



Ghosts and Strong Ghosts in the Stable Category

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Abstract. Suppose that G is a finite group and k is a field of characteristic $p > 0$. A *ghost map* is a map in the stable category of finitely generated kG -modules which induces the zero map in Tate cohomology in all degrees. In an earlier paper we showed that the thick subcategory generated by the trivial module has no nonzero ghost maps if and only if the Sylow p -subgroup of G is cyclic of order 2 or 3. In this paper we introduce and study variations of ghost maps. In particular, we consider the behavior of ghost maps under restriction and induction functors. We find all groups satisfying a strong form of Freyd’s generating hypothesis and show that ghosts can be detected on a finite range of degrees of Tate cohomology. We also consider maps that mimic ghosts in high degrees.

1 Introduction

Suppose that G is a finite group and k is a field whose characteristic divides the order of G . A ghost map is a map between kG -modules that induces the zero map in Tate cohomology in all degrees. There is an extensive literature on ghost maps in the stable module category [1, 5, 7, 8, 10–14] and in other triangulated categories [17–20]. Most of this literature was inspired by a famous conjecture in homotopy theory due to Peter Freyd [15] which goes under the name of the generating hypothesis. This conjecture asserts that there are no nontrivial ghost maps in the category of finite spectra. In the category of spectra a ghost map is a map which induces the zero map in stable homotopy in all degrees. Although not much progress has been made on this conjecture, analogues of ghost maps and the generating hypothesis have been introduced and studied in other triangulated categories in the aforementioned papers.

Motivated by the above work, we introduce and study some variations of ghost maps in the stable module category of a modular group algebra. Our analysis includes a complete characterization of the finite groups where a strong version of the generating hypothesis holds. The relevant definitions are as follows. Throughout the paper we assume that G is a finite group and that k is a field of characteristic p dividing the order of G .

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Definition 1.1 Let M and N be finitely generated kG -modules and let $\varphi: M \rightarrow N$ be a kG -homomorphism. We say that φ is a *ghost* if it induces the zero map in Tate cohomology in all degrees. That is, for all i , the induced map $\varphi_*: \widehat{H}^i(G, M) \rightarrow \widehat{H}^i(G, N)$ is the zero map. The map φ is a *strong ghost* if it is a ghost and remains a ghost on restriction to all subgroups H of G . That is, for all i and all subgroups H of G , the induced map $\varphi_*: \widehat{H}^i(H, M_{\downarrow H}) \rightarrow \widehat{H}^i(H, N_{\downarrow H})$ is the zero map.

The map φ is an *eventual ghost* if it induces the zero map in Tate cohomology in all sufficiently large degrees. That is, φ is an eventual ghost provided there is an integer n such that $\varphi_*: \widehat{H}^i(G, M) \rightarrow \widehat{H}^i(G, N)$ is zero for all integers $i > n$.

In [7, 12] it was shown that every ghost map between kG -modules in the thick subcategory of the stable category generated by the trivial module is zero if and only if the Sylow p -subgroup of G is C_2 or C_3 . This settled Freyd's generating hypothesis for modular group algebras. As for strong ghosts, there is only one more case in which all strong ghosts vanish. A main result of this paper proves that every strong ghost is zero if and only if the Sylow p -subgroup of G is cyclic of order 2, 3, or 4. The result can be viewed as saying that a strong form of Freyd's generating hypothesis holds only for the groups mentioned. The theorem was used in the work [14], which cited an early version of this manuscript. The proof of the theorem on strong ghosts is constructive, using Auslander and Reiten's theory of almost split sequences together with standard induction and restriction methods. These results are proved in Sections 2 and 3.

In Section 4, we demonstrate that the property of being a ghost for a map $\varphi: M \rightarrow N$ is detected in a finite range of cohomology degrees, which depend on M and N . An important step in the development is a proof that the dual of any ghost is again a ghost. This also applies to strong ghosts.

