radical square zero.*

PROOF. Let T/Z be the right socle of R/Z. Then it is obvious that $S \leq T$. Moreover T/S is contained in the right socle of R/S. Since R/S is a domain by Lemma 1.3, $T = S$ or $T = R$.

(i) Consider the case $T = S$. Let A/Z be a right ideal of R/Z such that $A/Z \nleq S/Z$. Then $A \nleq S$ and hence $A = aR$ for some $a \in R$ with $a^0 = 0$. Since $Z = aZ$ by Lemma 1.4(6), we obtain

$$R/Z \cong aR/aZ = A/Z.$$

Thus R/Z is right pseudo-semisimple.

Next we prove that $Z(R/Z) = 0$. If $Z = S$, then R/Z is a domain by Lemma 1.3, and the result holds trivially. So, assume that $Z \neq S$. Then by Lemma 1.4(3), $S = Z \oplus I$ for a non-zero ideal I of R. Consider $x + Z \in Z(R/Z)$. Then $xS \leq Z$, and hence $xI \leq Z \cap I = 0$. Consequently $xR \ncong R$, and therefore $x \in S$. Then $xZ = 0$ by Lemma 1.4(6), and $xS = x(Z \oplus I) = 0$. Thus $x \in Z$.

(ii) Now, assume that $T = R$. Then R/Z is semisimple, and clearly $Z(R/Z) = 0$. We claim that Z is a maximal ideal. Let u be a central idempotent in R/Z. Since Z is a nil ideal by Lemma 1.4(2), we may assume that $u = e + Z$ for some idempotent $e \in R$. According to Lemma 1.1, $e \in S$ or $(1 - e) \in S$; we may assume that $e \in S$. Then $Z \cap eR = 0$. Since $eR(1 - e) \leq Z$, $eR(1 - e) = 0$. It then follows by Corollary 1.2(2) that $e = 0$ or $e = 1$. Thus R/Z has no non-trivial central idempotents, and is therefore simple artinian. This proves our claim. Since $Z \leq S \cap J$ by Lemma 1.4(2), we obtain $S = J = Z$. It then follows by Lemma 1.3 that

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R/J is a division ring. Hence R is a local ring with $J^2 = 0$. The rest is obvious.

2. Maximal socle

We turn to the second type in Lemma 1.5. Here we do not know of an effective criterion for pseudo-simplicity. However, right pseudo-semisimple rings R of this type are characterized in the special case where S is a maximal right ideal. In view of Lemma 1.3, this additional assumption is automatically satisfied if R is regular.

We start by listing some properties of rings R with maximal socle. Note that such rings are local if and only if $S^2 = 0$. The proofs are straightforward, and hence are omitted.

LEMMA 2.1. *Let R be a non-local ring with maximal socle (that is, R/S is a division ring and $S^2 \neq 0$). Then R has the following properties:*

(1) *S is the only proper essential right ideal;*

(2) *every right ideal is semisimple or a direct summand;*

(3) *if $R = A \oplus B$ for right ideals A and B, then precisely one of them is semisimple;*

(4) *${}^0S = Z \leq J < S$, and $J^2 = 0$;*

(5) *$J \leq A$ for every right ideal A not contained in S;*

(6) *R is regular if and only if $J = 0$ if and only if R is semiprime.*

Consider an idempotent g in the socle of an arbitrary ring R. It is well known that $(1-g)R \cong R$ holds if and only if $R \cong R \oplus gR$ if and only if there exist $t, t^* \in R$ such that $t^*t = 1$ and $tt^* = 1 - g$ (hence $R(1-g) \cong R$ also holds). We call t a *shift* for g.

Now assume that for every isomorphism type of indecomposable idempotents f in S, there is a representative f' for which there exists a shift. Then $R \oplus fR \cong R \oplus f'R \cong R$. It follows that R has a shift for every idempotent $e \in S$. Indeed, $eR = \bigoplus_{i=1}^n e_iR$ with e_i indecomposable, and hence

$$R \oplus eR = R \oplus e_1R \oplus \cdots \oplus e_nR \cong R.$$

Such a ring R is said to have *enough shifts.*

THEOREM 2.2. *Let R be a non-local ring with maximal socle. Then R is right pseudo-semisimple if and only if R has enough shifts.*

PROOF. From (1) and (2) of Lemma 2.1, the proof is obvious.

