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Game interrupted: The rationality of considering the future

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Abstract

The "problem of points", introduced by Paccioli in 1494 and solved by Pascal and Fermat 160 years later, inspired the modern concept of probability. Incidentally, the problem also shows that rational decision-making requires the consideration of future events. We show that naïve responses to the problem of points are more future oriented and thus more rational in this sense when the problem itself is presented in a future frame instead of the canonical past frame. A simple nudge is sufficient to make decisions more rational. We consider the implications of this finding for hypothesis testing and predictions of replicability.

Keywords: rationality, probabilistic reasoning, distributive justice, framing effects.

1 Introduction

At the height of the Italian Renaissance, a Franciscan monk known as *Paccioli* (1494) published a comprehensive book on mathematics. Among other ideas (e.g., double-entry bookkeeping), Paccioli pondered the "problem of points". How should the stakes (say 20 *ducats*) be divided between two players when one has won 5 rounds of a game, where each event has an even chance of success, while the other has won 3 rounds? The players had agreed that whoever is the first to win 6 rounds gets the whole sum. When the game is interrupted by forces unrelated to the game, the question of fair division arises. Regarding an equal division indefensible, Paccioli granted each player a share corresponding to the proportion of rounds won. In his example, this decision rule allots 62.5% of the money to the leading player.

Upon reflection, this strategy is problematic. Why, for instance, should a player who is ahead 1:0 receive all the money when a player who leads 5:1 only receives 83.33%? In a famous exchange of letters, Blaise Pascal and Pierre de Fermat proposed to grant each player the proportion of the stake corresponding to the probability that this player would win the game if it were allowed to play to the agreed criterion. In Paccioli's exemplary game, the leading player receives 7/8 or 87.5% of the money. In this particular game, it is easy to see that the trailing player could win only by scoring three consecutive successes, which has a probability of $.5^3$ or .125. All other outcomes yield the leading player as the final winner. For other interrupted games, the calculations are rather difficult, which is perhaps why it took 160 years for the elegant solution to emerge. The story of the correspondence between Pascal and Fermat can be found in Devlin (2008).

The rule of dividing the stake according to the probability of winning gives each player his or her expected value at the time of interruption. If the game were to be sold to two other players, in its present state, each of the new players should be willing to pay no more than this expected value. The expected value is rational in the sense that it gives each player what the game is worth, in this sense, at the point of interruption. In addition, this principle uses all available information in a relevant way, without the sort of problems just described. This is a weak sense of rationality, because it is easy to imagine that the players could have agreed to some other rule at the outset. But they did not agree on any other rule, and this one at least has something to be said for it. We shall assume that the future-probability rule is rational henceforth.

To statisticians, the Paccioli-Pascal episode is the creation story of probability theory. To psychologists, it strikes at the core of rationality. Paccioli's question is, after all, a normative one. How *should* the money be divided? Proposing a general normative principle, Dawes (1988) suggested that rational judgment considers the future, that which has not yet occurred.[1] Rational judgment is about predictions; it draws on past events only as a source of data to make these predictions. Many irrationalities emerge, to Dawes, from the tendency of trying to understand the past in terms of a causal narrative. It is an interesting empirical question of how naïve respondents approach the problem of points. Do they show too much respect for the past within the context of this classi-

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[1]We are indebted to the late Robyn Dawes for many stimulating ideas and conversations over the years.

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cal decision task? In an initial series of studies, university undergraduates were presented with different versions of Paccioli's game. Most viewed the game as Paccioli had, with little concern for what might have lain ahead. Interestingly, their past-oriented reasoning was of a unique kind. Although respondents also showed evidence of the well-known fallacies of outcome bias and the illusion of control, these phenomena did not account for the past orientation in their distributive judgments (Krueger, 2000). The adherence to Paccioli's rule appears to be a unique and pure form of misplaced respect for the past.

Before concluding that past-oriented thinking is an autonomous property of mind, it is necessary to ask if the presentation of the game discourages future-oriented thinking. The canonical representation frames Paccioli's game with reference to the past. The decision makers are told how many rounds each player has already won, not how many more wins they would still need before winning according to the agreed criterion. This latter information is available only implicitly and requires the decision makers to do the subtractions themselves. By focusing attention on past events, the canonical presentation of the game may facilitate irrational decisions by making the past events unduly salient. Indeed, Hastie and Dawes (2010) suggested that it is the pull of perceptual salience that compromises good judgment because it detracts from important but not readily accessible information. To deconfound temporal perspective and salience, we use the canonical past-oriented frame along with a game reframed with respect to the future. When the respondent's attention is directed toward the part of the game that would yet have to be played, distributive decisions may be more in line with Pascal's decision rule. We tested this hypothesis in the experiment reported below.