In the final section of this paper we study eventual ghosts. It is clear that every ghost map is also an eventual ghost, so the converse seems to be a natural question. The answer is that this happens if and only if G has periodic cohomology. This question is related to the finite generation of Tate cohomology studied in [8].

2 Preliminaries: Ghosts Under Restriction, Induction and Duality

Throughout the paper we let G be a finite group and let k be a field of characteristic p . Recall that kG is a self-injective algebra, meaning that projective modules are injective and vice versa. The modules that we consider are all finitely generated. If M is a kG -module and $\varphi: P \rightarrow M$ is a projective cover, then the kernel of φ is denoted $\Omega(M)$. Dually, if $\theta: M \rightarrow Q$ is the injective hull of M , then the cokernel of θ is denoted $\Omega^{-1}(M)$. Inductively, we write $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ and $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{1-n}(M))$.

Most of the objects of this study reside in the *stable module category* $\text{stmod}(kG)$. It is the category whose objects are finitely generated left kG -modules. The set of morphisms between kG -modules M and N in $\text{stmod}(kG)$ is denoted $\underline{\text{Hom}}_{kG}(M, N)$. It is the quotient of the k -vector space of kG -module homomorphisms by the subspace of those maps that factor through a projective module. Thus, projective modules are zero in this category. A *stably trivial* map is a map between kG -modules which factors through a projective. The stable module category is a triangulated category in which

the triangles come from short exact sequences of kG -modules. The translation functor is Ω^{-1} . The stable module category is the natural home for Tate cohomology. A fact that we often use is that, for M a kG -module, the Tate cohomology group $\widehat{H}^i(G, M)$ is isomorphic to $\underline{\text{Hom}}_{kG}(\Omega^i(k), M)$. For more details on group cohomology and the stable category, see [6, 9].

Let H be a subgroup of a group G . There are restriction and induction functors between the corresponding stable categories: $\text{Res}_{G,H}: \text{stmod}(kG) \rightarrow \text{stmod}(kH)$ which remembers only the action of H on a module M and

$$\text{Ind}_H^G: \text{stmod}(kH) \rightarrow \text{stmod}(kG)$$

which takes a kH -module M to $M^{\uparrow G} = kG \otimes_{kH} M$. We denote the restriction of M to H by $M_{\downarrow H}$ or just M_H . The Eckmann–Shapiro lemma [6] says that these two functors are adjoint to each other. In particular, for M , a kH -module,

$$\widehat{H}^i(H, M) \cong \widehat{H}^i(G, M^{\uparrow G})$$

for all i .

We now develop some tools using the induction and restriction functors used in the proof of Theorem 3.3.

Lemma 2.1 *Let P be a Sylow p -subgroup of G . The restriction functor $\text{Res}_{G,P}$ is faithful. That is, if $\varphi: M \rightarrow N$ is a map of kG -modules such that $\text{Res}_{G,P}(\varphi): M_{\downarrow P} \rightarrow N_{\downarrow P}$ is zero in $\text{stmod}(kP)$, then φ is zero in $\text{stmod}(kG)$.*

Proof Suppose that $\text{Res}_{G,P}(\varphi)$ factors through a projective kP -module T :

$$M_{\downarrow P} \xrightarrow{\beta} T \xrightarrow{\gamma} N_{\downarrow P}.$$

Consider the diagram of kG -modules

$$M \xrightarrow{\widehat{\beta}} T^{\uparrow G} \xrightarrow{\widehat{\gamma}} N,$$

where $\widehat{\beta}$ and $\widehat{\gamma}$ are the adjoints of the maps β and γ . Let n be the index of P in G . It is easy to verify that $\widehat{\gamma}\widehat{\beta} = n\varphi$. Since P is a Sylow p -subgroup, the integer n is coprime to p , and therefore it is invertible in k . Replacing $\widehat{\beta}$ by $(1/n)\widehat{\beta}$, we get a factorization of φ through $T^{\uparrow G}$, a projective kG -module. This means φ is zero in $\text{stmod}(kG)$. ■