COROLLARY 2.3. *Let R be a ring with maximal socle. The following are equivalent*:

(1) *R is right and left pseudo-semisimple and regular*;
(2) *R is right pseudo-semisimple and $J = 0$*;
(3) *R is semiprime and has enough shifts.*

PROOF. That (1) implies (2) and that (2) implies (3) are obvious.

Assume (3). Since R is semiprime, $S^2 \neq 0$ and therefore R is non-local. Then (3) implies (1) follows from Lemma 2.1(6) and Theorem 2.2.

The next proposition effectively reduces the study of pseudo-simplicity for rings with maximal socle to the non-singular case.

PROPOSITION 2.4. *Let R be a non-local ring with maximal socle. Then R/Z is right pseudo-semisimple if and only if R is right pseudo-semisimple or $R = A \oplus B$, where A is a local ring with radical square zero and B is semisimple.*

PROOF. It is clear that R/Z is semisimple for any ring $R = A \oplus B$ as described above; the 'if' part then follows from Proposition 1.10. Conversely, assume that $\overline{R} = R/Z$ is right pseudo-semisimple. The right socle of $\overline{R}$ can either be $\overline{S}$ or $\overline{R}$.

In the first case, for any right ideal $C \nleq S$ we have $\overline{C} = \overline{cR}$ with $c^0 \leq Z$. Since $Z \leq C$ by Lemma 2.1 ((4) and (5)), we obtain $C = cR$. Also C is a direct summand of R, and hence is projective. Thus c^0 is a direct summand of R, and consequently $c^0 = 0$. Therefore $C \cong R$, and R is right pseudo-semisimple.

In the second case, we have $Z = J$ and $J^2 = 0$. Since $\overline{R}$ is semisimple, $\overline{R} = \bigoplus_{i=1}^n T_i$, where each T_i is a simple artinian ring. Then $1 = \sum_{i=1}^n e_i$ where the e_i are orthogonal idempotents of R, $\overline{e}_i$ is central in $\overline{R}$ and $\overline{e}_i\overline{R} = T_i$. Since for $i \neq j$, $e_ie_j = 0$ and R/S is a domain, all the e_i, except possibly one, are in S. We denote the exceptional one by e. By Lemma 2.1(3), $(1-e)R \leq S$ and therefore $(1-e)Re \leq Z \cap (1-e)R = 0$. Also

$$eR(1-e) \leq Z = {}^0S \leq {}^0(1-e) = Re\,,$$

and hence $eR(1-e) = 0$. Thus e is a central idempotent in R. Let $A = eRe$ and $B = (1-e)R(1-e)$. Then B is semisimple. As $A/J(A) = \overline{eR}$ is simple artinian, $J(A)$ is a maximal ideal in A. However $J(A) = A \cap J \leq A \cap S$; and so $J(A) = A \cap S$. Now $A/J(A) = A/(A \cap S) \cong A + S/S = R/S$, a division ring. It is obvious that $J(A) \neq 0$ and $J(A)^2 = 0$. Hence A is a local ring with radical-square zero.

COROLLARY 2.5. *Let R be a non-local ring with homogeneous maximal socle. Then R is right pseudo-semisimple if and only if R/Z is.*

We end this section by showing that any non-trivial right pseudo-semisimple ring R with $Z = 0$ can be embedded in one with maximal socle.

