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Graphical trades

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Given a simple graph G, a G-trade is a pair $\{T_1, T_2\}$, where T_1 and T_2 are two entirely different G-decompositions of some graph H with no isolated vertices, with no repeated copies of G in either T_1 or in T_2. The copies of G are known as *blocks*. The order of H is the *foundation* of the trade, and the number of blocks in T_1 and in T_2 is the *volume* of the trade. If H is a simple graph, meaning that no edge occurs in more than one block of T_1 or of T_2, then the trade is *Steiner*; we assume all trades are Steiner unless otherwise stated. A K_k-trade is conventionally regarded as a $(k, 2)$ trade, which is equivalent but written in block design notation rather than graphical notation.

The trade spectrum problem for unspecified foundation is to determine, for a given graph G, the set of integers s for which there exists a G-trade of volume s. The trade spectra problem for fixed or specified foundation is to determine, for a given G and for each possible foundation v, the set of integers s for which there exists a G-trade of volume s and foundation v.

The trade spectra problem with fixed foundation has previously been solved for $(3, 2)$ trades (Bryant), C_4-trades (Bryant, Granell, Griggs and Maenhaut), and C_5-trades (Maenhaut). In Chapters 2 to 7 we completely solve the trade spectra problem with fixed foundation for $(K_4 - e)$-trades, C_6-trades, $\Theta(1, 3, 3)$-trades, $(K_3 + e)$-trades and, for every integer $n \geqslant 2$, $K_{1,n}$-trades (the graph $K_{1,n}$ is known as the n-star). We also give partial results for $(4, 2)$ trades, including a tight lower bound on the trade volume for each foundation, and, for paths of arbitrary length, we solve the trade spectrum problem for every odd foundation.

The results on $(K_4 - e)$-trades and $(4, 2)$ trades are given in Chapters 2 and 3 respectively. These chapters include lengthy non-existence proofs to establish a tight lower bound on the trade volume for each foundation, as well as iterative constructions based on a number of small trades. The necessary $(K_4 - e)$-trade examples are listed in Appendix A. If a connected graph G contains a vertex x such that $G - \{x\}$

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is a disconnected graph, then x is a *cut-vertex* of G. If G contains non-adjacent vertices x and y such that $G - \{x, y\}$ is a disconnected graph, then $\{x, y\}$ is an *independent cut-pair* of G. In Chapter 4, we give a result which provides a tight lower bound on the G-trade volume for each foundation, provided that G has a cut-vertex or an independent cut-pair. This result is applicable to C_6-trades, $\Theta(1,3,3)$-trades, and $(K_3 + e)$-trades. In Chapter 5, we complete the trade spectra problem for these graphs by giving the required constructions, together with some additional necessary conditions. The necessary trade examples are listed in Appendices B, C and D.

The lower bound result presented in Chapter 4 is also applicable to trades on stars and paths of arbitrary size; thus the results in Chapters 6 and 7, in which we respectively complete the trade spectra problem for stars and (for odd foundation) paths, are also mostly constructive. However, the star and path trades require a somewhat different approach to the earlier trade constructions, since the stars and paths are of arbitrary size. Although in general we only determine the path trade spectra for odd foundations, we also include a short construction in Chapter 7 which completes the P_4 trade spectrum for even foundations as well.

In 2002, Billington and Hoffman solved the trade spectrum problem (with unspecified foundation) for all complete multipartite graphs, except that in the case where all parts are the same size, some volumes less than 6 were unresolved. In Chapter 8, we complete this problem, by providing a construction in one case and proving non-existence in all others.

In Chapter 9, we also complete an earlier result, in this case determining the maximum volume non-Steiner $(3, 2)$ trade for each possible foundation. Khosrovshahi and Torabi solved this problem in 1999, with one exception: they did not provide a general construction when the foundation is congruent to 5 modulo 6. We give the required construction, thus completing this problem.

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