Corollary 2.2 *A map $\varphi: M \rightarrow N$ is a ghost whenever its restriction to a Sylow p -subgroup of G is a ghost.*

Proof Let $\varphi: M \rightarrow N$ be a map such that its restriction to a Sylow p -subgroup of G is a ghost. To show that φ is ghost, we must show that the composition

$$\Omega^i(k) \xrightarrow{f} M \xrightarrow{\varphi} N$$

is zero in the stable module category for all integers i and all f . Since restriction to a Sylow p -subgroup P was shown to be faithful, it is enough to show that the restriction of this composition to P is zero. But the latter is true because φ restricted to P is a ghost by assumption. ■

Proposition 2.3 *A map $\varphi: M \rightarrow N$ is a strong ghost if and only if it is a ghost on restriction to every p -subgroup of G .*

Proof The “only if” part is obvious from the definition. So suppose that $\text{Res}_{G,Q}(\varphi)$ is a ghost for every p -subgroup Q of G . Suppose that H is any subgroup of G and that Q is a Sylow p -subgroup of H . Since $\text{Res}_{G,Q}(\varphi) = \text{Res}_{H,Q}(\text{Res}_{G,H}(\varphi))$ is a ghost, $\text{Res}_{G,H}(\varphi)$ is a ghost by the last lemma. So φ is a strong ghost. ■

Proposition 2.4 *Suppose that H is a subgroup of G that contains a Sylow p -subgroup of G . Let $\varphi: M \rightarrow N$ be a kG -homomorphism. Then φ is a strong ghost if and only if $\text{Res}_{G,H}(\varphi)$ is a strong ghost.*

Proof The “only if” part is obvious from the definition. Suppose that $\text{Res}_{G,H}(\varphi)$ is a strong ghost. Any p -subgroup Q of G is conjugate to a subgroup of H and hence $\text{Res}_{G,Q}(\varphi)$ is a ghost by the last proposition. Therefore, again by the last proposition, φ is a strong ghost. ■

For the induction functor we get an even stronger result. We also need this in the proof of Theorem 3.3.

Proposition 2.5 *Suppose that H is a subgroup of G and that $\varphi: M \rightarrow N$ is a homomorphism of kH -modules. Then φ is a strong ghost if and only if $\text{Ind}_H^G(\varphi)$ is a strong ghost.*

Proof Let Q be a p -subgroup of G . Then by the Mackey decomposition theorem,

$$(M^{\uparrow G})_{\downarrow Q} \cong \bigoplus_{x \in Q \backslash G/H} ((x \otimes M)_{\downarrow Q \cap x H x^{-1}})^{\uparrow Q},$$

where the sum is over a set of representatives of the Q - H double cosets. Note that for $x \in G$ and $m \in M$, the map $\text{Ind}_H^G(\varphi)$ on $M^{\uparrow G}$ is given by $\varphi(x \otimes m) = x \otimes \varphi(m)$. The point of this observation is that $\text{Ind}_H^G(\varphi)$ commutes with the Mackey decomposition. Hence $\text{Res}_{G,Q}(\text{Ind}_H^G(\varphi))$ is a direct sum of maps

$$\varphi_x: ((x \otimes M)_{\downarrow Q \cap x H x^{-1}})^{\uparrow Q} \longrightarrow ((x \otimes N)_{\downarrow Q \cap x H x^{-1}})^{\uparrow Q},$$

where, again, the sum is taken over a set of representative of the Q - H -double cosets. It follows that $\text{Res}_{G,Q}(\text{Ind}_H^G(\varphi))$ is a ghost if and only if every φ_x is a ghost.