PROPOSITION 2.6. *Let R be a non-trivial right pseudo-semisimple ring with $Z = 0$. Then R is isomorphic to a subring of a right pseudo-semisimple ring R_* with $Z(R_*) = 0$ and $S(R_*)$ maximal.*

PROOF. Let $\Sigma = \{c \in R: c^0 = 0 \text{ and } cS = S\}$. Clearly Σ is multiplicatively closed and $1 \in \Sigma$. If $xc = 0$ for $x \in R$ and $c \in \Sigma$, then $xS = xcS = 0$; hence $x \in Z = 0$. Thus Σ consists of regular elements. Now we prove that Σ is a right Öre set. Let $c \in \Sigma$ and $r \in R$. If $r \in S$, then $r \in cS$; consequently $r1 = cr'$ with $r' \in R$. Assume that $r \notin S$, and let $B = \{b \in R: rb \in cR\}$. It is clear that $S \leq B$. If $S = B$, then $cR \cap rR = rB \leq S$. This implies $\overline{cR} \cap \overline{rR} = 0$ in $\overline{R}$, in contradiction of the fact that $\overline{R}$ is a principal right ideal domain. Thus $S < B$, and therefore $B = c'R$ with $c' \in \Sigma$ (see Lemma 1.4(5)). Then $rc'R \leq cR$, and hence $rc' = cr'$ for $r' \in R$.

Let $R_* = R_\Sigma$, the localization of R with respect to Σ, and identify R with its image in R_*. One can easily check that S_* is an essential right ideal in R_*, and is semisimple as a right R_*-module. Thus S_* is the right socle of R_*. We prove that S_* is a maximal right ideal. Clearly $S_* \neq R_*$. If $S_* < M$ for some right ideal M of R_*, then $M = DR_*$ for a right ideal D of R with $S < D$. Hence $D = dR$ with $d \in \Sigma$, and $M = dRR_* = dR_* = R_*$.

Next we prove that R_* is right pseudo-semisimple. Let A be a right ideal of R. Then $A \leq S$ or $A = aR$ with $a^0 = 0$. Thus $AR_* \leq S_*$ or $AR_* = aRR_* = aR_* \cong R_*$.

Let $x \in Z(R_*)$. Then $xS_* = 0$, and hence $xS = 0$. Since $x = rc^{-1}$ for some $c \in \Sigma$, $rS = xcS = xS = 0$. Thus $r \in Z = 0$, and hence $x = 0$.

REMARK. We note that in Proposition 2.6, Σ is actually the largest right Öre set of R, and hence R_* is the maximal right classical ring of fractions of R.

3. Subrings of full linear rings

If R is a ring with $Z = 0$, then the maximal right quotient ring of R is a regular right self-injective ring having R as a subring. Moreover if S is essential in R, then $Q =$ End S_R, and is therefore a product of full linear rings (compare [5, Chapter 12]); it is just one full linear ring if and only if S is homogeneous.

In this section we discuss the existence of pseudo-semisimple rings which are subrings of full linear rings; in view of Lemma 1.5(2), non-trivial examples can only occur with linear rings of infinite dimensional vector spaces.

Throughout this section, Q will stand for the endomorphism ring of a vector space V of infinite dimension over a division ring D. We shall call an element $t \in Q$ a *shift endomorphism* if it is an isomorphism onto a subspace of codimension one. For such t we choose a complement U of tV, so that $V = tV \oplus U$ and $\dim U = 1$. We define $t^* = t^{-1}$ on tV and $t^* = 0$ on U. Let $e = 1 - tt^*$. Then e is the projection onto U along tV, and hence is of rank one. A subring R of Q is said to *contain a shift*, if $t, t^* \in R$ for some shift t; it is clear that $e \in R$ and $(1-e)R \cong R$ (also $R(1-e) \cong R$).

LEMMA 3.1. *Let L be a non-zero left ideal of Q consisting of linear transformations of finite rank, and let T be a subring of Q having L as a two sided ideal. If T contains a shift and T/L is a division ring, then T is a right pseudo-semisimple ring with $S(T) = L$ and $Z(T) = 0$; moreover T is regular if and only if $\bigcap\{\ker x : x \in L\} = 0$.*

PROOF. Let f be an indecomposable idempotent in L. Since L is a left ideal in Q, f stays indecomposable in Q, and therefore fQ is a minimal right ideal. Let $ft \neq 0$ for some $t \in T$. Then there exists $q \in Q$ with $ftq = f$. Consequently $ftqf = f$ and $qf \in L \leq T$. This proves that fT is a minimal right ideal in T.