The canonical representation of Paccioli's game involves a second asymmetry. Whereas Paccioli's rule calls for the calculation of a simple ratio (the number of rounds won by one player divided by the total number of rounds played), Pascal's rule requires the calculation of a probability, which involves the sums of binomial coefficients. To remove this asymmetry, we devised two additional decision rules. One rule considers only past events (i.e., completed rounds of the game) to compute the probability of the observed result (e.g., 5:3), or a result more extreme. This past-oriented probability rule recalls the logic of significance testing. The other additional rule considers the future but produces a ratio by dividing the number of wins needed by one player by the total number of wins needed by both players. (Appendix A presents the computational formulas of all four rules.) The introduction of the past-probability rule and the future-ratio rule removes the confound between the temporal frame of the rule (past vs. future) and the method of calculation (ratio vs. probability). Statistically, the confound is rather small. Over 15 incomplete games with a criterion value of 6 wins, the allotments offered by the two past-oriented rules (r = .92) and by the two future-oriented rules (r = .96) are highly correlated with each other. This quantitative alignment reinforces Dawes's (1988) view that what matters is the psychological difference between looking into the past and looking to the future.

2 Method

2.1 Participants

One hundred and twenty-five participants (mean age = 23.91 years, 46% women) completed the study online. These participants were recruited through the first author's Facebook site or Brown University e-mail lists. Another 156 participants (mean age = 19.43; 66% women) were Brown University students, who completed the study in a classroom setting.

2.2 Materials and procedure

Participants were presented with all 15 incomplete games possible with a criterion of six wins. In each game, one player was ahead by at least one round. Participants were randomly assigned to the past or the future frame of the game. In the past frame, the description noted how many rounds a player had won (Appendix B). In the future frame, the description noted how many rounds each player was short of reaching the criterion (Appendix C). The same full set of numerical information was available in both conditions. Using random.org, two sequences of the 15 games were created, and participants were randomly assigned to one of these series.

The online version of the experiment was conducted with Limesurvey™ software. When participants entered the site, they were randomly placed into one of the four conditions resulting from combining each temporal frame with each of the two random sequences of games (Appendix D). Data were collected confidentially and stored on a secure server. A cookie setting prevented repeat participation. To control for the inability to go back to the introduction page on the online version, a brief summary of the frame based on the condition to which the participant was assigned was provided at the top of each allotment page. The offline part of the study was conducted in a classroom setting. The experimenter introduced the survey and was available for questions. The dependent variable was the amount of money, with a possible range from $0.00 to $20.00, allocated to the leading player.

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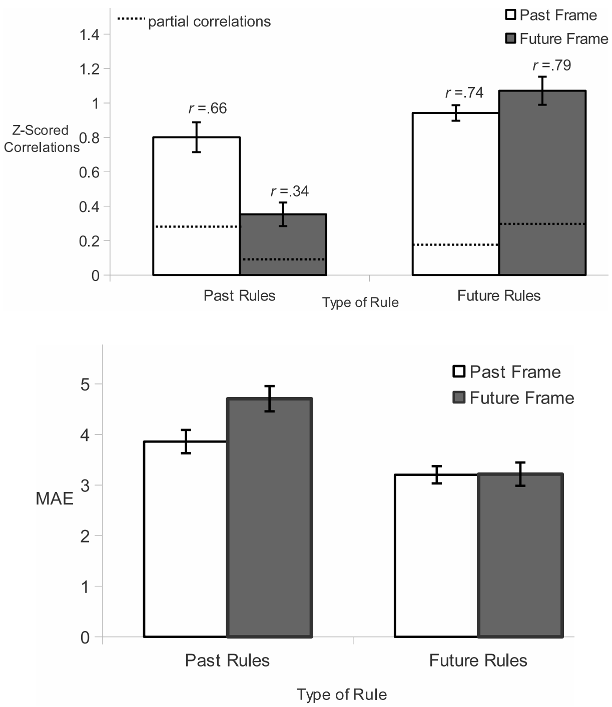
Figure 1: Measures of judgment-theory fit: Correlations (top panel) and MAE (bottom panel), pooled over method of calculation (ratio vs. probability).



3 Results

A total of 281 respondents responded to at least five of the fifteen games. The data of 59 respondents were excluded. Of these, 41 consistently allotted \$10 to each player and 10 consistently allotted \$20 to the leading player. The remaining eight respondents consistently allotted \$10, \$20, or \$0 across the number of games they answered. The effective sample size was 222. The method of data collection did not qualify the results. The mean allotments in the on- and offline versions of the study were highly correlated over the 15 incomplete games in both the past frame (r = .99) and in the future frame (r = .97). We therefore pooled the data.