Suppose that $\text{Ind}_H^G(\varphi)$ is a strong ghost. If Q is a p -subgroup of H , then

$$\text{Res}_{G,Q}(\text{Ind}_H^G(\varphi))$$

is a ghost and φ_x is a ghost for every x . In the case that $x = 1$, we have that $\varphi_1 = \text{Res}_{H,Q}(\varphi)$ which is a ghost. Proposition 2.4 implies that φ is a strong ghost.

On the other hand, if φ is a strong ghost, then for any p -subgroup Q of G , we have that

$$\varphi_x = \text{Res}_{x H x^{-1}, Q \cap x H x^{-1}}(x \otimes \varphi): (x \otimes M)_{\downarrow Q \cap x H x^{-1}} \longrightarrow (x \otimes N)_{\downarrow Q \cap x H x^{-1}}$$

is a ghost. Then the naturality of the Eckmann–Shapiro isomorphism

$$\widehat{H}^*(Q, ((x \otimes M)_{\downarrow Q \cap xHx^{-1}})^{\uparrow Q}) \cong \widehat{H}^*(Q \cap xHx^{-1}, (x \otimes M)_{\downarrow Q \cap xHx^{-1}})$$

asserts that each φ_x is a ghost, and hence $\underline{\text{Ind}}_H^G(\varphi)$ is a strong ghost by Proposition 2.4. ■

The last two propositions give us the following corollary which is useful in Theorem 3.3.

Corollary 2.6 *Let P be a Sylow p -subgroup of a group G . The strong generating hypothesis holds for $\text{stmod}(kG)$ if and only if it holds for $\text{stmod}(kP)$.*

3 Groups With No Strong Ghosts

In this section we consider groups whose stable module categories have no strong ghosts. Recall, from [7, Theorem 1.1], that the thick subcategory of $\text{stmod}(kG)$ generated by the trivial module has no nontrivial ghosts if and only if the Sylow p -subgroup of G is C_2 or C_3 . If the Sylow p -subgroup of G is either C_2 with $p = 2$ or C_3 with $p = 3$, then every ghost is a strong ghost, and hence there are no nontrivial strong ghosts in $\text{stmod}(kG)$. So we consider C_4 .

Proposition 3.1 *Suppose that k is a field of characteristic 2. Then $\text{stmod}(kC_4)$ has no nontrivial strong ghosts.*

Proof Let $G \cong C_4$ be a cyclic group of order 4. The group algebra has exactly three isomorphism classes of indecomposable modules represented by modules M_i of dimension i for $i = 1, 2, 3$. Observe that $M_1 \cong k$, $M_2 \cong k_H^{\uparrow G}$, and $M_3 = \Omega(k)$. Here H is the subgroup of G of order 2 and k_H denotes the trivial kH -module. All three of these modules are self-dual. Clearly, no nonzero (in the stable category) map from M_i to M_j can be ghost if either i or j is 1 or 3. This follows from the definition and Proposition 4.2. Consequently, any possible nonzero strong ghost maps M_2 to itself. However, the restriction of M_2 to H is $(M_2)_{\downarrow H} \cong k_H \oplus k_H$. Because any strong ghost from M_2 to itself induces the zero map on $\widehat{H}^0(H, (M_2)_{\downarrow H})$, it is actually the zero map. ■

Now notice that in the cases examined thus far, whenever G is a p -group and $\text{stmod}(kG)$ has no strong ghosts, the only indecomposable modules are either syzygies of the trivial module or induced modules from proper subgroups. This, in fact, is the whole story.

Proposition 3.2 *Let G be a finite group. If there exists an indecomposable nonprojective kG -module M such that*

- (i) *for every nontrivial p -subgroup Q of G , Q not a Sylow p -subgroup, the module M is not a direct summand of a module induced from Q and*
- (ii) *$M \not\cong \Omega^i(k)$ for any i ,*

then there exists a nontrivial strong ghost in $\text{stmod}(kG)$.