Given $x \in L$, there exists $p \in Q$ such that $xpx = x$. Then $g = px$ is an idempotent in L, and $xT = xgT \cong gT$. One may write $g = g_1 + \cdots + g_n$ where the $g_i \in Q$ are orthogonal idempotents of rank one. However $g_i = g_i g \in L$, and it follows by the preceding argument that $g_i T$ is a minimal right ideal. Therefore $xT \cong gT$ is semisimple. Hence L is contained in the right socle of T. Since T contains a shift, T is not semisimple. Then T/L is a division ring implies that L is the right socle of T.

Our argument also shows that any minimal right ideal in L is generated by a rank one idempotent. Since rank one idempotents are isomorphic in Q, they are also isomorphic in T (again since L is a left ideal in Q). Thus L is homogeneous. Let t be the given shift in T. Then it is clear that the rank one idempotent $e = 1 - tt^*$ is in L, and hence all rank one idempotents in L are isomorphic to e. Then T is right pseudo-semisimple by Theorem 2.2.

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Given $0 \neq y \in Q$, then $0 \neq yQe \leq L \leq T$ and $Qe \leq L \leq T$. Hence T_T is essential in Q_T, and therefore $Z(T) = 0$. (This also proves that Q is the maximal quotient ring of T.)

Now we prove the last statement of the theorem. In view of Lemma 1.5(2) and Lemma 2.1(6), T is regular if and only if $L^0 = 0$. Let $W = \bigcap\{\ker x : x \in L\}$. If $W \neq 0$, then there exists $q \in Q$ such that $0 \neq qeV \leq W$ (since $QeV = V$), and therefore $0 \neq qe \in L^0$; thus $L^0 \neq 0$. Conversely, assume that $W = 0$ and let $r \in L^0$. Then $rV \leq \ker x$ for every $x \in L$, and therefore $r = 0$. This proves that $L^0 = 0$ holds if and only if $W = 0$.

At this point it is convenient to discuss some examples. We start with [2, Example 4.26], which is originally due to G. M. Bergman, and represents a regular, but not unit-regular ring, in which perspectivity is transitive. This example was suggested to us by K. R. Goodearl through a communication by K. M. Rangaswami. Similar examples can be obtained from the more general construction to be discussed in Proposition 3.5.

EXAMPLE 3.2. (A regular right and left pseudo-semisimple ring which is not semisimple.) Let $V = F[[t]]$, the power series ring over a field F considered as an F-space, $Q =$ End V_F, and $F((t))$ the Laurent series ring, that is, the quotient field of $F[[t]]$. Let

$$L = \{x \in Q : \exists n \in \mathbb{N}\ (xt^n V = 0)\},$$
$$T = \{x \in Q : \exists n \in \mathbb{N},\ a \in F((t))((x - a)t^n V = 0)\}.$$

It is obvious that t is a shift and $t, t^* \in T$. One can verify that L is a left ideal of Q consisting of linear transformations of finite rank, T is a subring of Q having L as a two sided ideal and $T/L \cong F((t))$. Moreover $\bigcap\{\ker x : x \in L\} = \bigcap_{n \in \mathbb{N}} t^n V = 0$. Thus T is right pseudo-semisimple and regular by Lemma 3.1. According to Corollary 2.3, T is also left pseudo-semisimple.

EXAMPLE 3.3. (A non-singular right pseudo-semisimple ring which is not left pseudo-semisimple.) Modifying the above example by taking $V = F[[t]] \oplus F((t))$, one obtains a right pseudo-semisimple ring T with $Z(T) = 0$. However

$$J(T) = \bigcap\{\ker x : x \in L\} = F((t)).$$

Thus T is not left pseudo-semisimple in view of Corollary 2.3.

A right pseudo-semisimple ring R in which $Z \neq S$ satisfies $0 \leq Z \leq J < S$. Examples 3.2 and 3.3 correspond to the cases $0 = Z = J$ and $0 = Z < J$, respectively. Examples of the other two cases can be obtained using split extensions.

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Let A be any right pseudo-semisimple ring with $Z(A) = 0$ and $A/S(A)$ a division ring. Let $R = A \rtimes A/S(A)$. Then R has right singular ideal $0 \rtimes A/S(A)$, right socle $S(A) \rtimes A/S(A)$ and Jacobson radical $J(A) \rtimes A/S(A)$; and R is right pseudo-semisimple by Proposition 2.4.