For each respondent, we calculated two sets of indices to represent the alignment of their judgments with the allotments derived from each of the four decision rules. Correlations capture the profile similarity over games between allotments and theoretical values. They indicate to what extent respondents were sensitive to differences between games in way that theoretical division rules are. The other index of fit, mean absolute error (MAE), is not only sensitive to profile similarity, but also to overall tendencies of over- or under-allocating awards and to the dispersion of allotments over games (Cronbach & Gleser, 1953).

We first identified for each respondent's judgments the decision rule that provided the best fit. The decisions of 179 of the 222 participants (81%, SE = 5%) were best described by a future-oriented rule. When using correlations, we found that the majority of respondents favoring future-oriented as opposed to past-oriented rules was larger in the future frame (89%) than in the past frame (72%), χ^2(1, N = 222) = 10.36, $p < .01$, Φ = .22. The same pattern emerged for MAE. For 90% of the respondents, future-oriented rules provided the best fit in the future frame, as opposed to 67% in the past frame, χ^2(1, N = 222) = 17.06, $p < .01$, Φ = .28. When comparing ratio-based with probability-based rules, no significant interaction with temporal frame emerged when using correlations, χ^2(1, N = 222) = .512 p =. 50, Φ = .05, or MAE, χ^2(1, N = 222) = 1.35, p = .28, Φ = .08.

The means and the standard errors of the z-scored correlations are displayed in the top panel of Figure 1. A 2 (type of rule: past- vs. future-oriented) by 2 (calculation: ratio vs. probability) by 2 (frame: past vs. future) mixed-model analysis of variance (ANOVA), where the last factor varied between participants, showed main effects of rule type, F(1, 220) = 31.84, $p < .01$, η_p^2 = .13, and frame, F(1, 220) = 5.26, p = .02, η_p^2 = .02. More importantly, the critical interaction between type and frame was also significant, F(1, 220) = 14.39 $p < .01$, η_p^2 = .06. Simple comparisons revealed that past-oriented rules provided a better fit in the past frame than in the future frame F(1, 220) = 10.41, $p < .01$, but there was no significant difference in future-oriented rules between the frames F(1, 220) =. 87, p = .35. Although the three-way interaction involving the calculation method was also significant, F(1, 220) = 9.73, $p < .01$, η_p^2 = .04, we found that the critical interaction between type and frame had the same shape and was significant for both ratios and probabilities.

We computed partial correlations between each respondent's allotment decisions and individual decision rules, controlling for the three other rules. The data of four participants, who only judged five games, were excluded. The data of two participants were omitted because at least one of the correlations was unit, and thus not transformable. We replicated the ANOVA performed on the zero-order correlations and recovered the critical interaction between type and frame, F(1, 214) = 10.43, $p < .01$, η_p^2 = .05. Past-oriented rules provided a better fit in the past frame than in the future frame F(1, 214) = 4.98, p = .02, while there was no significant difference for future rules, F(1, 214) = 1.80, p = .18. The mean partial correlations are displayed as dashed lines in the top panel of Figure 1.[2]

[2] Although the findings obtained with the partial correlations are consistent with the other results, we view them with caution because some of the correlations involving partialled variables were very high.

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When evaluating fit with MAE, we found a main effect of type, $F(1, 220) = 179.84$, $p < .01$, $\eta_p^2 = .45$, and calculation, $F(1, 220) = 161.07$, $p < .01$, $\eta_p^2 = .42$. More importantly, the interaction between type and frame was again significant, $F(1, 220) = 26.94$, $p < .01$, $\eta_p^2 = .11$. As the pattern of means displayed in the bottom panel of Figure 1 shows, past-oriented rules offered a particularly poor fit with the judgments in the temporal frame with which the rules were inconsistent (i.e., in the future frame). There was no such difference for future-oriented rules. Although the type-by-frame interaction was further qualified by the method of calculation, $F(1, 220) = 18.88$, $p < .01$, $\eta_p^2 = .08$, we recovered a significant two-way interaction when considering ratio and probability rules separately. Two remaining interaction effects were statistically significant but without theoretical interest (calculation by frame: $F(1, 220) = 7.23$, $p < .01$, $\eta_p^2 = .03$; type by calculation; $F(1, 220) = 193.73$, $p < .01$, $\eta_p^2 = .47$).