Proof Consider the almost split sequence which ends in M [2]:

$$0 \longrightarrow \Omega^2(M) \longrightarrow X \longrightarrow M \longrightarrow 0.$$

This sequence is represented by a map $\varphi: M \rightarrow \Omega(M)$ in the stable module category. We claim that φ is a strong ghost in $\text{stmod}(kG)$. From the second condition on M , we know that φ is a nontrivial ghost in $\text{stmod}(kG)$. See [7].

The first condition on M implies that the above sequence splits on restriction to any p -subgroup Q . Let $\theta: (M_{\downarrow Q})^{\uparrow G} \rightarrow M$ be a homomorphism. This map cannot be a split epimorphism by the first condition. By the definition of an almost split sequence, there is a map μ such that the diagram

$$\begin{array}{ccccccc} & & & & (M_{\downarrow Q})^{\uparrow G} & & \\ & & & & \downarrow \theta & & \\ & & \mu & \nearrow & & & \\ 0 & \longrightarrow & \Omega^2(M) & \longrightarrow & X & \longrightarrow & M \longrightarrow 0 \end{array}$$

commutes. Hence, the map $\underline{\text{Hom}}_{kG}((M_{\downarrow Q})^{\uparrow G}, X) \rightarrow \underline{\text{Hom}}_{kG}((M_{\downarrow Q})^{\uparrow G}, M)$ is surjective. However, the Eckmann–Shapiro lemma then tells us that

$$\underline{\text{Hom}}_{kQ}(M_{\downarrow Q}, X_{\downarrow Q}) \rightarrow \underline{\text{Hom}}_{kQ}(M_{\downarrow Q}, M_{\downarrow Q})$$

is surjective and the almost split sequence splits on restriction to Q . This means that $\text{Res}_{G,Q}(\varphi) = 0$, and by Proposition 2.3, φ is a strong ghost. ■

We are now prepared to prove the main theorem of this section.

Theorem 3.3 *Let G be a finite group and k a field of characteristic p . In the stable module category $\text{stmod}(kG)$ every strong ghost is zero if and only if the Sylow p -subgroup of G is C_2 , C_3 , or C_4 .*

Proof By Lemma 2.1, Propositions 2.4 and 2.5, we may assume that G is a p -group. The “if” part is a consequence of [7, Theorem 1.1] and Proposition 3.1. By Proposition 3.2, it remains only to show that if G is a p -group that is not cyclic of order 2, 3, or 4, then G has an indecomposable nonprojective module M that is not a syzygy of the trivial module and not a direct summand of a module induced from a proper nontrivial subgroup of G . A p -group that is not cyclic of order 2, 3, or 4 belongs to exactly one of the following three disjoint cases. In each of these cases, we show that there exists a module M with the above-mentioned properties. We use the fact that direct summands of modules induced from proper subgroups have dimension divisible by p .

Assume that G is cyclic of order at least 5. In this case we let M be any indecomposable module of dimension n , where n is not 1 or $|G| - 1$ and not divisible by p . More specifically, we take $n = 2$ when p is odd and $n = 3$ when $p = 2$. Since G is cyclic and has order at least 5, the unique indecomposable module of this dimension has the desired properties.

Next, assume that G is not cyclic and has order at least 5. Consider a composition series $0 \subseteq A_1 (= k) \subseteq A_2 \subseteq \dots \subseteq A_s (= kG)$ of submodules of kG such that each

successive quotient is isomorphic to k . Such a series exists because G , being a p -group, has only one simple module, which is equal to k . Observe that since G is not cyclic, $s > p + 1$. Let $N = A_{p+1}$ and let $M = kG/N$. Note that $M^G = k$ and hence M is indecomposable, and moreover M has dimension not a multiple of p . We claim that M is not a syzygy of the trivial module. To see this, first note that a syzygy of the trivial module has dimension ± 1 modulo $|G|$, whereas the module M has dimension $-(p+1)$ modulo $|G|$. So if M is a syzygy of k , then either $-p-1-1$ or $-p$ is a multiple of $|G|$. The first possibility implies that $|G|$ divides 4, which contradicts the assumption that $|G| \geq 5$. The second possibility cannot occur, because G is not cyclic and hence it does not have order p . So we are done.