For the case $0 < Z = J$ (respectively $0 < Z < J$), take $R = T \rtimes T/L$ where T is the ring of Example 3.2 (respectively 3.3).

Lemma 3.4. *Let $t \in Q = \operatorname{End} V_D$ be a shift. If q is a non-zero polynomial over the centre of D, then $q(t)$ has infinite rank.*

Proof. Let $q = a_m X^m + \cdots + a_n X^n$ where $m \leq n$ and $a_m \neq 0$. Without loss of generality we may assume that $a_m = 1$. Let $K = \ker q(t)$. Then clearly $t^i K \leq K$ for every $i \in \mathbb{N}$. Now

$$0 = t^{*m} q(t) K = (1 + a_{m+1} t + \cdots + a_n t^{n-m}) K,$$

and so $K \leq tK$. Thus $K = t^i K \leq t^i V$ for every $i \in \mathbb{N}$. Writing $V = tV \oplus U$, we get

$$V = t^i V \oplus t^{i-1} U \oplus \cdots \oplus tU \oplus U.$$

Therefore $K \cap \bigoplus_{i=0}^{\infty} t^i U = 0$, and hence K has infinite codimension. Thus $q(t)$ is of infinite rank.

Let A denote the prime subring of D, that is, the subring of D generated by the identity element. We shall say that the pair (t, L) is *permissible* if t is a shift endomorphism and L a non-zero left ideal of Q consisting of linear transformations of finite rank such that:

(P_1) $\forall 0 \neq q \in A[X] \forall x \in Q \ (x \in L \Leftrightarrow x q(t) \in L)$;

(P_2) $\forall 0 \neq q \in A[X] \exists y \in L \ (\ker q(t) \cap \ker y = 0)$.

Remark. One particular choice of L is the ideal consisting of all linear transformations of finite rank. For this choice, a shift $t \in Q$ is such that (t, L) is permissible if and only if $\ker q(t)$ has finite dimension and $\operatorname{Im} q(t)$ has finite codimension for all $0 \neq q \in A[X]$.

A shift t satisfying the above requirements exists in every full linear ring $Q = \operatorname{End} V_D$. Indeed $A[X]$ is countable, and therefore the central localization $D[X, X^{-1}]_*$ at the non-zero elements of $A[X]$ is a countable dimensional D-space. Consequently $V \cong D[X] \oplus D[X, X^{-1}]_*^{(\dim V)}$. Define t as componentwise multiplication by X. This yields a shift with $\ker q(t) = 0$ for every $0 \neq q \in A[X]$. Moreover, if q is of degree n, then by the Euclidean Algorithm

$$D[X] = q(X) D[X] \oplus D \oplus DX \oplus \cdots \oplus DX^{n-1}.$$

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Clearly $q[X]D[X, X^{-1}]_* = D[X, X^{-1}]_*$. Therefore $\operatorname{Im} q(t)$ has finite codimension.

The following proposition ensures that subrings as described in Lemma 3.1 exist in every infinite-dimensional full linear ring.

PROPOSITION 3.5. *Let (t, L) be a permissible pair. Then*

$$T = \{x \in Q : \exists p, 0 \neq q \in A[X] \; (xq(t) - p(t) \in L)\}$$

is a non-singular right pseudo-semisimple ring with L as its right socle and $T/L \cong A(X)$, the quotient field of $A[X]$.

PROOF. From Lemma 3.4 and condition (P_1), a routine verification establishes that T is a subring of Q, and $\varphi: x \to p/q$ is a well defined ring homomorphism of T into $A(X)$. To show that φ is surjective consider any $p/q \in A(X)$. By (P_2) we have $y \in L$ such that $\ker q(t) \cap \ker y = 0$. Hence $q(t)_{|\ker y}$ is one-to-one. Let $V = q(t)\ker y \oplus C$. Then the mapping x given by

$$x(q(t)v) = p(t)v, \qquad v \in \ker y\,; \; x|_C = 0$$

is a well defined element in Q. Let $\alpha = xq(t) - p(t)$. Then $\alpha|_{\ker y} = 0$. Write $V = \ker y \oplus W$. Since $yW \cong W$, there exists $\beta \in Q$ such that $\beta y|_W = 1$. It then follows that $\alpha = \alpha\beta y \in L$. This shows that $x \in T$ as well as $\varphi(x) = p/q$.