4 Discussion

Our first main finding is that suggested allotments in the problem of points are overall more future- than past-oriented, thus qualifying earlier pessimism regarding rationality in this context (Krueger, 2000). Pascal's normative view of fair distribution is a better descriptor of our participants' decision than is Paccioli's. Our second main finding is that the dominance of rational decision is qualified by the presentational frame. Once the game is transformed to be viewed from a future perspective, the comparative fit for future-oriented decision rules is improved further. Interestingly, this effect occurs mainly because past-oriented rules provide a particularly poor fit with the data in the future frame. In other words, a future frame provides improvements not so much because it enhances rational decision-making but because it reduces irrational decision-making. We see this finding as being broadly consistent with Hastie and Dawes's (2010) proposal that perceptual salience plays a critical role in modulating rationality.

These general conclusions are further reinforced by the comparative lack of effects associated with the method of calculation. The distinction between ratio-based and probability-based rules appears to be a mathematical matter with little psychological consequence. When the temporal frame is held constant, the differences between the two mathematical indices small.

In psychology, rationality is often evaluated along with morality. In many strategic situations modeled by game theory, the two are at odds. It is often argued, for example, that self-interested rationality appears to demand defection in the prisoner's dilemma (Binmore, 2007), whereas social responsibility requires cooperation (Van Lange, Joireman, Parks, & Van Dijk, 2013). In contrast, Paccioli's game exemplifies a situation with no obvious conflict between rationality and morality. Here, the same arguments that support rationality (as we define it) also support morality and fairness, and players can begin to see this more readily when given a future-oriented perspective on the problem.

In contrast to a game such as the prisoner's dilemma, Paccioli's game does not afford morally motivated decision rules to demand a distribution that is more Pareto-efficient than its alternatives. Paccioli's game is zero-sum. Hence, arguments for other moral principles can be made. Our suggestion is that in such a situation, the demonstration of a rational division rule can break the impasse among moral arguments.[3]

Pascal and Fermat left a legacy to psychology. Thinking smart is more than being able to do math. Providing a future-oriented representation of a decision problem may hold promise as a general strategy to reduce the prevalence of irrational and hurtful decisions (e.g., outcome biases, Baron & Hershey, 1988; hindsight biases, Fischhoff, 2007; sunk cost biases; Arkes & Blumer, 1985). As no new information is introduced or required, a shift towards a future frame is an example of benign nudging (Thaler & Sunstein, 2008). It is these general implications that make the study of Paccioli's problem interesting and its study worthwhile. This is important because unlike many other problems, such as the prisoner's dilemma or the ultimatum game, the problem of points has few direct analogues in the social world.[4]

We close by considering two specific questions regarding the relevance and generalizability of Paccioli's game and the behavioral results it has produced. The first question is whether future-oriented reasoning is a special case of a flexible mental attitude. The answer appears to be "yes". Recall that Hastie and Dawes (2010) suggest that rational thinking (and indeed: "thinking") requires overcoming the dominance of salient stimuli and considering that which is not immediately available to the senses. A future orientation in decision-making naturally fits this definition, but a past-orientation may also incorporate

[3]Paccioli's game might be turned into an ultimatum game (Güth, Schmittberger, & Schwarze, 1982) for further study. In the ultimatum game most proposers offer more than the minimally divisible amount of the available funds. When they do this, it is difficult to distinguish morality from rationality. They may feel generous, but they may also rationally fear their offer to be rejected. In Paccioli's game, either the leading or the trailing player could make a proposal for fair division, with the other player holding veto power. Fully rational players would propose and accept Pascal's division. If we accept Pascal's solution, we must conclude that any other division underpays one of the players, which can be seen as immoral. A trailing player who rejects a Pascal portion of the stake (e.g., in the classical 5:3 game), but proposes it when in the lead, is being incoherent. He or she is also being immoral because the shift of strategy reveals greed.

[4]We do not take the view that a decision problem is important if, and only if, there is a direct applied incarnation of it.

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events that did not happen but could have, or incorporate past information that was available all along, but overlooked (e.g., base rates; Koehler 1996).

The second question is whether empirically working psychologists (or other social scientists) face Paccioli's tension between the past and the future, although in disguised form. We believe the answer to this question is also "yes". In a typical experimental study, the researcher makes a prediction about the future twice. The first prediction is explicit, as revealed by the articulation of a research hypothesis. The second prediction is implicit, consisting of the suggestion (or mere hope) that the findings will replicate. The tension is that statistical analysis is asked to speak both to the evaluation of the hypothesis (typically the rejection of a null hypothesis, which is now in the past) and to the replicability of that same evaluation (which lies in the future). The p value provided by significance testing is a poor basis for judgments of replicability (Cumming, 2008). To make a rational forecast, researchers must look beyond this salient stimulus and also consider what they regard background knowledge, such as the general plausibility or riskiness of a hypothesis (Krueger, 2001).