Finally, assume that G is not cyclic and has order at most 4. This means G is the Klein four group V_4 . We can take M to be any indecomposable module of dimension $2n$ for $n > 2$ (see [16]). The syzygies of the trivial kV_4 -module are all odd dimensional, so M is not one of them. In this case, proper subgroups are cyclic, there are only four isomorphism classes of modules induced from proper subgroups, and they all have dimension 2 or 4; but our module has dimension $2n$ where $n > 2$. ■

Remark 3.4. The reader might note that in [7], the main theorem characterizes the groups in which all ghosts between modules in the thick subcategory of $\text{stmod}(kG)$ generated by k are zero. This restriction to this subcategory is not necessary for strong ghosts, because as noted in Proposition 2.3, the property of being a strong ghost is detected by restrictions to p -subgroups, and for a p -group, the stable category is generated by the trivial module.

4 Cohomology in a Bounded Range Determines Ghosts

Our first proposition shows that in order to verify that a map is a ghost, it is enough to check the induced map in cohomology in finitely many degrees.

Proposition 4.1 *Let M and N be two finitely generated kG -modules. There exists a nonnegative integer d such that if $\varphi: M \rightarrow N$ is a kG -homomorphism with the property that $\widehat{H}^i(G, \varphi): \widehat{H}^i(G, M) \rightarrow \widehat{H}^i(G, N)$ is zero for all $-d \leq i \leq d$, then φ is a ghost.*

Proof Let $\mathcal{G}(M, N)$ denote the set of all ghost maps from M to N . Let

$$S_i = \{ \varphi: M \rightarrow N \mid \widehat{H}^j(G, \varphi) = 0 \text{ for all } j \text{ such that } -i \leq j \leq i \}.$$

Consider the descending sequence of subspaces $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$. This sequence stabilizes, because all the subspaces live in a finite-dimensional vector space. So there exists an integer d such that $S_d = S_{d+1} = \dots = \mathcal{G}(M, N)$. This equation is equivalent to the assertion in the the proposition. ■

Note that the integer d in the statement of this proposition depends only on M and N and not on the map φ between them.

We now prove a duality result that is used to show that ghosts can be detected in bounded non-negative degrees. Let $M^* = \text{Hom}_k(M, k)$ be the k -dual of a kG -module M . If $\varphi: M \rightarrow N$ is a kG -homomorphism, then the naturality of the functor Hom_k yields a map $\varphi^*: N^* \rightarrow M^*$.

Proposition 4.2 *The dual of a ghost is a ghost, and the dual of a strong ghost is a strong ghost.*

Proof Suppose that $\varphi: M \rightarrow N$ is a ghost. Recall that Tate duality gives a natural isomorphism $\widehat{H}^{-i-1}(G, L) \cong (\widehat{H}^i(G, L^*))^*$ for any finite-dimensional module L . Thus, for each i , we have the following commutative diagram, where the vertical maps are induced by φ :

$$\begin{array}{ccc} \widehat{H}^{-i-1}(G, M) & \xrightarrow{\cong} & (\widehat{H}^i(G, M^*))^* \\ \downarrow & & \downarrow \\ \widehat{H}^{-i-1}(G, N) & \xrightarrow{\cong} & (\widehat{H}^i(G, N^*))^* \end{array}$$

Since the two horizontal maps are isomorphisms, the right vertical map is zero, because the left vertical map is zero. Consequently, φ^* is also a ghost. The statement about strong ghosts follows from the fact that the dual operation and Tate duality commute with restriction to a subgroup. ■

The next result is a corollary of the above proof. The point is that the second condition in the corollary is equivalent (by the previous diagram) to the statement that $\widehat{H}^i(G, \varphi)$ is zero for all $i \leq 0$.