It is clear that $\ker\varphi = \{x \in Q : \exists 0 \neq q \in A[X] \; (xq(t) \in L)\}$. Then by (P_1), $L = \ker\varphi$. Hence L is an ideal in T and $T/L \cong A(X)$. Now $t1 - t = 0$ and $t^*t - 1 = 0$ imply that $t, t^* \in T$. The result now follows from Lemma 3.1.

THEOREM 3.6. *A ring R is a non-singular right pseudo-semisimple ring with homogeneous maximal socle if and only if*

(1) *R is a subring of a full linear ring Q,*

(2) *there exists a permissible pair (t, L) in Q with L an ideal in R and R/L a division ring, and*

(3) *the ring T corresponding to (t, L), as in Proposition 3.5, is a subring of R.*

PROOF. The 'if' part follows from Lemma 3.1 as $t, t^* \in T \leq R$.

'Only if'. Since R is right non-singular and S is homogeneous, the maximal quotient ring Q of R is a full linear ring; $Q = \operatorname{End} V_D$. Also $Q \cong \operatorname{End} S_R$, and hence S is a left ideal of Q consisting of linear transformations of finite rank. By Theorem 2.2, R has a shift for some indecomposable idempotent $e \in S$. As e is a rank one projection, t is a shift

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endomorphism in Q. We verify conditions (P_1) and (P_2) for the pair (t, S).

Let $0 \neq q \in A[X]$. Then $q(t)$ is not of finite rank by Lemma 3.4 and hence not in S. It follows by Lemma 2.1 ((2) and (3)) that $R = fR \oplus (1-f)R$ such that $q(t)R = fR$ and $1 - f \in S$. Let $f = q(t)r$, $r \in R$. Then $rq(t)$ is also an idempotent, and $rq(t) \notin S$; otherwise $q(t) = fq(t) = q(t)rq(t) \in S$, a contradiction.

For the non-trivial implication of (P_1), assume that $xq(t) \in S$ for some $x \in Q$. Then $xf = xq(t)r \in S$. Also $x(1-f) \in S$. Thus $x \in S$.

To prove (P_2), note that $1 - rq(t) \in S$ by Lemma 2.1(3). Clearly $\ker q(t) \cap \ker(1 - rq(t)) = 0$.

Now (P_1) and (P_2) being established, we may form the subring T of Q according to Proposition 3.5. Let $x \in T$. Then $xq(t) - p(t) = s \in S$, for some p, $0 \neq q \in A[X]$. With f and r as before, we obtain

$$xf = xq(t)r = (p(t) + s)r \in R,$$

and $x(1-f) \in S \leq R$. Hence $x \in R$.

Added in Proof

Using Lemma 3.1, the referee suggested the following example of a regular pseudo-semisimple ring which is not semisimple (a similar example was suggested by Mark L. Teply). Let Q be the ring of $\aleph_0 \times \aleph_0$ column-finite matrices over a field F, let $L = \operatorname{Socle} Q$ (set of matrices with a finite number of non-zero rows), and let M be the subset of Q consisting of all matrices of the form.

$$\begin{bmatrix} a_0 & a_1 & a_2 & & & & \\ b_1 & a_0 & a_1 & a_2 & \cdot & \cdot & \cdot \\ b_2 & b_1 & a_0 & a_1 & a_2 & & \\ \cdot & b_2 & b_1 & a_0 & & & \\ \cdot & & b_2 & & & & \end{bmatrix},$$

where only a finite number of the b_i are non-zero. Let $T = L + M$. It is clear that T is a ring which contains the standard shift

$$t = \begin{bmatrix} 0 & & & & & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix},$$

L is a two sided ideal in T, and $T/L \cong F((t^*))$ where

$$t^* = \begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & & \ddots \end{bmatrix}.$$

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