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Appendix A: Rule formulas

Past Ratio Rule:

$$\left(\frac{A}{A+B}\right) \times P$$

Where A is the number of wins player A has, B is the number of wins for player B, and P is the purse.

Past Probability Rule:

$$\left(1 - \left(\frac{\sum_{A+1}^{N} \binom{N}{r}}{2^N}\right)\right) \times P$$

Where $N = (A + B)$, and $A + 1$ is the number of wins player A has plus one, and P is the purse.

Future Ratio Rule:

$$\left(1 - \left(\frac{C-A}{(C-A)+(C-B)}\right)\right) \times P$$

Where A is the number of wins player A has, B is the number of wins player B has, C is the criterion and P is the purse.

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Future Probability Rule:

$$\left(1-\left(\frac{\sum_{C-B}^{N}\binom{N}{r}}{2^N}\right)\right)\times P$$

Where $N = 2C - A - B - 1$ (maximum number of coin tosses remaining). The sum ranges from the smallest number of wins $(C - B)$ to N, and P is the purse.

Appendix B: Past frame introduction

Consider the following situation: Two individuals, let's call them A and B, have found 20 dollars. Instead of splitting the money evenly, they decide to play a game of chance to see who can keep the entire amount. They agree to flip a coin and to record one point for A if it comes up Heads or record one point for B if it comes up Tails. They further agree to play until one of them has won six rounds. That player will then take the entire amount of 20 dollars.

After they have played a few rounds, but before the game is completed, the coin falls into a storm drain, and there is no replacement. The question is: how should A and B divide the $20?

Please keep this scenario in mind as you answer the following questions.

Be sure to read the directions carefully below:

In the pages that follow you will be given a series of 15 questions. Each allotment question represents a different point at which the game was interrupted.

For each question you will be given the **number of rounds each player has won at the point when the game was cut short**, in the form of **A:X----B:Y**.

For example, A:5---B:3 means player A has won 5 rounds and player B has won 3 rounds.

Please make a judgment of how much of the twenty dollars **player A** should receive for the following questions by writing down a dollar amount.

Appendix C: Future frame introduction

Consider the following situation: Two individuals, let's call them A and B, have found 20 dollars. Instead of splitting the money evenly, they decide to play a game of chance to see who can keep the entire amount. They agree to flip a coin and to record one point for A if it comes up Heads or record one point for B if it comes up Tails. They further agree to play until one of them has won six rounds. That player will then take the entire amount of 20 dollars.

After they have played a few rounds, but before the game is completed, the coin falls into a storm drain, and there is no replacement. The question is: how should A and B divide the $20?

Please keep this scenario in mind as you answer the following questions.

Be sure to read the directions carefully below:

In the pages that follow you will be given a series of 15 questions. Each allotment question represents a different point at which the game was interrupted.

For each question you will be given the **number of rounds each player still needs to reach the criterion of six when the game was cut short**, in the form of **A:X----B:Y**.

For example, A:1----B:3 means player A is **one round away from six rounds**, whereas player B is **three rounds away**.

Please make a judgment of how much of the twenty dollars **player A** should receive for the following questions by writing down a dollar amount.

Appendix D: Randomized series (numbers refer to rounds won)

Question Order; Series 1:

1. Player A = 3, Player B = 0
2. Player A = 1, Player B = 0
3. Player A = 2, Player B = 1
4. Player A = 4, Player B = 1
5. Player A = 2, Player B = 0
6. Player A = 4, Player B = 3
7. Player A = 5, Player B = 2
8. Player A = 3, Player B = 2
9. Player A = 4, Player B = 2
10. Player A = 5, Player B = 1
11. Player A = 5, Player B = 4
12. Player A = 3, Player B = 1
13. Player A = 5, Player B = 3
14. Player A = 4, Player B = 0
15. Player A = 5, Player B = 0

Question Order; Series 2:

1. Player A = 3, Player B = 2
2. Player A = 4, Player B = 0
3. Player A = 5, Player B = 3
4. Player A = 3, Player B = 1
5. Player A = 4, Player B = 3
6. Player A = 2, Player B = 0
7. Player A = 5, Player B = 0
8. Player A = 5, Player B = 4
9. Player A = 4, Player B = 2
10. Player A = 5, Player B = 2
11. Player A = 4, Player B = 1
12. Player A = 1, Player B = 0
13. Player A = 2, Player B = 1
14. Player A = 5, Player B = 1
15. Player A = 3, Player B = 0

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