Corollary 4.3 *A map $\varphi: M \rightarrow N$ between finitely generated kG -modules is a ghost if and only if the following two conditions hold.*

- (i) $\widehat{H}^i(G, \varphi): \widehat{H}^i(G, M) \rightarrow \widehat{H}^i(G, N)$ is zero for all $i \geq 0$.
- (ii) $\widehat{H}^i(G, \varphi^*): \widehat{H}^i(G, N^*) \rightarrow \widehat{H}^i(G, M^*)$ is zero for all $i \geq 0$.

Now recall that the Evens–Venkov theorem states that for any finitely generated kG -module M , the ordinary cohomology $H^*(G, M)$ is finitely generated as a module over $H^*(G, k)$. Moreover, the ring $H^*(G, k)$ is a finitely generated k -algebra. This can be used to show that ghosts are detected on non-negative cohomology.

Theorem 4.4 *Let M and N be finitely generated kG -modules. Let positive integers m and n be the least upper bounds for the degrees of the generators of $H^*(G, M)$ and $H^*(G, N^*)$, respectively. If $\varphi: M \rightarrow N$ is any map such that $\widehat{H}^i(G, \varphi) = 0$ for $0 \leq i \leq m$ and $\widehat{H}^i(G, \varphi^*) = 0$ for $0 \leq i \leq n$, then φ is a ghost.*

Proof By Corollary 4.3, to show that φ is a ghost, it is enough to show that $\widehat{H}^i(G, \varphi)$ and $\widehat{H}^i(G, \varphi^*)$ are both zero maps for all $i \geq 0$. Since $\widehat{H}^i(G, \varphi)$ is zero for all i with $0 \leq i \leq m$, it is zero in all of the degrees where the generators are located. Thus, it is zero in all nonnegative degrees. The same also holds for $H^i(G, \varphi^*)$. ■

5 Eventual Ghosts and Groups With Periodic Cohomology

We say that a map $\varphi: M \rightarrow N$ between finitely generated kG -modules is an *eventual ghost* if there exists an integer n such that $\widehat{H}^i(G, \varphi) = 0$ for all $i \geq n$. Clearly, every ghost is also an eventual ghost. In this section we show that the converse holds if and

only if G has periodic cohomology. We begin with a lemma which gives a sufficient condition for eventual ghosts.

Lemma 5.1 *Let M be a finitely generated kG -module. Assume that $H^*(G, k)$ is generated in degrees at most d and that $H^*(G, M)$ is generated as a right $H^*(G, k)$ -module in degrees at most m . Let $\varphi: M \rightarrow N$ be a homomorphism, and suppose that for some $t > m$, $H^i(G, \varphi) = 0$ for all i such that $t + 1 \leq i \leq t + d$. Then $H^i(G, \varphi)$ is zero for all $i \geq t + 1$.*

Proof Let $\{\zeta_1, \zeta_2, \dots, \zeta_u\}$ be a set of generators in positive degrees for $H^*(G, k)$, and let $d_i \leq d$ denote the degree of ζ_i . Then for any n with $n > m$,

$$H^n(G, M) = \sum_i H^{n-d_i}(G, M)\zeta_i.$$

Taking the induced map in cohomology, we have that $H^n(G, \varphi) = \sum_i H^{n-d_i}(G, \varphi)\zeta_i$. By hypothesis $H^i(G, \varphi) = 0$ for $t + 1 \leq i \leq t + d$. Inductively, assume that $H^i(G, \varphi) = 0$ for $t + 1 \leq i < n$ and $n > t + d$. Then by the last equation, $H^n(G, \varphi) = 0$, since $n - d_i \geq n - d \geq t + 1$. Thus, the proof follows by induction. ■

Let G be a finite group and let k be a field of characteristic p . A group G is said to have periodic cohomology if there exists a class η in $H^d(G, k)$ such that for $i \geq 0$, multiplication by η gives an isomorphism $H^i(G, k) \cong H^{i+d}(G, k)$. Groups with periodic cohomology play an important role in representation theory and topology. It is well known that G has periodic cohomology if and only if the Sylow p -subgroup of G is a cyclic group or a generalized quaternion group. In [8] we proved that for every finitely generated kG -module M , the Tate cohomology $\widehat{H}^*(G, M)$ is finitely generated as a graded module over $\widehat{H}^*(G, k)$ if and only if G has periodic cohomology.

Theorem 5.2 *Let M be a finitely generated kG -module. If every eventual ghost map from M is a ghost, then $\widehat{H}^*(G, M)$ is a finitely generated module over $\widehat{H}^*(G, k)$.*

Proof Suppose that $H^*(G, k)$ is generated in degrees at most d and that $H^*(G, M)$ as a module over $H^*(G, k)$ is generated in degrees at most m . Choose a k -basis $\{\theta_j\}$ for the finite-dimensional space $V = \sum_{i=m+1}^{m+d} H^i(G, M)$. Each is represented by a cocycle $\theta_j: \Omega^{e_j}(k) \rightarrow M$, where e_j is the degree. We assemble them to form a map $\eta: \sum \Omega^{e_j}(k) \rightarrow M$, which is completed to a triangle in $\text{stmod}(kG)$

$$\bigoplus_j \Omega^{e_j}(k) \xrightarrow{\eta} M \xrightarrow{\varphi} L.$$

Because the θ_j 's generate V , the map φ has the property that $H^i(G, \varphi) = 0$ for $m + 1 \leq i \leq m + d$. Thus, by Lemma 5.1, φ is an eventual ghost, and hence a ghost.

Let γ be an arbitrary homogeneous element in $\widehat{H}^*(G, M)$ in degree t . In the diagram

$$\begin{array}{ccccc}
 \oplus_j \Omega^{e_j}(k) & \xrightarrow{\eta} & M & \xrightarrow{\varphi} & L \\
 & & \uparrow \gamma & \nearrow 0 & \\
 & & \Omega^t(k) & &
 \end{array}$$

φ is a ghost and $\varphi\gamma$ is zero. Hence γ factors through η . This shows that the classes $\{\theta_j\}$ generate $\widehat{H}^*(G, M)$ as a module over $\widehat{H}^*(G, k)$. ■

The next example shows that the converse of this theorem is not true.

Example 5.3. Let $G = C_2 \times C_2$. Consider the generator η of $\widehat{H}^{-1}(G, k)$ which is the Tate dual of the identity in $\widehat{H}^0(G, k)$. This can be represented as $\eta: \Omega^{-1}(k) \rightarrow k$. The domain of η is $\Omega^{-1}(k)$, whose Tate cohomology is just a suspension of the Tate cohomology ring $\widehat{H}^*(G, k)$. In particular, it is finitely generated over $\widehat{H}^*(G, k)$. Thus, η is an eventual ghost but not a ghost. In fact, it follows from the multiplicative structure of the Tate cohomology ring of the Klein group that $\widehat{H}^i(G, \eta)$ is nonzero only in degree 0.

Theorem 5.4 *Let G be a finite group. Then every eventual ghost map in $\text{stmod}(kG)$ is a ghost map if and only if G has periodic cohomology.*

Proof The “only if” part is clear because if G has periodic cohomology, say of period d , then we can pick d consecutive integers sufficiently large where the induced map in Tate cohomology is zero. But then periodicity of Tate cohomology implies that they induce the zero map in Tate cohomology in all degrees.

If every eventual ghost map is a ghost map, then the above theorem tells us that every finitely generated kG -module has finitely generated Tate cohomology. By [8, Theorem 4.1], G has periodic cohomology. ■

Note that in the case when G does not have periodic cohomology, this theorem helps us construct an eventual-ghost map between kG -modules that is not a ghost.

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